**Notes on Intuitionistic Fuzzy Sets**
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# Living on the perimeter: A new directional operation on intuitionistic fuzzy sets

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**Abstract:** In the present paper, we introduce a new directional operation on intuitionistic fuzzy pairs and study some of its properties. For distinct inputs, the operation extrapolates from the first through the second to the triangle's perimeter. We provide an explicit formula and illustrate the possible geometric configurations. An accompanying interactive software application supports visualization and numerical exploration of the construction.
**Keywords:** Intuitionistic fuzzy pair, Intuitionistic fuzzy interpretational triangle.
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## 1 Introduction

Over the past almost 40 years, Intuitionistic Fuzzy Interpretational Triangle (IFIT, see, e.g., [1]) has established itself as a useful tool not only for illustrating the degrees of membership

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(validity, correctness, etc.) and of non-membership (non-validity, incorrectness, etc.), but also as a generator for the emergence of new ideas for defining operations and operators on intuitionistic fuzzy objects.

The Intuitionistic Fuzzy Pair (IFP, see [2]) is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Let us have two IFPs $A = \langle a, b\rangle$ and $B = \langle c, d\rangle$. Following [2], we define the relations

$$\begin{array}{lll} A < B & \text{if and only if} & a < c \text{ and } b > d \\ A \leq B & \text{if and only if} & a \leq c \text{ and } b \geq d \\ A > B & \text{if and only if} & a > c \text{ and } b < d \\ A \geq B & \text{if and only if} & a \geq c \text{ and } b \leq d \\ A = B & \text{if and only if} & a = c \text{ and } b = d \end{array}$$

Operations from different types are defined over two IFPs, for instance: union $A \wedge B = \langle \min(a, c), \max(b, d)\rangle$, intersection $A \vee B = \langle \max(a, c), \min(b, d)\rangle$, sum $A+B = \langle a+c-ac, bd\rangle$, multiplication $A.B = \langle ac, b + d - bd\rangle$, average $A@B = \langle \frac{a+c}{2}, \frac{b+d}{2}\rangle$, and others.

In this study, we will introduce a new operation between two IFPs, which, as a result of directed extrapolation of these IFPs, juxtaposes an IFP over the perimeter of the IFIT. This research is a continuation of a series of recent authors' investigations [3, 4] on the geometry of the intuitionistic fuzzy triangle and its application in decision making under uncertainty.

# 2 Main results

Initially, we will construct 16 IFITs corresponding to the different possible positions of two (different from each other) IFPs that do not lie on the side of the IFIT, and an operation "$*$" that matches these IFPs with the IFP that belongs to the perimeter of the IFIT (for a schematic representation, see Figure A and Figures 1–16). Then, we will show 6 cases in which both IFPs lie on one side of the IFIT (see Figures 17–22), and finally, we will show 3 cases in which the first IFP coincides with one of the vertices of the IFIT and the other IFP is inside the IFIT (see Figures 23–25).

Let $A * B = z = \langle x, y\rangle$. For each IFIT we will indicate the coordinates of the IFP $z$.

The essence of our constructions is that if $x$ is the first IFP and $y$ is the second IFP, we extend the segment from $x$ through $y$ until it reaches a side of the IFIT. The result is a new IFP on the corresponding side. The order of the IFPs $\langle x, y\rangle$ determines the direction, so there is no need for a third argument to the operation. Reversing the order of the arguments $\langle y, x\rangle$ selects the other end of the same segment.

For each configuration we will show the relationships between $a, b, c, d$.

Let us formulate the following four conditions that the constants $a, b, c, d$ must satisfy in the different cases:

$$c(1-b) > a(1-d) \tag{1}$$

$$d(1-a) > b(1-c) \tag{2}$$

$$d(1-a) \leq b(1-c) \tag{3}$$

$$c(1-b) \leq a(1-d) \tag{4}$$

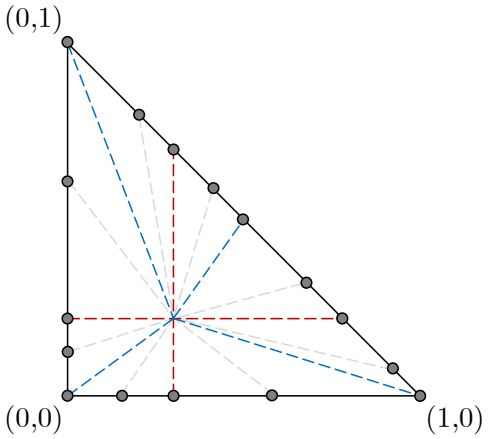



Figure A. Directional extrapolation from a fixed interior point to the boundary of the IFIT. Red lines indicate horizontal and vertical directions, blue lines indicate directions towards the vertices and the point $\langle \frac{1}{2}, \frac{1}{2} \rangle$, and grey lines illustrate other directions in between.

**Theorem 1.** *The newly defined directional operation is correct in intuitionistic fuzzy sense, i.e., the result of its application is an intuitionistic fuzzy pair.*

*Proof.* We will check the correctness of the formulas for $x$ and $y$ in the case shown in Figure 1, where both input points lie in the interior of the IFIT. First, by $(1)$, we see that

$$c - a + ad - bc = c(1-b) - a(1-d) > 0.$$

Also, $c + d - a - b > 0$, since $a + b < c + d$. Moreover,

$$\begin{aligned} c + d - a - b - c + a - ad + bc &= d - b - ad + bc \\ &= d - ad - b(1-c) \\ &> d - ad - (c + d - a)(1-c) \\ &= d - ad - c - d + a + c^2 + cd - ac \\ &= a(1-c-d) - c(1-c-d) \\ &= (a-c)(1-c-d) > 0. \end{aligned}$$

Therefore, $x \in (0,1)$. Hence, $y = 1 - x \in (0,1)$, since the formulas give $x + y = 1$.

The checks of the remaining formulas for $x$ and $y$ are similar. □

Here we will give consequently all 16 situations for the placement of IFPs $\langle a, b \rangle$ and $\langle c, d \rangle$, and the result $\langle x, y \rangle$ of the application of the directional operation over them.

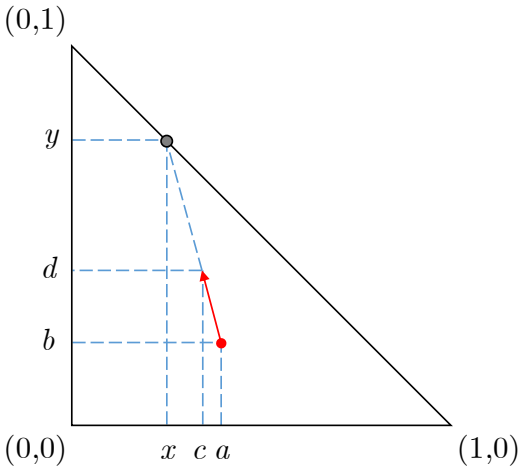



Figure 1. $a > c, b < d, a + b < c + d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

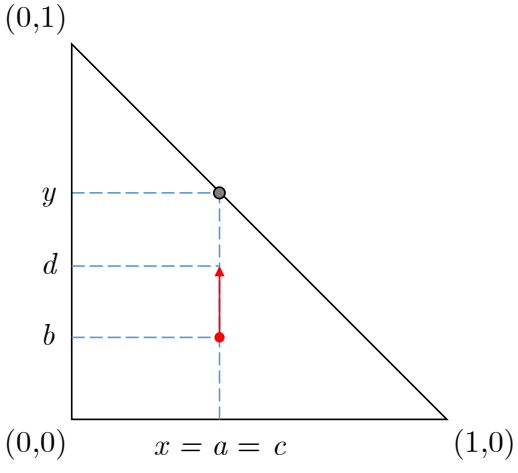



Figure 2. $a = c, b < d$, (1), (2) and $z = \langle a, 1 - a \rangle$

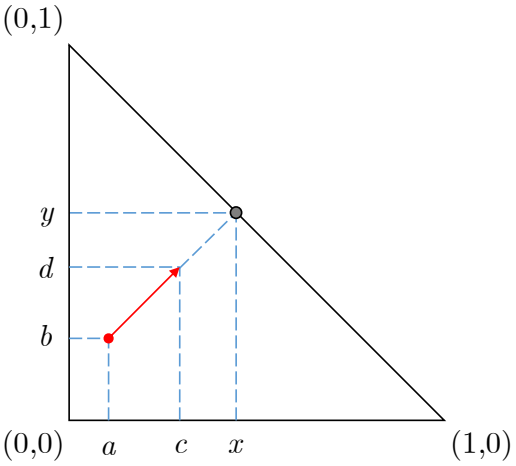



Figure 3. $a < c, b < d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

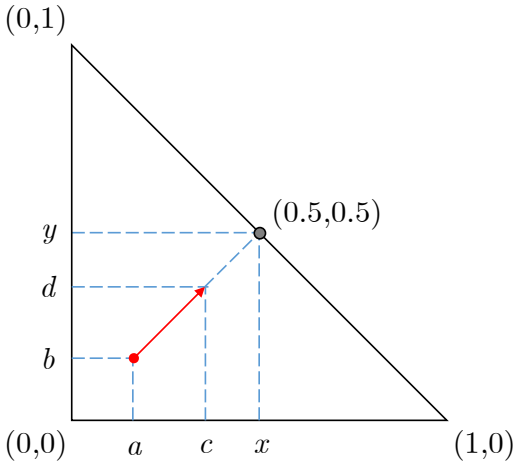



Figure 4. $a < c, b < d, a = b, c = d$, (1), (2) and $z = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$

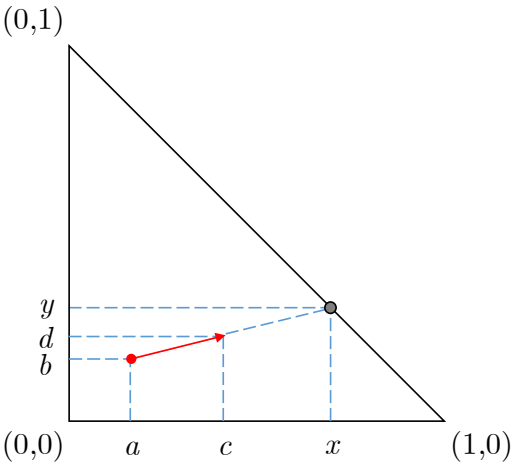



Figure 5. $a < c, b < d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

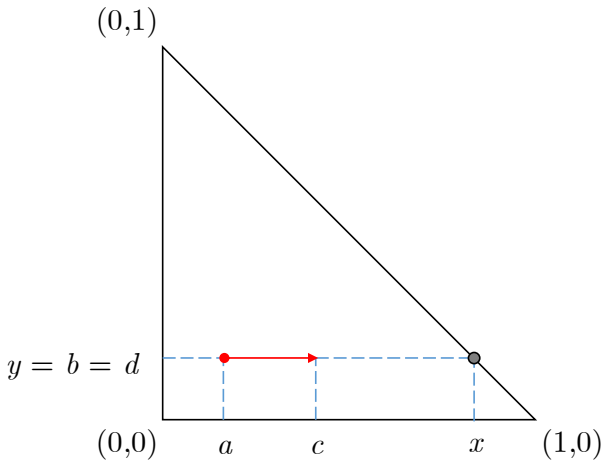



Figure 6. $a < c, b = d$, (1), (2) and $z = \langle 1 - b, b \rangle$

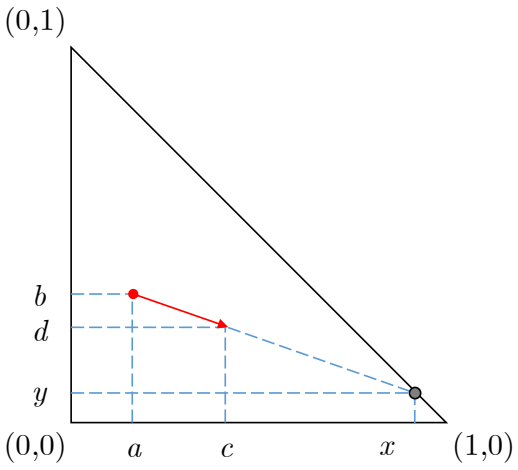



Figure 7. $a < c, b > d$, (1), (2) and $z = \left\langle \frac{c-a+ad-bc}{c+d-a-b}, \frac{d-b-ad+bc}{c+d-a-b} \right\rangle$

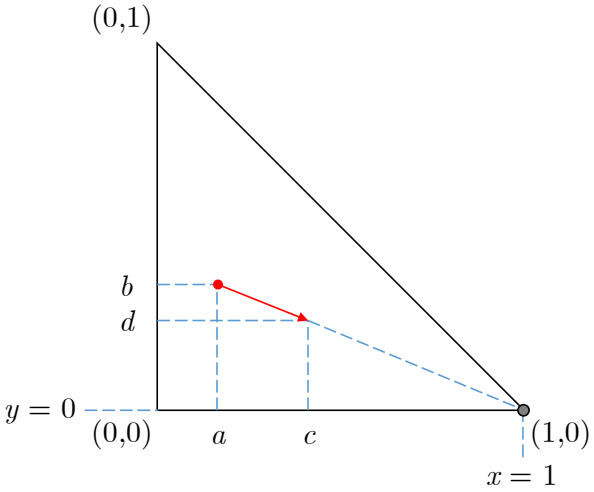



Figure 8. $a < c, b > d$, $\frac{1-a}{1-c} = \frac{b}{d}$, and $z = \langle 1, 0 \rangle$

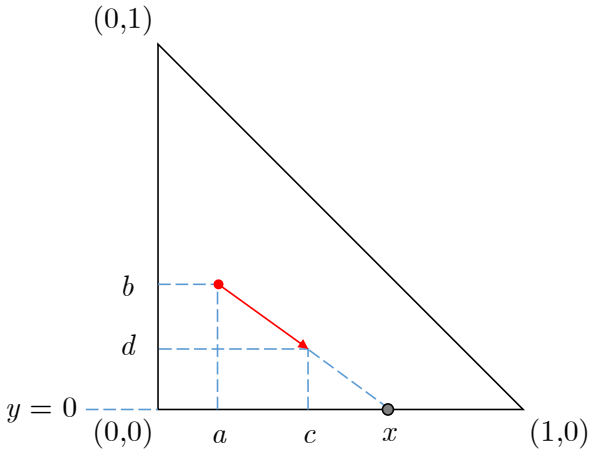



Figure 9. $a < c, b > d$, $\frac{1-a}{1-c} < \frac{b}{d}$, (3) and $z = \left\langle \frac{bc-ad}{b-d}, 0 \right\rangle$

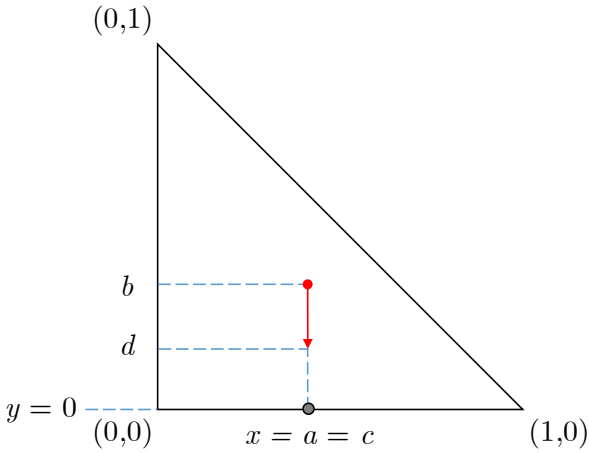



Figure 10. $a = c, b > d$, and $z = \langle a, 0 \rangle$

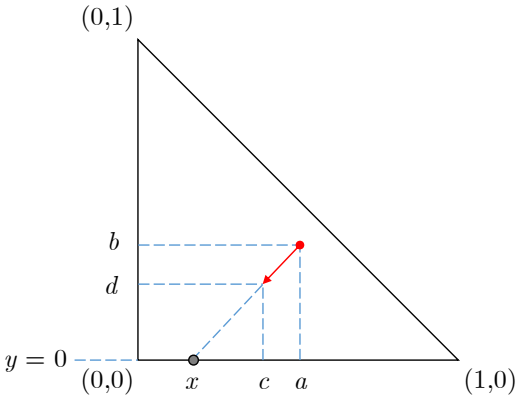



Figure 11. $a > c, b > d, bc > ad$, and $z = \left\langle \frac{bc-ad}{b-d}, 0 \right\rangle$

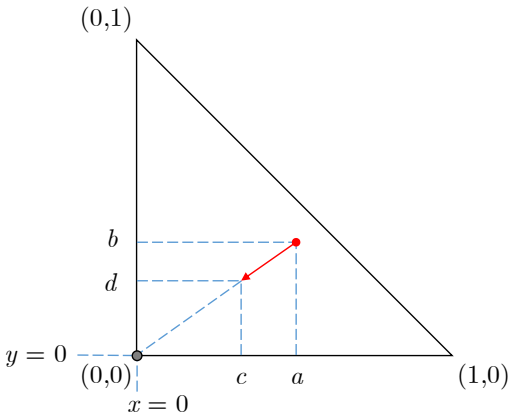



Figure 12. $a > c, b > d$, $\frac{a}{c} = \frac{b}{d}$ and $z = \langle 0, 0 \rangle$

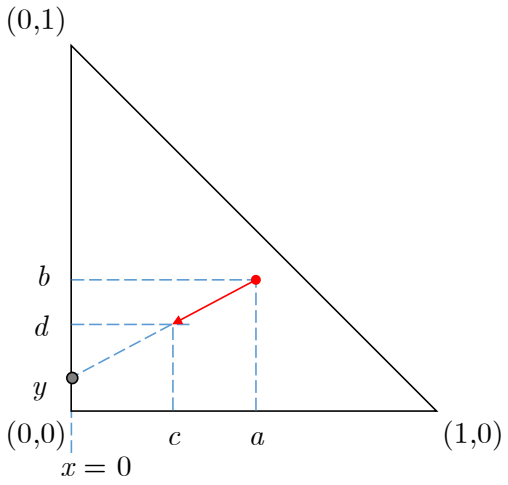



Figure 13. $a > c$, $b > d$, $ad > bc$, (4) and $z = \langle 0, \frac{ad-bc}{a-c} \rangle$

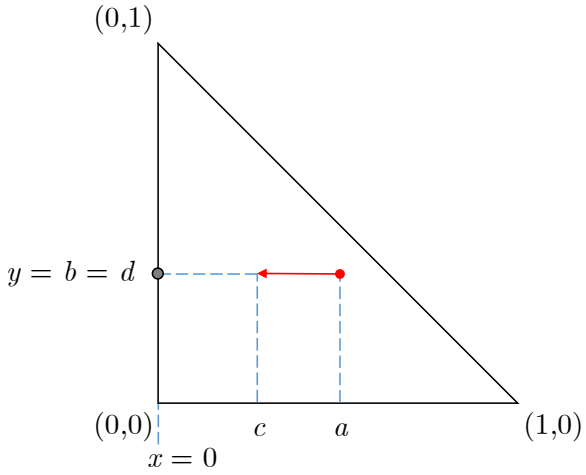



Figure 14. $a > c$, $b = d$, and $z = \langle 0, b \rangle$

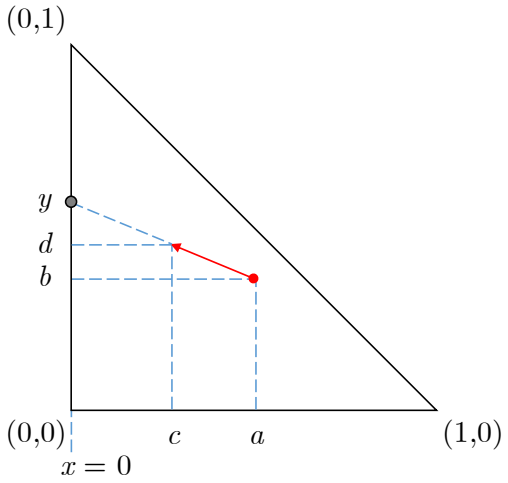



Figure 15. $a > c$, $b < d$, (4) and $z = \langle 0, \frac{ad-bc}{a-c} \rangle$

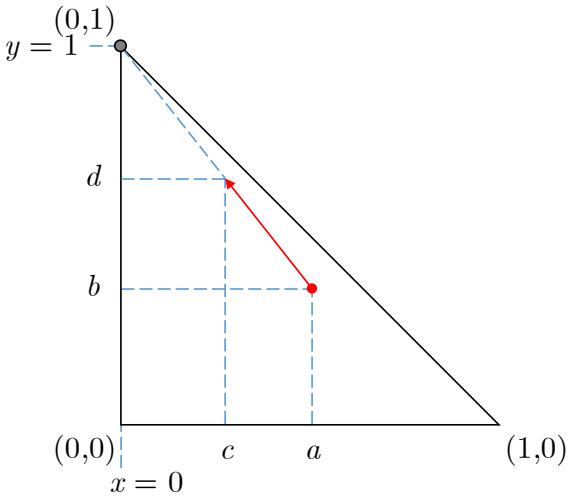



Figure 16. $a > c$, $b < d$, $\frac{a}{c} = \frac{1-b}{1-d}$ and $z = \langle 0, 1 \rangle$

The cases when the segment belongs to the IFIT's sides (respectively, hypotenuse, abscissa and ordinate) are illustrated on Figures 17–22.

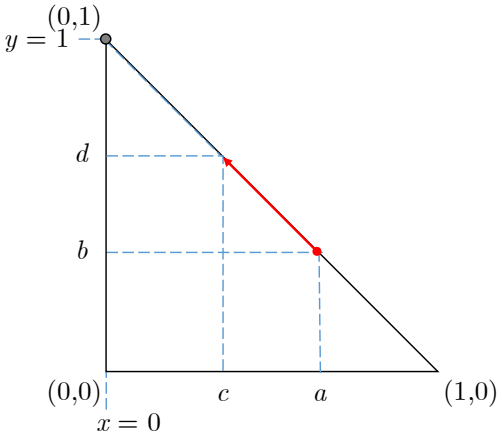



Figure 17. $a > c$, $b < d$, $a + b = c + d = 1$ and $z = \langle 0, 1 \rangle$

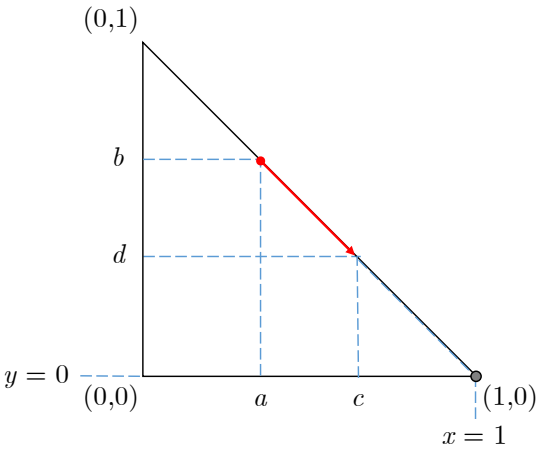



Figure 18. $a < c$, $b > d$, $a + b = c + d = 1$ and $z = \langle 1, 0 \rangle$

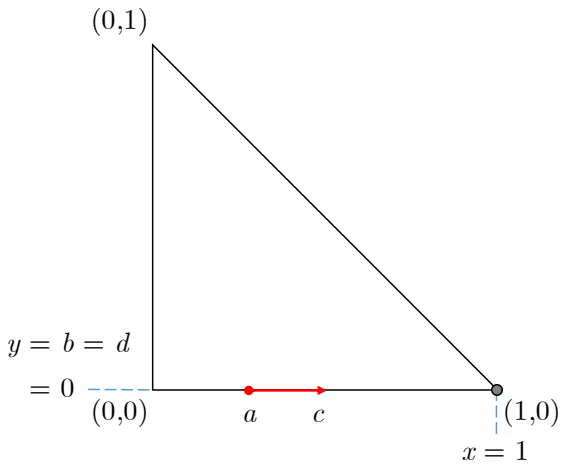



Figure 19. $a < c, b = d = 0$ and $z = \langle 1, 0 \rangle$

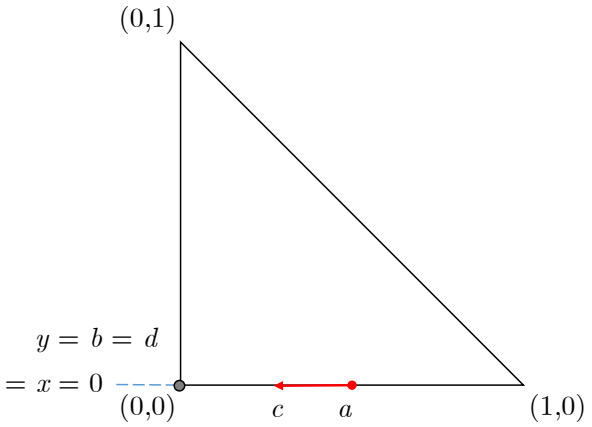



Figure 20. $a > c, b = d = 0$ and $z = \langle 0, 0 \rangle$

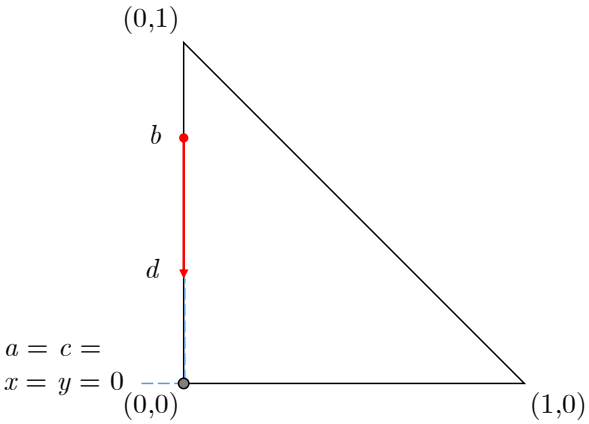



Figure 21. $a = c = 0, b > d$ and $z = \langle 0, 0 \rangle$

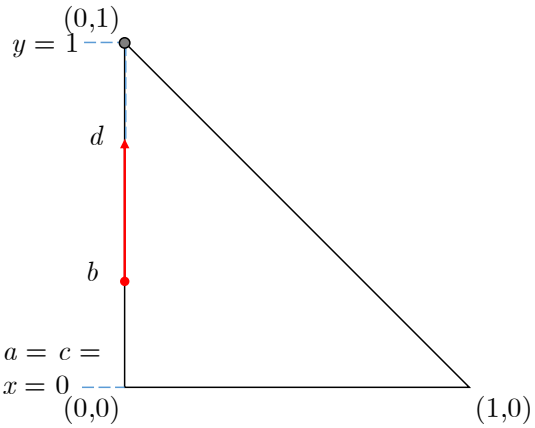



Figure 22. $a = c = 0, b < d$ and $z = \langle 0, 1 \rangle$

The cases when the segment starts from an IFIT's vertex (respectively, $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, and $\langle 0, 0 \rangle$) are illustrated on Figures 23–25.

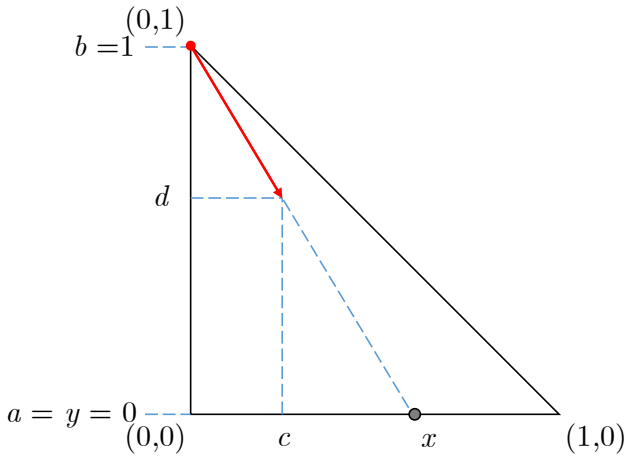



Figure 23. $a = 0, b = 1, c > 0, d > 0,$ $c + d < 1$ and $z = \left\langle \frac{c}{1-d}, 0 \right\rangle$

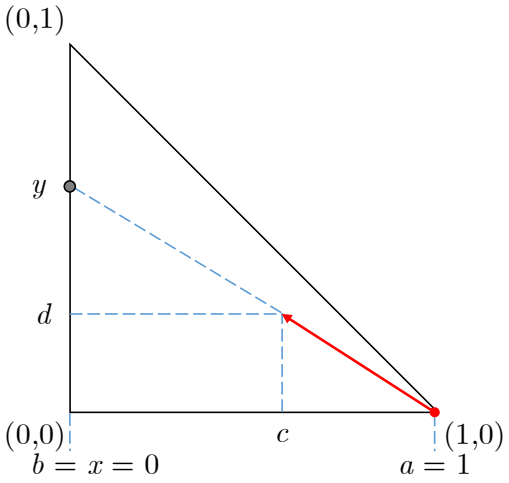



Figure 24. $a = 1, b = 0, c > 0, d > 0,$ $c + d < 1$ and $z = \left\langle 0, \frac{d}{1-c} \right\rangle$

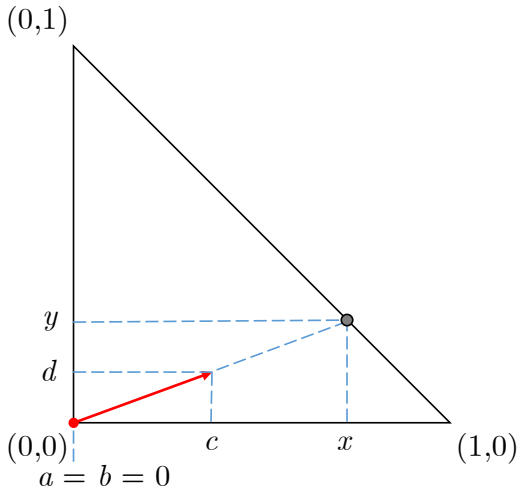



Figure 25. $a = b = 0$, $c > 0$, $d > 0$, $c + d < 1$ and $z = \left\langle \frac{c}{c+d}, \frac{d}{c+d} \right\rangle$

From these formulas, as particular cases we see that the following equalities are valid:

- if $a, b, a + b \in (0, 1)$ and $c + d = 1$, then $x * y = \langle c, d \rangle$,
- if $a, b, a + b \in (0, 1)$ and $c \leq 1, d = 0$, then $x * y = \langle c, 0 \rangle$,
- if $a, b, a + b \in (0, 1)$ and $c = 0, d \leq 1$, then $x * y = \langle 0, d \rangle$,

Finally, we must define

$$\langle a, b \rangle * \langle a, b \rangle = \langle a, b \rangle.$$

Now, we can give an explicit formula for $x * y$.

For $x = \langle a, b \rangle \neq y = \langle c, d \rangle$, put

$$s^* = \min\left( \left\{ \frac{c}{a-c} \;\middle|\; a > c \right\} \cup \left\{ \frac{d}{b-d} \;\middle|\; b > d \right\} \cup \left\{ \frac{1-c-d}{c+d-a-b} \;\middle|\; c+d > a+b \right\} \right),$$

where a term is included only when its stated condition holds. Then

$$x * y = \langle c + s^*(c-a),\ d + s^*(d-b) \rangle .$$

For coincident inputs, we define $x * x = x$.

Obviously, operation "*" is not commutative, but for it the following assertion is valid.

Recall that $\neg\langle a, b \rangle = \langle b, a \rangle$.

**Theorem 2.** *For every two IFPs $x$ and $y$:*

$$\neg(x * y) = \neg x * \neg y.$$

*Proof.* Let the two IFPs $x$ and $y$ satisfy conditions $(1)$ and $(2)$. Then

$$\begin{aligned} \neg(x * y) &= \neg(\langle a, b \rangle * \langle c, d \rangle) \\ &= \neg \left\langle \tfrac{c-a+ad-bc}{c+d-a-b}, \tfrac{d-b-ad+bc}{c+d-a-b} \right\rangle \\ &= \left\langle \tfrac{d-b-ad+bc}{c+d-a-b}, \tfrac{c-a+ad-bc}{c+d-a-b} \right\rangle \end{aligned}$$

$$= \langle b, a\rangle * \langle d, c\rangle$$
$$= \neg\langle a, b\rangle * \neg\langle c, d\rangle$$
$$= \neg x * \neg y.$$

The checks of the remaining cases are done in a similar manner.

For $x = y$, the equality follows directly from the convention $x * x = x$. □

**Remark 1.** *We should mention that among the operations listed in the Introduction, the arithmetic mean operation @ also has this property. Its formula is:*

$$x@y = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle.$$

**Remark 2.** *The proposed operation is not associative. Indeed, for* $x = \left\langle \frac{1}{5}, \frac{1}{5} \right\rangle$, $y = \left\langle \frac{1}{5}, \frac{2}{5} \right\rangle$, $w = \left\langle \frac{1}{5}, \frac{3}{5} \right\rangle$, *we obtain*

$$(x * y) * w = \left\langle \frac{1}{5}, 0 \right\rangle \neq \left\langle \frac{1}{5}, \frac{4}{5} \right\rangle = x * (y * w).$$

The directional operation extends pointwise to intuitionistic fuzzy sets. For two IFSs $A$ and $B$ on a common universe $E$, such that $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}$, then $A * B = \{\langle x, \mu_{A*B}(x), \nu_{A*B}(x)\rangle \mid x \in E\}$, where $\mu_{A*B}(x), \nu_{A*B}(x)$ are calculated according to the formulas above. The resulting pair belongs to the IFIT for every $x \in$ and lies on its perimeter whenever the two input pairs at $x$ are distinct.

# 3 Software implementation

To support the geometric exploration of the proposed operation, we developed the *Intuitionistic Fuzzy Directional Operation Explorer*, a standalone interactive application implemented in HTML, CSS and JavaScript. The application runs locally in a web browser and allows several ordered pairs of IFPs to be examined simultaneously. The input points can be specified numerically or positioned directly in the IFIT. For distinct inputs, the application displays the forward boundary result $E(P, Q) = P * Q$, the reverse result $E(Q, P) = Q * P$, their coordinates and the equation of the line through the inputs. Presets provide access to the vertices, side midpoints and centroid of the triangle.

The boundary point is calculated as the last point of the ray that remains in the IFIT, subject to the nonnegativity of the membership, non-membership and hesitation components. A synchronized linear view represents each IFP by the values $\mu$ and $1 - \nu$, whose difference is the hesitation degree $\pi$. An additional view plots these values against cumulative Euclidean distance along the trajectory. The application supports graphical export in SVG and PNG formats and numerical export in CSV format.

The development of the application also provided an opportunity to implement the billiards-inspired operators introduced in [5]. A separate mode visualizes their reflected trajectories through animation, allowing the constructions from that earlier study to be explored alongside the directional operation proposed here.

# 4 Conclusion

In this study, we introduced a new—directional—operation on intuitionistic fuzzy pairs based on extending the ray from one intuitionistic fuzzy pair through another to the boundary of the IFIT. For distinct inputs, the resulting pair has at least one zero component among its membership, non-membership and hesitation degrees. The illustrated configurations show how the relative positions and order of the inputs determine the resulting boundary point.

The operation is noncommutative and nonassociative, while it commutes with intuitionistic fuzzy negation. The accompanying software provides an interactive environment for exploring the construction and visualizing its dependence on the input pairs. Its development also enabled a software implementation of the billiards-inspired operators introduced in 2024 in [5].

Further work will investigate additional algebraic properties of the operation, its relationships with modal operators, and its connections with the geometrically motivated constructions studied in [5, 6].


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