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Intuitionistic fuzzy probability and convergence of intuitionistic fuzzy observables

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Abstract: The aim of this contribution is to define a convergence in distribution, a convergence in measure and an almost everywhere convergence with respect to an intuitionistic fuzzy probability. We prove a version of Central limit theorem, a version of Weak law of large numbers and a version of Strong law of large numbers for intuitionistic fuzzy observables with respect to the intuitionistic fuzzy probability. We study a connection between convergence of intuitionistic fuzzy observables with respect to the intuitionistic fuzzy probability and a convergence of random variables, too.
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1 Introduction

The convergence of different types of intuitionistic fuzzy observables with respect to a different types of intuitionistic fuzzy probabilities was studied by many authors. In [15] B. Riečan and K. Lendelová proved the Weak law of large numbers for intuitionistic fuzzy observables with respect to a separating intuitionistic fuzzy probability. Later in [18] the authors proved the Central

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limit theorem for separating intuitionistic fuzzy probability. In papers [12, 13] was studied a convergence of intuitionistic fuzzy observables generated by MV-algebra and the Strong law of large numbers using an intuitionistic fuzzy probability was proved. The almost everywhere convergence with respect to an intuitionistic fuzzy probability was defined in paper [8]. Another versions of Strong law of large numbers and the Central limit theorems were studied by P. Nowak and O. Hryniewicz in [16, 17]. They used a M-probability with the Zadeh connectives and an intuitionistic fuzzy probability with the Lukasiewicz connectives.

In this paper we introduce the notion of a convergence in distribution, a convergence in measure and an almost everywhere convergence with respect to an intuitionistic fuzzy probability. We prove a version of Central limit theorem, a version of Weak law of large numbers and a version of Strong law of large numbers for intuitionistic fuzzy observables with respect to the intuitionistic fuzzy probability as an example. We study a connection between convergence of intuitionistic fuzzy observables with respect the intuitionistic fuzzy probability and a convergence of random variables, too.

We remark that further in the text we use the denotation "IF" to signify the phrase "intuitionistic fuzzy".

2 Basic notions

In this section, we present the basic notions from intuitionistic fuzzy probability theory. Recall that the notion of intuitionistic fuzzy sets was introduced by K. T. Atanassov in 1983 as a generalization of Zadeh's fuzzy sets (see [1–3]).

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \le 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$. The function $\mu_A : \Omega \longrightarrow [0, 1]$ will be called the membership function and the function $\nu_A : \Omega \longrightarrow [0, 1]$ will be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)) \end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \le \mathbf{B} \Longleftrightarrow \mu_A \le \mu_B, \nu_A \ge \nu_B.$$

In paper we use max-min connectives defined by

$$\begin{aligned} \mathbf{A} \vee \mathbf{B} &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \\ \mathbf{A} \wedge \mathbf{B} &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \end{aligned}$$

and the De Morgan rules

$$(a \vee b)^* = a^* \wedge b^*,$$
$$(a \wedge b)^* = a^* \vee b^*,$$

where $a^* = 1 - a$.

Let $\mathcal{J}$ be the family of all compact intervals. The notion of an IF-probability was defined axiomatically by B. Riečan in [19].

Definition 2.3. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ is called an IF-probability, if the following conditions hold:*

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1, 1]$ *,* $\mathcal{P}((0_\Omega, 1_\Omega)) = [0, 0]$*;*

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$*, then* $\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$*;*

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$*, then* $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.
(Recall that $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ *means that* $\alpha_n \nearrow \alpha$*,* $\beta_n \nearrow \beta$*, but* $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ *means* $\mu_{A_n} \nearrow \mu_A$*,* $\nu_{A_n} \searrow \nu_A$*.)*

IF-probability $\mathcal{P}$ is called **separating**, if

$$\mathcal{P}\big((\mu_A, \nu_A)\big) = [\mathcal{P}^\flat(\mu_A), 1 - \mathcal{P}^\sharp(\nu_A)],$$

where the functions $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{T} \to [0, 1]$ are probabilities.

Of course, each $\mathcal{P}(\mathbf{A})$ is an interval, denote it by $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$. In this way we obtain two functions

$$\mathcal{P}^\flat : \mathcal{F} \to [0, 1], \mathcal{P}^\sharp : \mathcal{F} \to [0, 1]$$

and some properties of $\mathcal{P}$ can be characterized by some properties of $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see [20].

Theorem 2.4 ([20, Theorem 2.3])**.** *Let* $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ *and* $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ *for each* $\mathbf{A} \in \mathcal{F}$*. Then* $\mathcal{P}$ *is an IF-probability if and only if* $\mathcal{P}^\flat$ *and* $\mathcal{P}^\sharp$ *are IF-states.*

Recall that by an **intuitionistic fuzzy state m** we understand each mapping $\mathbf{m} : \mathcal{F} \to [0, 1]$ which satisfies the following conditions (see [21]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e., $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Now we introduce the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted by $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets (see [25]).

Definition 2.5. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) *if $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(R)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;*

(iii) *if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R)$, $n \in N$, then $x(A_n) \nearrow x(A)$.*

Similarly we can define the notion of n-dimensional IF-observable.

Definition 2.6. *By an n-dimensional IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) *if $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(R^n)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;*

(iii) *if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R^n)$, $n \in N$, then $x(A_n) \nearrow x(A)$.*

Now we explain the definition of an IF-mean value with the help of an IF-probability. First, the notion of integrable observable in the sense of IF-probability appeared in [15]. There $\mathcal{P}_x^\flat = \mathcal{P}^\flat \circ x$ and $\mathcal{P}_x^\sharp = \mathcal{P}^\sharp \circ x$.

Definition 2.7. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. An IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ is called integrable, if there exist*

$$\mathbf{E}^\flat(x) = \int_R t \, d\mathcal{P}_x^\flat(t), \ \mathbf{E}^\sharp(x) = \int_R t \, d\mathcal{P}_x^\sharp(t).$$

Later, in [13] was defined the notion of a square integrable IF-observable and the notion of IF-dispersion and IF-mean value in the sense of IF-probability.

Definition 2.8. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states $\big($i.e., $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$, $\mathbf{A} \in \mathcal{F}\big)$ and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. We say that IF-observable x is integrable, if the integrals $\int_R t \, d\mathcal{P}_x^\flat(t)$, $\int_R t \, d\mathcal{P}_x^\sharp(t)$ exist. Then the IF-mean values are defined by*

$$\mathbf{E}^\flat(x) = \int_R t \, d\mathcal{P}_x^\flat(t) \quad , \quad \mathbf{E}^\sharp(x) = \int_R t \, d\mathcal{P}_x^\sharp(t).$$

We say that IF-observable x is square integrable, if the integrals $\int_R t^2 \, d\mathcal{P}_x^\flat(t)$, $\int_R t^2 \, d\mathcal{P}_x^\sharp(t)$ exist. Then the IF-dispersions are defined by

$$\begin{aligned}
\mathbf{D}_\flat^2(x) &= \int_R (t - \mathbf{E}^\flat(x))^2 \, d\mathcal{P}_x^\flat(t) = \int_R t^2 \, d\mathcal{P}_x^\flat(t) - \big(\mathbf{E}^\flat(x)\big)^2, \\
\mathbf{D}_\sharp^2(x) &= \int_R (t - \mathbf{E}^\sharp(x))^2 \, d\mathcal{P}_x^\sharp(t) = \int_R t^2 \, d\mathcal{P}_x^\sharp(t) - \big(\mathbf{E}^\sharp(x)\big)^2.
\end{aligned}$$

Remark 2.9. *By Definition 2.8 the IF-mean value (or IF-dispersion) of IF-observable x induced by IF-probability consists of two IF-mean values (or IF-dispersions) induced by IF-state, i.e.*

$$\mathbf{E}(x) = [\mathbf{E}^{\flat}(x), \mathbf{E}^{\sharp}(x)] \ \textit{and} \ \mathbf{D}^2(x) = [\mathbf{D}^2_{\flat}(x), \mathbf{D}^2_{\sharp}(x)].$$

Similarly, we can define an IF-distribution function induced by an IF-probability, see [12].

Definition 2.10. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. Then a mapping $\mathbf{F} : R \to \mathcal{J}$ defined by formula*

$$\mathbf{F}(t) = \mathcal{P} \circ x((-\infty, t)) = \big[\mathcal{P}^{\flat}\big((x(-\infty, t))\big), \mathcal{P}^{\sharp}\big((x(-\infty, t))\big)\big] = [\mathbf{F}^{\flat}(t), \mathbf{F}^{\sharp}(t)]$$

for each $t \in R$ is called IF- distribution function, where $\mathbf{F}^{\flat}, \mathbf{F}^{\sharp} : R \to [0, 1]$ are the corresponding distribution functions.

Theorem 2.11 ([7, Theorem 4.5])**.** *For fixed IF-probability $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ define $\mathbf{F}^{\flat} : R \to [0, 1], \mathbf{F}^{\sharp} : R \to [0, 1]$ by the formulas*

$$\mathbf{F}^{\flat}(t) = \mathcal{P}^{\flat}\Big(x\big((-\infty, t)\big)\Big), \ \mathbf{F}^{\sharp}(t) = \mathcal{P}^{\sharp}\Big(x\big((-\infty, t)\big)\Big).$$

Then $\mathbf{F}^{\flat}, \mathbf{F}^{\sharp}$ are distribution functions, and

$$\mathbf{E}^{\flat}(x) = \int_R t \, d\mathbf{F}^{\flat}(t) \quad , \quad \mathbf{E}^{\sharp}(x) = \int_R t \, d\mathbf{F}^{\sharp}(t),$$
$$\mathbf{D}^2_{\flat}(x) = \int_R (t - \mathbf{E}^{\flat}(x))^2 \, d\mathbf{F}^{\flat}(t) \quad , \quad \mathbf{D}^2_{\sharp}(x) = \int_R (t - \mathbf{E}^{\sharp}(x))^2 \, d\mathbf{F}^{\sharp}(t).$$

In [14] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$.

Definition 2.12. *If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then their product $\mathbf{A} \cdot \mathbf{B}$ is defined by the formula*

$$\mathbf{A} \cdot \mathbf{B} = \big(\mu_A \cdot \mu_B, 1 - (1 - \nu_A) \cdot (1 - \nu_B)\big) = \big(\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B\big).$$

B. Riečan defined the notion of a joint IF-observable and he proved its existence, see [22].

Definition 2.13. *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\varnothing) = (0_\Omega, 1_\Omega)$;

(ii) *if $A, B \in \mathcal{B}(R^2)$ and $A \cap B = \varnothing$, then*

$$h(A \cup B) = h(A) \oplus h(B) \ \textit{and} \ h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) *if $A, A_n \in \mathcal{B}(R^2)$, $n \in N$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;*

(iv) *$h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in \mathcal{B}(R)$.*

Theorem 2.14 ([22, Theorem 3.3])**.** *For each two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF-observable.*

Remark 2.15. *The joint IF-observable of IF-observables x, y from Definition 2.13 is two-dimensional IF-observable.*

Very important notion in IF-probability theory is a notion of independence of IF-observables. First time this notion for a separating IF-probability was appear in [15]. Later B. Riečan introduced the notion of independence of IF-observables with respect to IF-probability in [23].

Definition 2.16. *Let $\mathcal{P}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states $\big($i.e., $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]\big)$. IF-observables $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent with respect to $\mathcal{P}$ if for the n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ there holds*

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \Big[\mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big)\Big]$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$. The map h_n is called the joint IF-observable of $x_1, x_2, \ldots, x_n$.

Recall that the **IF-observables** $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ **are independent with respect to IF-state** $\mathbf{m}$ if for the n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ there holds

$$\mathbf{m}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \mathbf{m}\big(x_2(A_2)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, see [11].

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. Regarding this we provide the following definition, see [6].

Definition 2.17. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n be their joint IF-observable and $g_n : R^n \to R$ be a Borel measurable function. Then we define the IF-observable $g_n(x_1, ..., x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big)$$

for each $A \in \mathcal{B}(R)$.

Example 2.18. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables and $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint IF-observable. Then*

1. *the IF-observable $y_n = g_n(x_1, \ldots, x_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} x_i - a\Big)$ is defined by the equality $y_n = h_n \circ g_n^{-1}$, where $g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} u_i - a\Big)$;*
2. *the IF-observable $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ is defined by the equality $y_n = h_n \circ g_n^{-1}$, where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$;*
3. *the IF-observable $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{n}\sum\limits_{i=1}^{n}(x_i - \mathbf{E}(x_i))$ is defined by the equality $y_n = h_n \circ g_n^{-1}$, where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n}(u_i - \mathbf{E}(x_i))$;*

4. *the IF-observable* $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ *is defined by the equality* $y_n = h_n \circ g_n^{-1}$, *where* $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big)$,

for all real numbers $u_1, \ldots, u_n$.

3 Convergence of IF-observables with respect to the IF-state

In papers [4–6, 9–11] we studied a convergence in distribution, a convergence in measure and an almost everywhere convergence with respect to the IF-state **m**.

Definition 3.1. *Let* $(y_n)_n$ *be a sequence of IF-observables in the IF-space* $(\mathcal{F}, \mathbf{m})$, *where* $\mathbf{m}$ *be an IF-state. We say that*

(i) the sequence $(y_n)_n$ *converges in distribution to a function* $\Psi : R \longrightarrow [0, 1]$, *if for each* $t \in R$

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\infty, t))\big) = \Psi(t);$$

(ii) the sequence $(y_n)_n$ *converges in measure* **m** *to* 0, *if for each* $\varepsilon > 0$, $\varepsilon \in R$

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\varepsilon, \varepsilon))\big) = 1;$$

(iii) the sequence $(y_n)_n$ *converges* $\mathbf{m}$*-almost everywhere to* 0, *if*

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathbf{m}\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

The above definition of convergences of IF-observables with respect to the IF-state **m** was inspired by classical definition for random variables.

Definition 3.2. *Let* $(\eta_n)_n$ *be a sequence of random variables in a probability space* $(\Omega, \mathcal{S}, P)$. *We say that*

(i) the sequence $(\eta_n)_n$ *converges in distribution to a function* $F : R \longrightarrow [0, 1]$, *if for each* $t \in R$

$$\lim_{n\to\infty} P\big(\eta_n^{-1}((-\infty, t))\big) = F(t);$$

(ii) the sequence $(\eta_n)_n$ *converges in measure* P *to* 0, *if for each* $\varepsilon > 0$, $\varepsilon \in R$

$$\lim_{n\to\infty} P\big(\eta_n^{-1}((-\varepsilon, \varepsilon))\big) = 1;$$

(iii) the sequence $(\eta_n)_n$ *converges* P*-almost everywhere to* 0, *if*

$$P\left(\bigcap_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1,$$

i.e.,

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

We proved the following three versions of limit theorems for independent IF-observables with respect to IF-state **m** in papers [10, 11].

Theorem 3.3 ([11, Theorem 5]). **(Central limit theorem)** *Let* $(\mathcal{F}, \mathbf{m})$ *be an IF-space,* $(x_n)_n$ *be a sequence of independent IF-observables with the same distribution* $\mathbf{m}_x$ *and such that* $\mathbf{D}^2(x_n) = \sigma^2$, $\mathbf{E}(x_n) = a, (n = 1, 2, \ldots)$ *and* $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)$. *Then for all* $t \in R$

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\infty, t))\big) = \Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u,$$

i.e., the sequence $(y_n)_n$ *converges in distribution to* $\Phi : R \longrightarrow [0, 1]$.

Theorem 3.4 ([10, Theorem 5.2]). **(Weak law of large numbers)** *Let* $(\mathcal{F}, \mathbf{m})$ *be an IF-space,* $(x_n)_n$ *be a sequence of independent IF-observables with the same distribution and such that* $\mathbf{E}(x_n) = a, (n = 1, 2, \ldots)$ *and* $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i - a$. *Then for each* $\varepsilon > 0$, $\varepsilon \in R$

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\varepsilon, \varepsilon))\big) = 1,$$

i.e., the sequence $(y_n)_n$ *converges in measure* **m** *to* 0.

Theorem 3.5 ([11, Theorem 6]). **(Strong law of large numbers)** *Let* $(\mathcal{F}, \mathbf{m})$ *be an IF-state space,* $(x_n)_n$ *be a sequence of independent IF-observables such that* $\mathbf{D}^2(x_n)$ *exists for every* $n \in N$ *and* $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2(x_n)}{n^2} < \infty$. *Then* $(y_n)_n$ *converges* $\mathbf{m}$*-almost everywhere to* 0*, i.e.,*

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} \mathbf{m}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big) = 1,$$

where $y_n = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mathbf{E}(x_i))$, $n \in \mathbf{N}$.

In paper [9] we studied a connection between a convergence of IF-observables induced by Borel measurable function and a convergence of random variables.

Proposition 3.1 ([9, Proposition 3]). *Let* $(x_n)_n$ *be a sequence of IF-observables in the IF-space* $(\mathcal{F}, \mathbf{m})$, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ *be the joint IF-observable of* $x_1, \ldots, x_n$ *and* $g_n : R^n \to R$ *be a Borel measurable function. Let IF-observable* $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ *be given by* $y_n = h_n \circ g_n^{-1}$ *and random variable* $\eta_n = g_n(t_1, \ldots, t_n) : R^N \to R$ *be defined by* $\eta_n = g_n \circ \pi_n$, *where* $\pi_n : R^N \to R^n$ *is the n-th coordinate random vector defined by* $\pi_n\big((t_n)_n\big) = (t_1, \ldots, t_n)$. *It follows that*

$$P_{\eta_n} = P \circ \eta_n^{-1} = \mathbf{m} \circ y_n = \mathbf{m}_{y_n} \text{ and}$$

(i) the sequence $(y_n)_n$ *converges in distribution to a function* F *if and only if so does the sequence* $(\eta_n)_n$*;*

(ii) the sequence $(y_n)_n$ *converges in measure* $\mathbf{m}$ *to* 0 *if and only if* $(\eta_n)_n$ *converges in measure* P *to* 0*;*

(iii) if the sequence $(\eta_n)_n$ *converges* P*-almost everywhere to* 0*, then the sequence* $(y_n)_n$ *converges* $\mathbf{m}$*-almost everywhere to* 0.

4 Convergence of IF-observables with respect to the IF-probability

In this section we study three types of convergence of IF-observables with respect to the IF-probability $\mathcal{P}$. Since the IF-probability $\mathcal{P}$ can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$ (see [20, 23]), so we can use the results from previous section.

4.1 Convergence in distribution

In this subsection we explain a convergence in distribution with respect to the IF-probability. We were inspired by definition of convergence in distribution for IF-observables with respect to the IF-state (see *Definition 3.1*).

Definition 4.1. *Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathcal{P})$, $\mathcal{P}$ be an IF-probability. We say that $(y_n)_n$ converges in distribution to a function $\Psi : R \longrightarrow [0, 1]$, if for each $t \in R$*

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \big[\Psi(t), \Psi(t)\big] = \big\{\Psi(t)\big\}.$$

Now we show a relation between a convergence in distribution with respect to the IF-probability $\mathcal{P}$ and a convergence in distribution with respect to the corresponding IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

Theorem 4.2. *A sequence $(y_n)_n$ of an IF-observables converges in distribution to a function $\Psi : R \longrightarrow [0, 1]$ with respect to the IF-probability $\mathcal{P}$ if and only if it converges in distribution to a function Ψ with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.*

Proof. "⇒" Let $\mathcal{P}$ be an IF-probability and let a sequence $(y_n)_n$ of an IF-observables converges in distribution to a function $\Psi : R \longrightarrow [0, 1]$ with respect to the IF-probability $\mathcal{P}$. Then by *Definition 4.1* we have

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \big[\Psi(t), \Psi(t)\big] = \big\{\Psi(t)\big\}$$

for each $t \in R$. Using Theorem 2.4 we obtain that

$$\mathcal{P}\big(y_n((-\infty, t))\big) = \big[\mathcal{P}^\flat\big(y_n((-\infty, t))\big), \mathcal{P}^\sharp\big(y_n((-\infty, t))\big)\big],$$

where $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are IF-states. Therefore

$$\lim_{n\to\infty} \mathcal{P}^\flat\big(y_n((-\infty, t))\big) = \Psi(t) \text{ and } \lim_{n\to\infty} \mathcal{P}^\sharp\big(y_n((-\infty, t))\big) = \Psi(t),$$

for each $t \in R$, i.e., a sequence $(y_n)_n$ of an IF-observables converges in distribution to a function Ψ with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

"⇐" The opposite direction can be proved by similar way. □

Now we present a Central limit theorem for independent IF-observables using IF-probability $\mathcal{P}$.

Theorem 4.3. (Central limit theorem) *Let $(\mathcal{F}, \mathcal{P})$ be an IF-space with IF-probability $\mathcal{P}$, $(x_n)_n$ be a sequence of independent IF-observables with the same distribution $\mathcal{P}_{x_n} = \mathcal{P} \circ x_n$ and such that $\mathbf{D}^2(x_n) = [\mathbf{D}^2_\flat(x_n), \mathbf{D}^2_\sharp(x_n)] = [\sigma^2, \sigma^2] = \{\sigma^2\}, \mathbf{E}(x_n) = [\mathbf{E}^\flat(x_n), \mathbf{E}^\sharp(x_n)] = [a, a] = \{a\}$, $n \in N$ and $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} x_i - a\Big)$. Then for all $t \in R$*

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u\right] = \left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u\right\},$$

i.e., the sequence $(y_n)_n$ converges in distribution to a function $\Psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u$.

Proof. Let $\mathcal{P}$ be an IF-probability. Then it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see *Theorem 2.4*. Hence

$$\mathcal{P}\big(y_n((-\infty, t))\big) = \big[\mathcal{P}^\flat\big(y_n((-\infty, t))\big), \mathcal{P}^\sharp\big(y_n((-\infty, t))\big)\big]. \tag{1}$$

Denote by $\Psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,\mathrm{d}u,\ t \in R$. Since a sequence $(x_n)_n$ is the sequence of independent IF-observables with respect to the IF-probability $\mathcal{P}$, then by *Definition 2.16* there exists a joint IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ of IF-observables $x_1, \ldots, x_n$ such that

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \Big[\mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big)\Big] \tag{2}$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$. But using *Theorem 2.4* we have

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \big[\mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big), \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big)\big] \tag{3}$$

Therefore by (2) and (3) we obtain

$$\begin{aligned} \mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \\ \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big), \end{aligned}$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$, i.e. the IF-observables $x_1, \ldots, x_n$ are independent with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$ in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$. Moreover

$$\mathbf{E}^\flat(x_n) = a = \mathbf{E}^\sharp(x_n), \mathbf{D}^2_\flat(x_n) = \sigma^2 = \mathbf{D}^2_\sharp(x_n)$$

and the sequence $(x_n)_n$ of independent IF-observables have the same distributions $\mathcal{P}^\flat_{x_n}$, $\mathcal{P}^\sharp_{x_n}$ in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$. Hence from a Central limit theorem for IF-state (see *Theorem 3.3*) we obtain

$$\lim_{n\to\infty} \mathcal{P}^\flat\big(y_n((-\infty, t))\big) = \Psi(t), \lim_{n\to\infty} \mathcal{P}^\sharp\big(y_n((-\infty, t))\big) = \Psi(t) \tag{4}$$

for all $t \in R$. Finally using (1) and (4) we have

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \lim_{n\to\infty} \big[\mathcal{P}^\flat\big(y_n((-\infty, t))\big), \mathcal{P}^\sharp\big(y_n((-\infty, t))\big)\big] = [\Psi(t), \Psi(t)] = \{\Psi(t)\}$$

for all $t \in R$. □

4.2 Convergence in measure

In this section we study a convergence in measure $\mathcal{P}$, where $\mathcal{P}$ is an IF-probability. Again we were inspired by definition of convergence in measure for IF-observables with respect to the IF-state (see *Definition 3.1*).

Definition 4.4. *Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathcal{P})$, $\mathcal{P}$ be an IF-probability. We say that $(y_n)_n$ converges in measure $\mathcal{P}$ to 0, if for each $\varepsilon > 0$, $\varepsilon \in R$*

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = [1, 1] = 1.$$

Theorem 4.5. *A sequence $(y_n)_n$ of an IF-observables converges in measure $\mathcal{P}$ to 0 if and only if it converges in measures $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ to 0.*

Proof. "⇒" Let $\mathcal{P}$ be an IF-probability and let a sequence $(y_n)_n$ of an IF-observables converges in measure $\mathcal{P}$ to 0. Then by *Definition 4.4* we have

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = [1, 1] = 1$$

for each $\varepsilon > 0$, $\varepsilon \in R$. Using *Theorem 2.4* we obtain that

$$\mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = \big[\mathcal{P}^\flat\big(y_n((-\varepsilon, \varepsilon))\big), \mathcal{P}^\sharp\big(y_n((-\varepsilon, \varepsilon))\big)\big],$$

where $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are IF-states. Therefore

$$\lim_{n\to\infty} \mathcal{P}^\flat\big(y_n((-\varepsilon, \varepsilon))\big) = 1 \text{ and } \lim_{n\to\infty} \mathcal{P}^\sharp\big(y_n((-\infty, t))\big) = 1,$$

for each $\varepsilon > 0$, $\varepsilon \in R$, i.e., a sequence $(y_n)_n$ of an IF-observables converges in measures $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ to 0.

"⇐" The opposite direction can be proved by similar way. □

Now we prove a Weak law of large numbers for independent IF-observables using IF-probability $\mathcal{P}$.

Theorem 4.6. (Weak law of large numbers) *Let $(\mathcal{F}, \mathcal{P})$ be an IF-space with IF-probability $\mathcal{P}$, $(x_n)_n$ be a sequence of independent IF-observables with the same distribution $\mathcal{P}_{x_n} = \mathcal{P} \circ x_n$ and such that $\mathbf{E}(x_n) = [\mathbf{E}^\flat(x_n), \mathbf{E}^\sharp(x_n)] = [a, a] = \{a\}$, $n \in N$ and $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i - a$. Then for each $\varepsilon > 0$, $\varepsilon \in R$*

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = [1, 1] = 1,$$

i.e., the sequence $(y_n)_n$ converges in measure $\mathcal{P}$ to 0.

Proof. Let $\mathcal{P}$ be an IF-probability. Then it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see *Theorem 2.4*. Hence

$$\mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = \big[\mathcal{P}^\flat\big(y_n((-\varepsilon, \varepsilon))\big), \mathcal{P}^\sharp\big(y_n((-\varepsilon, \varepsilon))\big)\big]. \tag{5}$$

Since a sequence $(x_n)_n$ is the sequence of independent IF-observables with respect to IF-probability $\mathcal{P}$, then by *Definition 2.16* there exists a joint IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ of IF-observables $x_1, \ldots, x_n$ such that

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \Big[\mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big)\Big] \quad (6)$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$. But using *Theorem 2.4* we have

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \big[\mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big), \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big)\big] \quad (7)$$

Therefore by (6) and (7) we obtain

$$\begin{aligned} \mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \\ \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big), \end{aligned}$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$, i.e. the IF-observables $x_1, \ldots, x_n$ are independent with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$ in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$.

Moreover $\mathbf{E}^\flat(x_n) = a = \mathbf{E}^\sharp(x_n)$ and the sequence $(x_n)_n$ of independent IF-observables have the same distributions $\mathcal{P}^\flat_{x_n}$, $\mathcal{P}^\sharp_{x_n}$ in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$. Hence from a Weak law of large numbers for IF-state (see *Theorem 3.4*) we obtain

$$\lim_{n\to\infty} \mathcal{P}^\flat\big(y_n((-\varepsilon, \varepsilon))\big) = 1, \lim_{n\to\infty} \mathcal{P}^\sharp\big(y_n((-\varepsilon, \varepsilon))\big) = 1 \quad (8)$$

for each $\varepsilon > 0$, $\varepsilon \in R$. Finally using (5) and (8) we have

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\varepsilon, \varepsilon))\big) = \lim_{n\to\infty} \big[\mathcal{P}^\flat\big(y_n((-\varepsilon, \varepsilon))\big), \mathcal{P}^\sharp\big(y_n((-\varepsilon, \varepsilon))\big)\big] = [1, 1] = 1$$

for each $\varepsilon > 0$, $\varepsilon \in R$. □

4.3 $\mathcal{P}$-almost everywhere convergence

In this section we study $\mathcal{P}$-almost everywhere convergence of IF-observables in IF-space $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is an IF-probability. Recall that the $\mathcal{P}$-almost everywhere convergence of IF-observables was studied in [8].

Definition 4.7. *Let $(y_n)_n$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathcal{P})$. We say that $(y_n)_n$ converges $\mathcal{P}$-almost everywhere to* 0*, if*

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathcal{P}\bigg(\bigwedge_{n=k}^{k+i} y_n\bigg(\bigg(-\frac{1}{p}, \frac{1}{p}\bigg)\bigg)\bigg) = [1, 1] = 1.$$

The following theorem says about a connection between $\mathcal{P}$-almost everywhere convergence and $\mathbf{m}$-almost everywhere convergence.

Theorem 4.8 ([8, Theorem 5])**.** *A sequence $(y_n)_n$ of an IF-observables converges $\mathcal{P}$-almost everywhere to* 0 *if and only if it converges $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere to* 0.

Now we formulate a Strong law of large numbers for independent IF-observables using IF-probability $\mathcal{P}$.

Theorem 4.9. (Strong law of large numbers) *Let* $(\mathcal{F}, \mathcal{P})$ *be an IF-space with IF-probability* $\mathcal{P}$*,* $(x_n)_n$ *be a sequence of independent IF-observables such that* $\mathbf{D}^2(x_n) = [\mathbf{D}^2_\flat(x_n), \mathbf{D}^2_\sharp(x_n)]$ *exists for every* $n \in N$ *and* $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2_\flat(x_n)}{n^2} < \infty$*,* $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2_\sharp(x_n)}{n^2} < \infty$ *and* $\mathbf{E}(x_n) = [\mathbf{E}^\flat(x_n), \mathbf{E}^\sharp(x_n)]$*,* $\mathbf{E}^\flat(x_n) = \mathbf{E}^\sharp(x_n)$*. Then a sequence* $(y_n)_n$ *of IF-observables converges* $\mathcal{P}$*-almost everywhere to* 0*, i.e.,*

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathcal{P}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = [1,1] = 1,$$

where $y_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mathbf{E}(x_i))$*,* $n \in N$*.*

Proof. Let $\mathcal{P}$ be an IF-probability. Then it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see *Theorem 2.4*. Hence

$$\mathcal{P}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = \Big[\mathcal{P}^\flat\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big), \mathcal{P}^\sharp\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big)\Big] \quad (9)$$

Since a sequence $(x_n)_n$ is the sequence of independent IF-observables with respect to IF-probability $\mathcal{P}$, then by *Definition 2.16* there exists a joint IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ of IF-observables $x_1, \ldots, x_n$ such that

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \Big[\mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big)\Big] \quad (10)$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$. But using *Theorem 2.4* we have

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \big[\mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big), \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big)\big] \quad (11)$$

Therefore by (10) and (11) we obtain

$$\begin{aligned} \mathcal{P}^\flat\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \\ \mathcal{P}^\sharp\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) &= \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big), \end{aligned}$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$, i.e. the IF-observables $x_1, \ldots, x_n$ are independent with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$ in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$.

Moreover $\mathbf{D}^2_\flat(x_n)$ and $\mathbf{D}^2_\sharp(x_n)$ exist for every $n \in N$ and $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2_\flat(x_n)}{n^2} < \infty$, $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2_\sharp(x_n)}{n^2} < \infty$, $\mathbf{E}(x_n) = \mathbf{E}^\flat(x_n) = \mathbf{E}^\sharp(x_n)$. Hence from a Strong law of large numbers for IF-state (see *Theorem 3.5*) we obtain

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathcal{P}^\flat\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = 1, \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathcal{P}^\sharp\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = 1. \quad (12)$$

for each $\varepsilon > 0$, $\varepsilon \in R$. Finally using (9) and (12) we have

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\mathcal{P}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) =$$

$$= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\left[\mathcal{P}^\flat\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big),\mathcal{P}^\sharp\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big)\right] = [1,1] = 1,$$

i.e., the sequence $(y_n)_n$ of IF-observables converges $\mathcal{P}$-almost everywhere to 0, where $y_n = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mathbf{E}(x_i))$, $n \in N$. □

4.4 Convergence of functions of several IF-observables

In this section we show a connection between a convergence of IF-observables induced by Borel measurable function with respect to the IF-probability and a convergence of random variables.

Proposition 4.1. *Let $(x_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F},\mathcal{P})$ with IF-probability $\mathcal{P}$, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be the joint IF-observable of $x_1,\dots,x_n$ and $g_n : R^n \to R$ be a Borel measurable function. Let IF-observable $y_n = g_n(x_1,\dots,x_n) : \mathcal{B}(R) \to \mathcal{F}$ be given by $y_n = h_n \circ g_n^{-1}$ and random variable $\eta_n = g_n(t_1,\dots,t_n) : R^N \to R$ be defined by $\eta_n = g_n \circ \pi_n$, where $\pi_n : R^N \to R^n$ is the n-th coordinate random vector defined by $\pi_n\big((t_n)_n\big) = (t_1,\dots,t_n)$. It follows that*

$$\big[P^\flat_{\eta_n}, P^\sharp_{\eta_n}\big] = \big[P^\flat \circ \eta_n^{-1}, P^\sharp \circ \eta_n^{-1}\big] = \mathcal{P} \circ y_n = \mathcal{P}_{y_n}$$

and

- *(i) the sequence $(y_n)_n$ converges in distribution to a function F with respect to the IF-probability $\mathcal{P}$ if and only if the sequence $(\eta_n)_n$ converges in distribution to F with respect to the probabilities $P^\flat$, $P^\sharp$;*
- *(ii) the sequence $(y_n)_n$ converges in measure $\mathcal{P}$ to 0 if and only if the sequence $(\eta_n)_n$ converges in measures $P^\flat$ and $P^\sharp$ to 0;*
- *(iii) if the sequence $(\eta_n)_n$ converges $P^\flat$-almost everywhere and $P^\sharp$-almost everywhere to 0, then the sequence $(y_n)_n$ converges $\mathcal{P}$-almost everywhere to 0.*

Proof. The proof follows from Theorem 2.4, Proposition 3.1, Theorem 4.2, Theorem 4.5 and Theorem 4.8. □

5 Conclusion

We defined a convergence in distribution, a convergence in measure with respect to the IF-probability and $\mathcal{P}$-almost everywhere convergence for IF-observables. We showed a connection between a convergence of IF-observables with respect to the IF-probability and a convergence of IF-observables with respect to the IF-state. We proved a modification of Central limit theorem, Weak law of large numbers and Strong law of large numbers for an independent sequence of IF-observables with using IF-probability. We studied a connection between convergence of IF-observables with respect to the intuitionistic fuzzy probability and a convergence of random variables, too.

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