Notes on Intuitionistic Fuzzy Sets
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About the L^p space of intuitionistic fuzzy observables

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Abstract: The aim of this paper is to define an L^p space of intuitionistic fuzzy observables. We work in an intuitionistic fuzzy space $(\mathcal{F}, \mathbf{m})$ with product, where $\mathcal{F}$ is a family of intuitionistic fuzzy events and $\mathbf{m}$ is an intuitionistic fuzzy state. We prove that the space L^p with corresponding intuitionistic fuzzy pseudometric ρ_{IF} is a pseudometric space.
Keywords: Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Joint intuitionistic fuzzy observable, Function of several intuitionistic fuzzy observables, Product, L^p space, Pseudometric space, Intuitionistic fuzzy pseudometric.
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1 Introduction

In paper [7], B. Riečan studied L^p space of fuzzy sets $\mathcal{M}$. He proved that this L^p space is a complete pseudometric space. A more general situation was studied in paper [8]. There, an L^p space was constructed for the observables of MV-algebra with product. In this case L^p is a complete pseudometric space, too.

In this paper, we define an L^p space of intuitionistic fuzzy observables and we prove that the space L^p with corresponding intuitionistic fuzzy pseudometric ρ_{IF} is a pseudometric space. Since the notion of intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$ is a generalization of the notion of random variable $\xi : \Omega \to R$ (more precisely $\xi : (\Omega, \mathcal{S}, P) \to (R, \mathcal{B}(R), P_\xi)$), we are inspired by L^p space of random variables. There

$$\int_\Omega |\xi|^p \, dP = \int_R |t|^p \, dP_\xi(t).$$

The distance in the L^p space of random variables is defined by the formula

$$\rho(\xi, \eta) = \left(\int_\Omega |\xi - \eta|^p \, dP \right)^{\frac{1}{p}} = \left(\iint_{R^2} |u - v|^p \ dP_{T(u,v)} \right)^{\frac{1}{p}},$$

where $T = (\xi, \eta) : \Omega \to R^2$, $P_T : \mathcal{B}(R^2) \to [0,1]$, $P_T(A) = P\big(T^{-1}(A)\big)$.

Remark that in the whole text we use the abbreviation “IF” for the term “intuitionistic fuzzy”.

2 Preliminaries and auxiliary notions

The year 2023 is the 40-th anniversary of the invention of the concept and theory of intuitionistic fuzzy sets by K. T. Atanassov in the paper [1]. As an *IF-set* $\mathbf{A}$ on Ω he understands a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.

In this paper we will work with a family of intuitionistic fuzzy events on $(\Omega, \mathcal{S})$ denoted by $\mathcal{F}$.

Recall that an *IF-event* is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that the functions $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable (see [3, 2]).

On this family we use the *Łukasiewicz binary operations* $\oplus, \odot$ given by

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),$$
$$\mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)),$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$. The *partial ordering* is given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In the papers [9, 11], B. Riečan defined the notion of an *IF-state* as a mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ with the following three conditions:

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e., $\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

and he defined the notion of an *IF-observable* as a mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$, where $\mathcal{B}(R)$ is a σ-algebra of the family $\mathcal{J}$ of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Similarly, we can formulate the notion of an *n-dimensional IF-observable* as a mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$ with the following conditions:

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$ for each $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$, we simply say that x is an IF-observable.

Remark that the composition of an IF-state $\mathbf{m}$ and an IF-observable x is a probability measure denoted $\mathbf{m}_x$, i.e., $\mathbf{m}_x(C) = \mathbf{m}(x(C))$ for each $C \in \mathcal{B}(R)$.

In [10], B. Riečan defined the notion of a joint IF-observable and proved its existence. The *joint IF-observable of the IF-observables* x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A, B \in \mathcal{B}(R^2)$ and $A \cap B = \emptyset$, then

$$h(A \cup B) = h(A) \oplus h(B) \text{ and } h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) if $A, A_n \in \mathcal{B}(R^2)$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in \mathcal{B}(R)$.

There $\cdot$ is a product operation on the family of IF-events $\mathcal{F}$ introduced in [6]. It is defined by

$$\mathbf{A} \cdot \mathbf{B} = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$$

for each $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$.

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. Regarding this, we provide the following definition, see [5].

Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n their joint IF-observable and $g_n : R^n \to R$ a Borel measurable function. Then we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

3 L^p space of IF-observables

In this section, we formulate L^p space of IF-observables. We can consider an IF-observable x instead of a random variable and a joint IF-observable h instead of a random vector.

Definition 3.1. *Fix a real number $p \geq 1$. Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with product. We say that an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ belongs to $L^p_{\mathbf{m}}$ if there exists the integral*

$$\int_R |t|^p \, d\mathbf{m}_x(t).$$

If $x, y : \mathcal{B}(R) \to \mathcal{F}$ are the IF-observables and $h_{xy} : \mathcal{B}(R^2) \to \mathcal{F}$ is their joint IF-observable, then we define the IF-observable $x - y : \mathcal{B}(R) \to \mathcal{F}$ by the formula

$$(x - y)(A) = h_{xy}\big(g^{-1}(A)\big)$$

for each $A \in \mathcal{B}(R)$, where $g : R^2 \to R$ is a Borel measurable function defined by $g(u, v) = u - v$.

Proposition 3.1. *Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with product. If the IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ are in $L^p_{\mathbf{m}}$, then the IF-observable $x - y : \mathcal{B}(R) \to \mathcal{F}$ is in $L^p_{\mathbf{m}}$.*

Proof. From definition of IF-observable $x - y$ we have

$$(x - y)(A) = h_{xy}\big(g^{-1}(A)\big)$$

for each $A \in \mathcal{B}(R)$, where $g(u, v) = u - v$ and h_{xy} is the joint IF-observable of IF-observables x, y.

Consider the probability space $(R^2, \mathcal{B}(R), P = \mathbf{m} \circ h_{xy})$ and the random variables $\xi, \eta : R^2 \to R$ defined by

$$\xi(u, v) = u, \quad \eta(u, v) = v.$$

Evidently,

$$\begin{aligned} P_\xi(A) &= P\big(\xi^{-1}(A)\big) \\ &= \mathbf{m} \circ h_{xy}\big(\xi^{-1}(A)\big) \\ &= \mathbf{m}\big(h_{xy}(A \times R)\big) \\ &= \mathbf{m}\big(x(A) \cdot y(R)\big) \\ &= \mathbf{m}\big(x(A) \cdot (1_\Omega, 0_\Omega)\big) \\ &= \mathbf{m}\big(x(A)\big) \\ &= \mathbf{m}_x(A) \end{aligned} \tag{1}$$

and

$$\begin{aligned} P_\eta(A) &= P\big(\eta^{-1}(A)\big) \\ &= \mathbf{m} \circ h_{xy}\big(\eta^{-1}(A)\big) \\ &= \mathbf{m}\big(h_{xy}(R \times A)\big) \\ &= \mathbf{m}\big(x(R) \cdot y(A)\big) \\ &= \mathbf{m}\big((1_\Omega, 0_\Omega) \cdot y(A)\big) \\ &= \mathbf{m}\big(y(A)\big) \\ &= \mathbf{m}_y(A). \end{aligned} \tag{2}$$

Since $x, y \in L^p_{\mathbf{m}}$, i.e., the integrals $\int_R |t|^p \, d\mathbf{m}_x(t)$, $\int_R |t|^p \, d\mathbf{m}_y(t)$ exist, then by (1), (2) we have

$$\begin{aligned}\iint_{R^2} |\xi|^p \, dP &= \int_R |t|^p \, dP_\xi(t) = \int_R |t|^p \, d\mathbf{m}_x(t) < \infty, \\ \iint_{R^2} |\eta|^p \, dP &= \int_R |t|^p \, dP_\eta(t) = \int_R |t|^p \, d\mathbf{m}_y(t) < \infty.\end{aligned}$$

Therefore, the random variables ξ, η belong to L^p_P and the random variable $\xi - \eta$ belong to L^p_P, too. Since $g(u, v) = u - v = \xi(u, v) - \eta(u, v)$, then we have

$$\begin{aligned}\mathbf{m}_{x-y} &= \mathbf{m} \circ (x - y) \\ &= \mathbf{m} \circ h_{xy} \circ g^{-1} \\ &= \mathbf{m} \circ h_{xy} \circ (\xi - \eta)^{-1} \\ &= P\big((\xi - \eta)^{-1}\big) \\ &= P_{(\xi - \eta)}\end{aligned}$$

and

$$\int_R |t|^p \, d\mathbf{m}_{x-y}(t) = \int_R |t|^p \, dP_{(\xi-\eta)}(t) = \iint_{R^2} |\xi - \eta|^p \, dP.$$

But $\xi - \eta \in L^p_P$, i.e., the integral $\iint_{R^2} |\xi - \eta|^p \, dP$ exists, hence the integral $\int_R |t|^p \, d\mathbf{m}_{x-y}(t)$ exists and $x - y \in L^p_{\mathbf{m}}$. □

Definition 3.2. *Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with product. For each IF-observables $x, y \in L^p_{\mathbf{m}}$ define the map $\rho_{IF} : L^p_{\mathbf{m}} \times L^p_{\mathbf{m}} \to R$ by*

$$\rho_{IF}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \left(\iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{xy}) \right)^{\frac{1}{p}} & \text{if } x \neq y, \end{cases}$$

where $h_{xy} : \mathcal{B}(R^2) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y and the Borel measurable function $g : R \to R$ is given by $g(u, v) = u - v$.

***Remark* 3.3.** *The map $\rho_{IF} : L^p_{\mathbf{m}} \times L^p_{\mathbf{m}} \to R$ given by*

$$\rho_{IF}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \left(\iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{xy}) \right)^{\frac{1}{p}} & \text{if } x \neq y, \end{cases}$$

can be rewritten in the following form

$$\rho_{IF}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \left(\int_R |t|^p \, d\mathbf{m}_{x-y}(t) \right)^{\frac{1}{p}} & \text{if } x \neq y. \end{cases}$$

Really

$$\begin{aligned}\iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{xy}) &= \int_R |t|^p \, d(\mathbf{m} \circ h_{xy} \circ g^{-1})(t) \\ &= \int_R |t|^p \, d(\mathbf{m} \circ (x - y))(t) \\ &= \int_R |t|^p \, d\mathbf{m}_{x-y}(t).\end{aligned}$$

Proposition 3.2. *The IF-space* $(L^p_{\mathbf{m}}, \rho_{IF})$ *is a pseudometric space.*

Proof. By the Definition 3.2, we have $\rho_{IF}(x,x) = 0$ and $\rho_{IF}(x,y) \geq 0$.

Now, we prove the symmetry. Consider any different IF-observables $x, y \in L^p_{\mathbf{m}}$. Let h_{xy} be the joint IF-observable of IF-observables x, y and h_{yx} be the joint IF-observable of IF-observables y, x. Put $\varphi(u,v) = (v,u)$, then $h_{yx} = h_{xy} \circ \varphi^{-1}$. Really,

$$\begin{aligned} h_{xy} \circ \varphi^{-1}(A \times B) &= h_{xy}(B \times A) \\ &= x(B) \cdot y(A) \\ &= y(A) \cdot x(B) \\ &= h_{yx}(A \times B). \end{aligned}$$

If we put $g(u,v) = u - v$ and $\psi(w) = -w$, then we obtain

$$\begin{aligned} \mathbf{m}_{y-x} &= \mathbf{m} \circ (y - x) \\ &= \mathbf{m} \circ h_{yx} \circ g^{-1} \\ &= \mathbf{m} \circ h_{xy} \circ \varphi^{-1} \circ g^{-1} \\ &= \mathbf{m} \circ h_{xy} \circ (g \circ \varphi)^{-1} \\ &= \mathbf{m} \circ h_{xy} \circ (\psi \circ g)^{-1} \\ &= \mathbf{m} \circ h_{xy} \circ g^{-1} \circ \psi^{-1} \\ &= \mathbf{m} \circ (x - y) \circ \psi^{-1} \\ &= \mathbf{m}_{x-y} \circ \psi^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \big(\rho_{IF}(y,x)\big)^p &= \iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{yx}) \\ &= \int_R |t|^p \, d\mathbf{m}_{y-x}(t) \\ &= \int_R |t|^p \, d(\mathbf{m}_{x-y} \circ \psi^{-1})(t) \\ &= \int_R |-t|^p \, d\mathbf{m}_{x-y}(t) \\ &= \int_R |t|^p \, d\mathbf{m}_{x-y}(t) \\ &= \iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{xy}) \\ &= \big(\rho_{IF}(x,y)\big)^p. \end{aligned}$$

Next we prove the triangle inequality. Let $x, y, z : \mathcal{B}(R) \to \mathcal{F}$ be three different IF-observables. Consider a joint IF-observable $h_{xyz} : \mathcal{B}(R^3) \to \mathcal{F}$ of IF-observables x, y, z. Then

$$h_{xyz}(A \times B \times C) = x(A) \cdot y(B) \cdot z(C)$$

for each $A, B, C \in \mathcal{B}(R)$.

Consider the probability space $(R^3, \mathcal{B}(R^3), P = \mathbf{m} \circ h_{xyz})$. Then the mappings $\xi, \eta, \zeta : R^3 \to R$ defined by

$$\xi(u,v,w) = u, \quad \eta(u,v,w) = v, \quad \zeta(u,v,w) = w$$

are the random variables and

$$\begin{aligned} P_\xi(A) &= P\big(\xi^{-1}(A)\big) \\ &= P(A \times R \times R) \\ &= \mathbf{m}\big(h_{xyz}(A \times R \times R)\big) \\ &= \mathbf{m}\big(x(A) \cdot y(R) \cdot z(R)\big) \\ &= \mathbf{m}\big(x(A) \cdot (1_\Omega, 0_\Omega) \cdot (1_\Omega, 0_\Omega)\big) \\ &= \mathbf{m}\big(x(A)\big) \\ &= \mathbf{m}_x(A). \end{aligned} \tag{3}$$

Similarly,

$$P_\eta(A) = \mathbf{m}_y(A), \quad P_\zeta(A) = \mathbf{m}_z(A) \tag{4}$$

for each $A \in \mathcal{B}(R)$. Using (3), (4) and $x, y, z \in L^p_{\mathbf{m}}$, we obtain that $\xi, \eta, \zeta \in L^p_P$.

Put $g(u,v) = u - v$ and $\pi_{xy}(u,v,w) = (u,v)$. Then $h_{xy} = h_{xyz} \circ \pi_{xy}^{-1}$ is a joint IF-observable of IF-observables x, y. Really,

$$\begin{aligned} h_{xy}(A \times B) &= h_{xyz}(A \times B \times R) \\ &= x(B) \cdot y(A) \cdot z(R) \\ &= x(A) \cdot y(B) \cdot (1_\Omega, 0_\Omega) \\ &= x(A) \cdot y(B). \end{aligned}$$

Since

$$\begin{aligned} \mathbf{m}_{x-y} &= \mathbf{m} \circ (x - y) \\ &= \mathbf{m} \circ h_{xy} \circ g^{-1} \\ &= \mathbf{m} \circ h_{xyz} \circ \pi_{xy}^{-1} \circ g^{-1} \\ &= \mathbf{m} \circ h_{xyz} \circ (g \circ \pi)^{-1} \\ &= P \circ (g \circ \pi_{xy})^{-1}, \end{aligned}$$

then

$$\begin{aligned} \rho_{IF}(x,y) &= \left(\iint_{R^2} |g|^p \, d(\mathbf{m} \circ h_{xy}) \right)^{\frac{1}{p}} \\ &= \left(\int_R |t|^p \, d\mathbf{m}_{x-y}(t) \right)^{\frac{1}{p}} \\ &= \left(\int_R |t|^p \, d(P \circ (g \circ \pi_{xy})^{-1})(t) \right)^{\frac{1}{p}} \\ &= \left(\iiint_{R^3} |g \circ \pi_{xy}|^p \, dP \right)^{\frac{1}{p}} \\ &= \left(\iiint_{R^3} |\xi - \eta|^p \, dP \right)^{\frac{1}{p}}. \end{aligned}$$

Analogously, we obtain

$$\mathbf{m}_{x-z} = P \circ (g \circ \pi_{xz})^{-1}, \quad \mathbf{m}_{y-z} = P \circ (g \circ \pi_{yz})^{-1}$$

and

$$\rho_{IF}(x,z) = \left(\iiint_{R^3} |\xi - \zeta|^p \, dP \right)^{\frac{1}{p}}, \quad \rho_{IF}(y,z) = \left(\iiint_{R^3} |\eta - \zeta|^p \, dP \right)^{\frac{1}{p}},$$

where $\pi_{xz}(u,v,w) = (u,w)$, $\pi_{yz}(u,v,w) = (v,w)$ and $h_{xz} = h_{xyz} \circ \pi_{xz}^{-1}$ is a joint IF-observable of IF-observables x, z and $h_{yz} = h_{xyz} \circ \pi_{yz}^{-1}$ is a joint IF-observable of IF-observables y, z.

Finally, using the triangle inequality and the symmetry in L_P^p and the symmetry in $L_{\mathbf{m}}^p$ we have

$$\begin{aligned} \rho_{IF}(x,y) &= \left(\iiint_{R^3} |\xi - \eta|^p \, dP \right)^{\frac{1}{p}} \\ &\leq \left(\iiint_{R^3} |\xi - \zeta|^p \, dP \right)^{\frac{1}{p}} + \left(\iiint_{R^3} |\zeta - \eta|^p \, dP \right)^{\frac{1}{p}} \\ &= \rho_{IF}(x,z) + \rho_{IF}(z,y). \end{aligned}$$

Therefore, the IF-space $(L_{\mathbf{m}}^p, \rho_{IF})$ is a pseudometric space. □

4 Conclusion

The paper is devoted to an L^p space of IF-observables with respect the IF-state $\mathbf{m}$. We proved that $(L_{\mathbf{m}}^p, \rho_{IF})$ is a pseudometric space. The presented results are the generalization of the results in [7], because if $\mu_A : \Omega \longrightarrow [0,1]$ is a fuzzy set, then $\mathbf{A} = (\mu_A, 1 - \mu_A) : \Omega \to [0,1]^2$ is the corresponding intuitionistic fuzzy set. The Definition 3.1 generalizes the notion of integrable and square integrable IF-observable introduced in [4].

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