Notes on Intuitionistic Fuzzy Sets
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Development of intuitionistic fuzzy data envelopment analysis model based on interval data with an application to MGNREGA 2018-19

Meena Yadav [1] **and Shiv Prasad Yadav** [2]

[1] Department of Applied Sciences and Humanities, Dronacharya College of Engineering
Farrukhnagar, Harayana, India
email: yadavmeenu3793@gmail.com

[2] Department of Mathematics, IIT Roorkee
Roorkee, India
e-mail: spyorfma@gmail.com

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Abstract: In modelling real life problems, intuitionistic fuzzy variables are best way of modelling linguistic variables. They can express the vagueness of variables to a greater extent. In this paper, we develop a new approach for measuring relative efficiency of decision making units (DMUs) with intuitionistic fuzzy inputs and outputs. Derived from data envelopment analysis (DEA) and interval DEA, the proposed model calculates the relative efficiency of a DMU in the form of an interval. The merit of the proposed model over the existing methods is justified comparison of units over the same production possibility set (PPS). We also develop a ranking algorithm for comparison of DMUs. Another merit of the proposed method is almost uniform ranking of DMUs for different α and β-cuts. We verify the proposed model using an example and apply our model to the scheme of MGNREGA. We check the efficiency intervals and calculate the ranking of Indian States and Union Territories. The states of Telangana, West Bengal and Jharkhand have emerged as the best performing states. The worst performers are the states of Karnataka, Nagaland, Bihar and Goa.

Keywords: Fuzzy sets, Intuitionistic fuzzy sets, Data envelopment analysis, Intuitionistic fuzzy data envelopment analysis, Ranking of intuitionistic fuzzy sets, Efficiency interval.
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1 Introduction

As a management and efficiency tool, the technique of data development analysis (DEA) gained wide popularity right after it was proposed by [10] due to its non-parametric nature and its ability to deal with multiple inputs and outputs simultaneously. After development of DEA, various models have been developed to find the efficiencies of decision making units (DMUs), BCC [8] being the second best favorite choice of the researchers, followed by SBM model [24]. While the CCR model was based on constant returns to scale (CRS), the BCC model focuses on variable returns to scale (VRS). Both CCR and BCC models are radial in nature, whereas SBM is non-radial model. All these models are extensively used in modelling various real life problems. But not all input or output variables can be exactly determined. There is always a doubt or a certain level of uncertainty with data of associated variables which led to the development of fuzzy set (FS) theory [32]. It generalized the crisp characteristic function and defined the membership function as the degree or extent of membership of an element on a scale of 0 to 1. Intuitionistic fuzzy set (IFS) is further generalization of the FS [6] on the basis of hesitation associated with degree of membership. It defines the non-membership function of an element on a scale of 0 to 1. The area of IFS theory has emerged as a highly researched topic in last few decades. Another branch of DEA, interval DEA (IDEA), deals with interval data, i.e., when information about inputs and outputs is not known exactly but a range is defined in the form of an interval. In the present study, we will develop an intuitionistic fuzzy DEA (IFDEA) model using IDEA model.

For data with crisp values of inputs and outputs, CCR model has been widely applied to measure the relative efficiencies of banks [17], schools [25], hospitals [15], academic departments [20, 26], coal plants [23], various schemes [31] etc. But practically, not all data is exact. The essence of uncertainty or vagueness present in input and output variables is mathematically represented by fuzzy variables [32]. Many of these studies are extended from crisp to fuzzy data using fuzzy DEA [1, 12, 16, 21]. With IF data, various models are developed to find the relative efficiencies of DMUs. [18] proposed optimistic and pessimistic DEA approach with IF data. [4] developed IFDEA models based on α- and β-cut approach and also proposed input-output targets for DMUs. Various ranking methods are also proposed for ranking of DMUs with IF data [4, 5, 22]. The ranking of DMUs with IF variables differ from decision maker (DM) to DM due to the fact that IF numbers are partially ordered. [11] developed IF TOPSIS to fully rank DMUs in uncertain environment. [3] also proposed IF super-efficiency SBM model for efficiency measurement and ranking of DMUs.

Based on interval arithmetic, [28] constructed IDEA models. They assessed the efficiencies of DMUs with interval data. The approach used by [28] to compare DMUs is different from the approach based on favorable and unfavorable inputs/ outputs as used by [16].

The rest of the paper is organized as follows. Section 2 includes some basic definitions and information. Section 3 presents the proposed methodology and the proposed ranking method, followed by merits of the proposed method over the existing methods in Section 4, a detailed example in Section 5 and concluding remarks in Section 6.

2 Preliminaries

This section defines some basic terminology and models that will be used later to establish a background for present research.

Model 2.1 (CCR Model, [10])**.** *Consider a set of n homogeneous DMUs using m inputs to produce s outputs. The CCR model is used to find the relative efficiency of the k-th DMU, (DMU$_k$) is given by*

$$\text{Max } E_k = \frac{\sum_{r=1}^{s} v_{rk}.y_{rk}}{\sum_{i=1}^{m} u_{ik}.x_{ik}} \tag{Model 2.1}$$

subject to

$$\begin{aligned} \frac{\sum_{r=1}^{s} v_{rk}.y_{rj}}{\sum_{i=1}^{m} u_{ik}.x_{ij}} &\leq 1, \quad j = 1, 2, \ldots, n, \\ u_{ik} &\geq 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\geq 0, \quad r = 1, 2, \ldots, s, \end{aligned}$$

where u_{ik} and v_{rk} are unknown weights associated with the i-th input and r-th output of DMU$_k$.

The CCR Model measures the relative efficiencies of homogeneous set of n DMUs in the interval (0,1]. DMU$_k$ is said to be **CCR-efficient** if the optimal value ${E_k}^*$ of Model 2.1 is 1, and the optimal weights ${u_{ik}}^*$ and ${v_{rk}}^*$ are all positive.

Definition 2.1 (Fuzzy set (FS), [32])**.** *A fuzzy set $\tilde{M}$ of a universal set X is defined by*

$$\tilde{M} = \{\langle x, \mu_{\tilde{M}}(x)\rangle \mid x \in X\}$$

where $\mu_{\tilde{M}} : X \to [0, 1]$ is called the membership function of $\tilde{M}$ and the value of $\mu_{\tilde{M}}(x)$ is called the degree of membership or association of x in $\tilde{M}$.

Definition 2.2 (Fuzzy Number (FN), [33])**.** *A fuzzy set $\tilde{M}$ in **R**, the set of real numbers, is called a fuzzy number if:*

1. *$\tilde{M}$ is a convex FS, i.e.,* $\mu_{\tilde{M}}(\lambda_1 x_1 + \lambda_2 x_2) \geq \min\{\mu_{\tilde{M}}(x_1),\ \mu_{\tilde{M}}(x_2)\}$ *for any x_1 and $x_2 \in$ **R**,* $\forall\ \lambda_1, \lambda_2 \geq 0$ *and* $\lambda_1 + \lambda_2 = 1$,
2. *$\tilde{M}$ is a normal fuzzy set, i.e.,* $\sup \mu_{\tilde{M}}(x) = 1$,
3. *there exists a unique $x_0 \in$ **R** such that* $\mu_{\tilde{M}}(x_0) = 1$,
4. *$\mu_{\tilde{M}}$ is piecewise continuous in **R**.*

x_0 is called modal or mean value of $\tilde{M}$.

Definition 2.3 (Triangular fuzzy number (TFN), [33]). *A TFN, $\tilde{M}$ is expressed as $\tilde{M}$= (a, b, c) and is defined by the membership function, $\mu_{\tilde{M}}$, given by*

$$\mu_{\tilde{M}}(x) = \begin{cases} \frac{x-a}{b-a}, & a < x \le b, \\ \frac{c-x}{c-b}, & b \le x < c, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 2.4 (Fuzzy CCR (FCCR) model, [16]). *When the input and output variables are fuzzy numbers then the fuzzy CCR model is given by:*

$$\text{Max } \tilde{E}_k = \frac{\sum_{r=1}^{s} v_{rk}.\tilde{y}_{rk}}{\sum_{i=1}^{m} u_{ik}.\tilde{x}_{ik}} \qquad \text{(Model 2.2)}$$

subject to

$$\begin{aligned} \frac{\sum_{r=1}^{s} v_{rk}.\tilde{y}_{rj}}{\sum_{i=1}^{m} u_{ik}.\tilde{x}_{ij}} &\le \tilde{1}, \quad j = 1, 2, \ldots, n, \\ u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s, \end{aligned}$$

where $\tilde{x}_{ik}$ $(i = 1, 2, \ldots, m)$ and $\tilde{y}_{rk}$ $(r = 1, 2, \ldots, s)$ are the i-th fuzzy inputs and r-th fuzzy outputs of DMU$_k$ $(k = 1, 2, \ldots, n)$. A DMU is said to be **FCCR efficient** if the optimal value, $\tilde{E}_k^*$, of Model 2.2 is $\tilde{1}$ and optimal weights u_{ik}^* and v_{rk}^* are all positive.

Definition 2.5 (Intuitionistic fuzzy set (IFS), [6]). *An IFS $\tilde{M}^I$ of a universal set X is defined by*

$$\tilde{M}^I = \{\langle x, \mu_{\tilde{M}^I}(x), \nu_{\tilde{M}^I}(x)\rangle \mid x \in X\}$$

where $\mu_{\tilde{M}^I} : X \to [0, 1]$ is called the membership function and $\nu_{\tilde{M}^I} : X \to [0, 1]$ is called the non-membership function. The value $\mu_{\tilde{M}^I}$ is called the degree of membership or acceptance of x in $\tilde{M}^I$ and the value $\nu_{\tilde{M}^I}$ is called the degree of non-membership or rejection of x in $\tilde{M}^I$. The function $\pi_{\tilde{M}^I} : X \to [0, 1]$ defined by $\pi_{\tilde{M}^I}(x) = 1 - \mu_{\tilde{M}^I}(x) - \nu_{\tilde{M}^I}(x)$ for all $x \in X$, is called the hesitation function and the value $\pi_{\tilde{M}^I}(x)$ is called the degree of hesitation associated with x in $\tilde{M}^I$. If $\pi_{\tilde{M}^I}(x) = 0$ for all $x \in X$, then the IFS $\tilde{M}^I$ reduces to the fuzzy set $\tilde{M}$.

Definition 2.6 (Intuitionistic fuzzy number (IFN), [7,9]). *An IFN $\tilde{M}^I = (a,\ b,\ c;\ a',\ b,\ c')$ is an IFS in **R** whose membership and non-membership functions, $\mu_{\tilde{M}^I}$ and $\nu_{\tilde{M}^I}$, are given by:*

$$\mu_{\tilde{M}^I}(x) = \begin{cases} g_1(x), & a \le x < b, \\ 1, & x = b, \\ g_2(x), & b < x \le c, \\ 0, & \textit{otherwise}, \end{cases} \quad \textit{and } v_{\tilde{M}^I}(x) = \begin{cases} h_1(x), & a' \le x < b, \\ 0, & x = b, \\ h_2(x), & b < x \le c', \\ 1, & \textit{otherwise}, \end{cases}$$

where g_1 and h_2 are piecewise continuous and increasing functions on $[a, b)$ and $(b, c']$, respectively, and g_2 and h_1 are piecewise continuous and decreasing functions on $(b, c]$ and $[a', b)$, respectively.

Definition 2.7 (Triangular intuitionistic fuzzy number (TIFN), [7]). *A TIFN* $\tilde{M}^I = (a, b, c; a^{'}, b, c^{'})$ *is an IFN whose membership and non-membership functions are given by:*

$$\mu_{\tilde{M}^I}(x) = \begin{cases} \frac{x-a}{b-a}, & a < x \leq b, \\ \frac{c-x}{c-b}, & b \leq x < c, \\ 0, & otherwise, \end{cases} \quad and \quad v_{\tilde{M}^I} = \begin{cases} \frac{b-x}{b-a^{'}}, & a^{'} < x \leq b, \\ \frac{x-b}{c^{'}-b}, & b \leq x < c^{'}, \\ 1, & otherwise. \end{cases}$$

Definition 2.8 (α-cut, [6]). *The α-cut of an IFS $\tilde{M}^I$, denoted by M^I_α, is defined by*

$$M^I_\alpha = \{x \in X : \mu_{\tilde{M}^I}(x) \geq \alpha\} \ \forall \alpha \in [0, 1].$$

Definition 2.9 (β-cut, [6]). *The β-cut of an IFS $\tilde{M}^I$, denoted by M^I_β, is defined by*

$$M^I_\beta = \{x \in X : \mu_{\tilde{M}^I}(x) \leq \beta\} \ \forall \beta \in [0, 1].$$

Definition 2.10 (Positive IFN, [6]). *An IFN* $\tilde{M}^I = (a, b, c; a^{'}, b, c^{'})$ *is said to be positive IFN if* $a^{'} > 0$.

3 Methodology

The vague character of variables in real life scenario is perfectly modelled by IF variables. It also includes the hesitation of decision maker in defining these variables. To measure the efficiency intervals of DMUs, we use α- and β-cuts for membership and non-membership functions of both the input and output variables. For IFN, the α- and β-cuts are in the form of intervals. For different values of α and β, we calculate different efficiency intervals of DMUs using the Interval DEA (IDEA) model presented below. Then using these efficiency intervals, we find the best and worst performers among the DMUs and rank them according to the ranking algorithm developed in Section 3.3.

3.1 Extension of DEA to IDEA

Model 2.1 is applicable for crisp data set, i.e., when input and output variables are crisp. Now, suppose the input and output variables are given in the form of intervals. Let the interval input data be $[x_{ij}{}^l, x_{ij}{}^u]$ and interval output data be $[y_{rj}{}^l, y_{rj}{}^u]$ for DMU$_j, j = 1, 2, \ldots, n$. Then the IDEA model as presented by [28] is given by

$$\text{Max } [\theta_k{}^l, \ \theta_k{}^u] = \frac{\sum_{r=1}^{s} v_{rk} \cdot [y_{rk}{}^l, y_{rk}{}^u]}{\sum_{i=1}^{m} u_{ik} \cdot [x_{ik}{}^l, x_{ik}{}^u]} \qquad \text{(Model 3.1)}$$

subject to

$$\frac{\sum_{r=1}^{s} v_{rk}.[y_{rj}{}^l, \ y_{rj}{}^u]}{\sum_{i=1}^{m} u_{ik}.[x_{ij}{}^l, \ x_{ij}{}^u]} \leq 1, \quad j = 1, 2, \ldots, n,$$
$$u_{ik} \geq 0, \quad i = 1, 2, \ldots, m,$$
$$v_{rk} \geq 0, \quad r = 1, 2, \ldots, s$$

Using interval arithmetic, Model 3.1 can be split into two models, namely, lower bound and upper bound optimization problems given by:

$$\text{Max } \theta_k^l = \frac{\sum_{i=1}^{s} v_{rk}.y_{rk}^l}{\sum_{i=1}^{m} u_{ik}.x_{ik}^u} \qquad \text{(Model 3.2L)}$$

subject to

$$\theta_j^u = \frac{\sum_{i=1}^{s} v_{rk}.y_{rj}^u}{\sum_{i=1}^{m} u_{ik}.x_{ij}^l} \leq 1, \quad j = 1, 2, \ldots, n,$$
$$u_{ik} \geq 0, \quad i = 1, 2, \ldots, m,$$
$$v_{rk} \geq 0, \quad r = 1, 2, \ldots, s.$$

and

$$\text{Max } \theta_k^u = \frac{\sum_{i=1}^{s} v_{rk}.y_{rk}^u}{\sum_{i=1}^{m} u_{ik}.x_{ik}^l} \qquad \text{(Model 3.2U)}$$

subject to

$$\frac{\sum_{i=1}^{s} v_{rk}.y_{rj}^u}{\sum_{i=1}^{m} u_{ik}.x_{ij}^l} \leq 1, \quad j = 1, 2, \ldots, n,$$
$$u_{ik} \geq 0, \quad i = 1, 2, \ldots, m,$$
$$v_{rk} \geq 0, \quad r = 1, 2, \ldots, s.$$

Model 3.2L and Model 3.2U are fractional models which can be easily changed into linear programming problems, as given by Model 3.3L and Model 3.3U below.

$$\theta_k^l = \sum_{i=1}^{s} v_{rk}.y_{rk}^l \qquad \text{(Model 3.3L)}$$

subject to

$$\sum_{i=1}^{m} u_{ik}.x_{ik}^u = 1$$
$$\sum_{i=1}^{s} v_{rk}.y_{rj}^u - \sum_{i=1}^{m} u_{ik}.x_{ij}^l \leq 0, \quad j = 1, 2, \ldots, n,$$
$$u_{ik} \geq 0, \quad i = 1, 2, \ldots, m,$$
$$v_{rk} \geq 0, \quad r = 1, 2, \ldots, s.$$

and

$$\theta_k^u = \sum_{i=1}^{s} v_{rk}.y_{rk}^u \qquad \text{(Model 3.3U)}$$

subject to

$$\sum_{i=1}^{m} u_{ik}.x_{ik}^l = 1$$
$$\sum_{i=1}^{s} v_{rk}.y_{rj}^u - \sum_{i=1}^{m} u_{ik}.x_{ij}^l \leq 0, \quad j = 1, 2, \ldots, n,$$
$$u_{ik} \geq 0, \quad i = 1, 2, \ldots, m,$$
$$v_{rk} \geq 0, \quad r = 1, 2, \ldots, s.$$

The interval $[\theta_k^l,\ {\theta_k}^u]$ forms the efficiency interval of $\text{DMU}_k, k = 1, 2, \ldots, n$.

3.2 Proposed intuitionistic fuzzy DEA model

Assume a set of n homogeneous DMUs with inputs $\tilde{x}^I_{ik}, i = 1,2,\ldots,m$ and outputs $\tilde{y}^I_{rk}$, $r = 1,2,\ldots,s$, where $\tilde{x}^I_{ik}$ and $\tilde{y}^I_{rk}$ are TIFNs for each $k = 1,2,\ldots,n$. Then the IF efficiency of the k-th DMU (DMU$_k$) is given by IFDEA model (Model 3.4).

$$\text{Max}\tilde{E}^I_k = \frac{\sum_{r=1}^{s} v_{rk}.\tilde{y}^I_{rk}}{\sum_{i=1}^{m} u_{ik}.\tilde{x}^I_{ik}} \qquad \text{(Model 3.4)}$$

subject to

$$\begin{aligned} \frac{\sum_{r=1}^{s} v_{rk}.\tilde{y}^I_{rj}}{\sum_{i=1}^{m} u_{ik}.\tilde{x}^I_{ij}} \leq \tilde{1}^I, &\quad 1 \leq j \leq n, \\ u_{ik} \geq 0, &\quad i = 1,2,\ldots,m, \\ v_{rk} \geq 0, &\quad r = 1,2,\ldots,s, \end{aligned}$$

where $\tilde{1}^I$ is the TIFN given by $\tilde{1}^I = (1,1,1;1,1,1)$.

Let $\tilde{x}^I_{ik} = (x^l_{ik}, x^m_{ik}, x^u_{ik}; x'^l_{ik}, x^m_{ik}, x'^u_{ik})$ and $\tilde{y}^I_{rk} = (y^l_{rk}, y^m_{rk}, y^u_{rk}; y'^l_{rk}, y^m_{rk}, y'^u_{rk})$. Then, Model 3.4 can be rewritten as

$$\text{Max}\tilde{E}^I_k = \frac{\sum_{r=1}^{s} v_{rk}.(\, y^l_{rk},\ y^m_{rk},\ y^u_{rk};\ y'^l_{rk},\ y^m_{rk},\ y'^u_{rk})}{\sum_{i=1}^{m} u_{ik}.(x^l_{ik},\ x^m_{ik},\ x^u_{ik};\ x'^l_{ik},\ x^m_{ik},\ x'^u_{ik})} \qquad \text{(Model 3.5)}$$

subject to

$$\begin{aligned} \frac{\sum_{r=1}^{s} v_{rk}.(y^l_{rj},\ y^m_{rj},\ y^u_{rj};\ y'^l_{rj},\ y^m_{rj},\ y'^u_{rj})}{\sum_{i=1}^{m} u_{ik}.(x^l_{ij},\ x^m_{ij},\ x^u_{ij};\ x'^l_{ij},\ x^m_{ij},\ x'^u_{ij})} \leq (1,1,1;1,1,1)\,, &\quad k = 1,2,\ldots,n, \\ u_{ik} \geq 0, &\quad i = 1,2,\ldots,m, \\ v_{rk} \geq 0, &\quad r = 1,2,\ldots,s. \end{aligned}$$

(i) **Interval efficiency model based on α-cut**

The α- cut for $\alpha \in (0,1]$, of the i-th IF input $\tilde{x}^I_{ik}$ is given by the interval $[\alpha x^m_{ik} + (1-\alpha)\, x^l_{ik}, \alpha x^m_{ik} + (1-\alpha) x^u_{ik}]$, $k = 1,2,\ldots,n$. Similarly, the α-cut for $\alpha \in (0,1]$, of the r-th IF output $\tilde{y}^I_{rk}$ is given by the interval $[\alpha y^m_{rk} + (1-\alpha)\, y^l_{rk}, \alpha y^m_{rk} + (1-\alpha) y^u_{rk}]$, $k = 1,2,\ldots,n$. Then Model 3.5 implies that

$$\text{Max}[E^l_{k,\alpha},\ E^u_{k,\alpha}] = \frac{\sum_{r=1}^{s} v_{rk}.[\alpha y^m_{rk} + (1-\alpha)\, y^l_{rk}, \alpha y^m_{rk} + (1-\alpha) y^u_{rk}]}{\sum_{i=1}^{m} u_{ik}.\ [\alpha x^m_{ik} + (1-\alpha)\, x^l_{ik},\ \ \alpha x^m_{ik} + (1-\alpha) x^u_{ik}]} \qquad \text{(Model 3.6)}$$

subject to

$$\begin{aligned} \frac{\sum_{r=1}^{s} v_{rk}.[\alpha y^m_{rj} + (1-\alpha)\, y^l_{rj}, \alpha y^m_{rj} + (1-\alpha) y^u_{rj}]}{\sum_{i=1}^{m} u_{ik}.\ [\alpha x^m_{ij} + (1-\alpha)\, x^l_{ij},\ \alpha x^m_{ij} + (1-\alpha) x^u_{ij}]} \leq [1,1]\,, &\quad j = 1,2,\ldots,n, \\ u_{rk} \geq 0, &\quad r = 1,2,\ldots,s, \\ v_{ik} \geq 0, &\quad i = 1,2,\ldots,m. \end{aligned}$$

Note that Model 3.6 is an interval DEA model, which can split into the following models:

$$\text{Max } E^l_{k,\alpha} = \frac{\sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rk} + (1-\alpha)\, y^l_{rk}\right)}{\sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ik} + (1-\alpha)\, x^u_{ik}\right)} \qquad \text{(Model 3.7L)}$$

subject to

$$\begin{aligned} \frac{\sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rj} + (1-\alpha)\, y^u_{rj}\right)}{\sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ij} + (1-\alpha)\, x^l_{ij}\right)} &\le 1, \quad j = 1, 2, \ldots, n, \\ u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s. \end{aligned}$$

and

$$\text{Max } E^u_{k,\alpha} = \frac{\sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rk} + (1-\alpha)\, y^u_{rk}\right)}{\sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ik} + (1-\alpha)\, x^l_{ik}\right)} \qquad \text{(Model 3.7U)}$$

subject to

$$\begin{aligned} \frac{\sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rj} + (1-\alpha) y^u_{rj}\right)}{\sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ij} + (1-\alpha)\, x^l_{ij}\right)} &\le 1, \quad j = 1, 2, \ldots, n \\ u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s. \end{aligned}$$

Models 3.7L and 3.7U are fractional DEA models which can be converted into linear programming problems given by Models 3.8L and 3.8U, respectively. For comparability of different α-cuts and to keep the production possibility set same for all α, we use $\alpha = 0$ in the constraint [28].

$$\text{Max } E^l_{k,\alpha} = \sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rk} + (1-\alpha) y^l_{rk}\right) \qquad \text{(Model 3.8L)}$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ik} + (1-\alpha)\, x^u_{ik}\right) &= 1, \\ \sum_{i=1}^{s} v_{rk}.y^u_{rj} - \sum_{i=1}^{m} u_{ik}.x^l_{ij} &\le 0, \quad j = 1, 2, \ldots, n, \\ u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s. \end{aligned}$$

and

$$E^u_{k,\alpha} = \sum_{i=1}^{s} v_{rk}.\left(\alpha y^m_{rk} + (1-\alpha) y^u_{rk}\right) \qquad \text{(Model 3.8U)}$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} u_{ik}.\left(\alpha x^m_{ik} + (1-\alpha)\, x^l_{ik}\right) &= 1, \\ \sum_{i=1}^{s} v_{rk}.y^u_{rj} - \sum_{i=1}^{m} u_{ik}.x^l_{ij} &\le 0, \quad j = 1, 2, \ldots, n, \\ u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m, \\ v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s. \end{aligned}$$

Models 3.8L and 3.8U give the lower and upper bounds of the membership function of the efficiency for each DMU for $\alpha \in (0, 1]$. The interval $[E^l_{k,\alpha},\ E^u_{k,\alpha}]$ forms the membership interval of the k-th DMU for each $\alpha \in (0, 1]$. Using α- cuts for $\alpha \in (0, 1]$, we can find the efficiency score for each DMU. Note that the interval $(E^l_{k,0}, E^u_{k,0})$ gives the support of the triangular fuzzy membership function of efficiency score of DMU$_k$, i.e., $E^l_{k.0} = a$, $E^u_{k,0} = c$. Also, $E^l_{k,1} = E^u_{k,1} = b$ is the mean value of the membership function.

(ii) **Interval efficiency model based on β-cut**

The β-cut of the i-th IF input $\tilde{x}^I_{ik}$ is given by the interval $[(1-\beta)x^m_{ik}+\beta x'^l_{ik}, (1-\beta)x^m_{ik}+\beta x'^u_{ik}]$ for $k = 1, 2, \ldots, n$. Similarly, the β-cut of the r-th IF output $\tilde{y}^I_{rk}$ is given by the interval $[(1-\beta)y^m_{rk} + \beta y'^l_{rk}, (1-\beta)y^m_{rk} + \beta y'^u_{rk}]$ for $k = 1, 2, \ldots, n$. Proceeding as in Models 3.6 and 3.7 with β- cuts of IF inputs and outputs, the non-membership interval efficiency $[E'^u_{k,\beta},\ E'^l_{k,\beta}]$ for each DMU$_k$, $k = 1, 2, \ldots, n$, and for $\beta \in [0, 1)$ is as given in Model 3.9.

$$E'^l_{k,\beta} = \textstyle\sum_{i=1}^{s} v_{rk}.((1-\beta)y^m_{rk} + \beta y'^l_{rk}) \qquad \text{(Model 3.9L)}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{m} u_{ik}.\left((1-\beta)x^m_{ik} + \beta x'^u_{ik}\right) &= 1,\\
\sum_{i=1}^{s} v_{rk}.y'^u_{rj} - \sum_{i=1}^{m} u_{ik}.x'^l_{ij} &\le 0, \quad j = 1, 2, \ldots, n,\\
u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m,\\
v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s,
\end{aligned}$$

and

$$E'^u_{k,\beta} = \textstyle\sum_{i=1}^{s} v_{rk}.((1-\beta)y^m_{rk} + \beta y'^u_{rk}) \qquad \text{(Model 3.9U)}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{m} u_{ik}.\left((1-\beta)x^m_{ik} + \beta x'^l_{ik}\right) &= 1,\\
\sum_{i=1}^{s} v_{rk}.y'^u_{rj} - \sum_{i=1}^{m} u_{ik}.x'^l_{ij} &\le 0, \quad j = 1, 2, \ldots, n,\\
u_{ik} &\ge 0, \quad i = 1, 2, \ldots, m,\\
v_{rk} &\ge 0, \quad r = 1, 2, \ldots, s.
\end{aligned}$$

Models 3.9L and 3.9U give the lower and upper bounds of the non-membership function of the efficiency for each DMU for each $\beta \in [0, 1)$. The interval $[E'^l_{k,\beta},\ E'^u_{k,\beta}]$ forms the non-membership interval of the k-th DMU for all $\beta \in [0, 1)$.

Using β-cuts for $\beta \in [0, 1)$, we can find the non-membership efficiency scores for each DMU. One can see that $E'^l_{k,1} = a'$, $E'^u_{k,1} = c'$ and $E'^l_{k,0} = E'^u_{k,1} = b$ for DMU$_k$.

Using Models 4.8L, 4.8U and 4.9L, 4.9U, we can obtain the efficiency intervals for different α- and β-cuts. Using these intervals, we can also represent the efficiency as TIFN and then compare the DMUs using any of the present methods for comparison of TIFNs [2, 13, 19]. In the present study, we will present a new ranking approach given below for comparison of efficiency intervals and ranking of DMUs.

3.3 Proposed ranking approach

Mostly ranking of IFNs are based on score and accuracy function of IFN [30]. The proposed ranking method is based on combined index of the membership and non-membership efficiency intervals for respective α and β. First, obtain the IF efficiency of each DMU.

(i) **Index based on α-cut**

For $\alpha \in (0, 1]$, define the set J_α by

$$J_\alpha = \{\text{DMU}_k : \; E^u_{k,\alpha} = 1\} \tag{3.10}$$

where $[E^l_{k,\alpha}, \; E^u_{k,\alpha}]$ is the efficiency interval of DMU$_k$. Now, define the ranking index, $R_{k,\alpha}$, by

$$R_{k,\alpha} = \begin{cases} E^l_{k,\alpha}, & \text{if } DMU_k \in J_\alpha, \\ \dfrac{E^l_{k,\alpha}}{1 - E^u_{k,\alpha}}, & \text{if } DMU_k \notin J_\alpha. \end{cases} \tag{3.11}$$

The interval $[E^l_{k,\alpha}, \; E^u_{k,\alpha}] \subseteq (0, 1] \;\; \forall \alpha \in (0, 1]$. Index $R_{k,\alpha}$ defines the ratio of distance of $E^l_{k,\alpha}$ from zero to the distance of $E^u_{k,\alpha}$ from 1.

(ii) **Index based on β-cut**

For $\beta \in [0, 1)$, define the set $J^{'}_\beta$ by

$$J^{'}_\beta = \{\text{DMU}_k : E^{'u}_{k,\beta} = 1\} \tag{3.12}$$

where $[E^{'l}_{k,\beta}, \; E^{'u}_{k,\beta}]$is the efficiency interval of DMU$_k$.

Now, define the ranking index $R^{'}_{k,\beta}$ by

$$R^{'}_{k,\beta} = \begin{cases} E^{'l}_{k,\beta}, & \text{if DMU}_k \in J^{'}_\beta, \\ \dfrac{E^{'l}_{k,\beta}}{1 - E^{'u}_{k,\beta}}, & \text{if DMU}_k \notin J^{'}_\beta. \end{cases} \tag{3.13}$$

Again, the interval $[E^{'l}_{k,\beta}, \; E^{'u}_{k,\beta}] \subseteq [0, 1] \;\; \forall \beta \in [0, 1)$ Index $R^{'}_{k,\beta}$ defines the ratio of distance of $E^{'l}_{k,\beta}$ from zero to the distance of $E^{'u}_{k,\beta}$ from 1.

(iii) **Overall index**

For $0 < \alpha + \beta \leq 1$ and $\alpha \in (0, 1]$, $\beta \in [0, 1)$ define the overall ranking index $I^{\alpha,\beta}_k$, by

$$I^{\alpha,\beta}_k = \eta R_{k,\alpha} + (1 - \eta) R^{'}_{k,\beta} \tag{3.14}$$

where $\eta \in [0, 1]$ is based on decision maker's choice.

Algorithm for ranking: For $\alpha \in (0,1]$ and $\beta \in [0,1)$, apply the following steps to obtain the overall ranking.

Step 1: For each DMU_k, find the efficiency interval $[E^l_{k,\alpha},\ E^u_{k,\alpha}]$ for $\alpha \in (0,1]$ using Model 3.8.

Step 2: Find the set J_α as defined in (3.10).

Step 3: Calculate the index $R_{k,\alpha}$ as given in (3.11).

Step 4: For each DMU$_k$, find the efficiency interval $[E'^l_{k,\beta},\ E'^u_{k,\beta}]$ for $\beta \in [0,1)$ using Model 3.9.

Step 5: Find the set J'_β as defined in (3.12).

Step 6: Calculate the index $R'_{k,\beta}$ as defined in (3.13).

Step 7: Calculate the overall ranking index $I^{\alpha,\beta}_k$ as defined in (3.14).

First, rank the DMUs $\in J_\alpha \cup J'_\beta$ in increasing order of $I^{\alpha,\beta}_k$. Then rank the DMUs $\notin J_\alpha \cup J'_\beta$ in increasing order of $I^{\alpha,\beta}_k$.

4 Merits of the proposed method over existing methods

The proposed model catches the imprecision or vagueness of input and output variables to calculate efficiency interval of DMUs. The main achievement of the proposed Models 4.8 and 4.9 is that the same production possibility set is used for comparison of DMUs. This provides a unified frontier for all DMUs and justifies the comparison of DMUs. The lack of a unified frontier for all DMUs had been a great drawback in almost all existing approaches. The proposed model overcomes this drawback. The proposed model takes into account both the membership and non-membership function values of input and output variables. Many studies use accuracy function to calculate the efficiency, but it leads to loss of information about data.

Table 1. IF Input and IF output data of 12 DMUs. Source: [18]

DMUs	Input 1	Input 2	Output 1	Output 2
DMU_1	$(17,20,23;15,20,25)$	$(148,151,153;145,151,154)$	$(97,100,103;95,100,105)$	$(86,90,95;84,90,97)$
DMU_2	$(15,19,22;12,19,26)$	$(129,131,134;127,131,136)$	$(148,150,151;146,150,153)$	$(48,50,53;45,50,55)$
DMU_3	$(22,25,28;20,25,30)$	$(158,160,163;154,160,165)$	$(157,160,163;154,160,165)$	$(53,55,57;50,55,59)$
DMU_4	$(23,27,29;22,27,33)$	$(165,168,170;163,168,172)$	$(178,180,183;176,180,185)$	$(70,72,74;67,72,75)$
DMU_5	$(20,22,25;18,22,27)$	$(155,158,161;153,158,163)$	$(91,94,97;89,94,99)$	$(60,66,68;60,66,71)$
DMU_6	$(51,55,58;49,55,60)$	$(252,255,258;250,255,260)$	$(228,230,231;226,230,232)$	$(87,87,90,93;85,90,95)$
DMU_7	$(31,33,36;29,33,39)$	$(233,235,236;230,235,238)$	$(217,220,223;215,220,225)$	$(83,88,92;81,88,95)$
DMU_8	$(29,31,34;27,31,36)$	$(202,203,208;200,206,210)$	$(150,152,154;148,152,155)$	$(78,80,83;76,80,84)$
DMU_9	$(27,30,32;24,30,35)$	$(241,244,246;238,244,248)$	$(187,190,194;183,190,196)$	$(97,100,102;94,100,104)$
DMU_{10}	$(47,50,54;44,50,55)$	$(262,268,271;260,268,273)$	$(246,250,253;244,250,255)$	$(97,100,104;95,100,105)$
DMU_{11}	$(51,53,55;48,53,57)$	$(302,306,308;300,306,309)$	$(258,260,261;256,260,262)$	$(142,147,149;140,147,152)$
DMU_{12}	$(34,38,40;32,38,41)$	$(281,284,286;280,284,287)$	$(248,250,256;243,250,258)$	$(119,120,123;117,120,125)$

Table 2. The interval efficiencies $[E^l_{k,\alpha},\ E^u_{k,\alpha}]$ for different values of α

α	0	0.25	0.5	0.75	1
DMU_1	$(0.888, 1)$	$[0.9, 0.984]$	$[0.912, 0.968]$	$[0.924, 0.952]$	0.936
DMU_2	$(0.944, 1)$	$[0.952, 0.994]$	$[0.961, 0.989]$	$[0.969, 0.984]$	0.978
DMU_3	$(0.823, 0.881)$	$[0.831, 0.875]$	$[0.838, 0.868]$	$[0.846, 0.861]$	0.854
DMU_4	$(0.934, 1)$	$[0.941, 0.99]$	$[0.948, 0.981]$	$[0.955, 0.971]$	0.962
DMU_5	$(0.659, 0.756)$	$[0.674, 0.747]$	$[0.689, 0.738]$	$[0.704, 0.729]$	0.719
DMU_6	$(0.779, 0.825)$	$[0.785, 0.819]$	$[0.791, 0.814]$	$[0.797, 0.808]$	0.803
DMU_7	$(0.811, 0.873)$	$[0.818, 0.864]$	$[0.826, 0.856]$	$[0.833, 0.848]$	0.84
DMU_8	$(0.736, 0.793)$	$[0.742, 0.785]$	$[0.747, 0.776]$	$[0.753, 0.767]$	0.758
DMU_9	$(0.776, 0.858)$	$[0.782, 0.831]$	$[0.789, 0.817]$	$[0.796, 0.81]$	0.802
DMU_{10}	$(0.81, 0.877)$	$[0.817, 0.867]$	$[0.824, 0.857]$	$[0.831, 0.847]$	0.837
DMU_{11}	$(0.882, 0.928)$	$[0.888, 0.923]$	$[0.895, 0.918]$	$[0.901, 0.913]$	0.907
DMU_{12}	$(0.849, 0.897)$	$[0.853, 0.888]$	$[0.856, 0.88]$	$[0.859, 0.871]$	0.863

5 Example

In this section, an example is discussed to illustrate an application of the proposed method. Consider 12 DMUs using two IF inputs to produce two IF outputs (TIFNs). The data of inputs and outputs is given in Table 1. For each DMU, the interval efficiencies $[E^l_{k,\alpha},\ E^u_{k,\alpha}]$, as determined by α- cuts (for $\alpha = 0.25,\ 0.5,\ 0.75,\ 1$ using Models 3.8L and 3.8U) are given in Table 2.

For $\alpha \neq 0$, all DMUs are contained inside the frontier which is fixed by $\alpha = 0$, and hence $E^u_{k,\alpha} < 1$. However, to find the relatively best performer for a particular $\alpha \neq 0$, we divide $E^u_{k,\alpha}$ by $\max_k\ E^u_{k,\alpha}$. The results are shown in Table 4. For $\alpha \neq 0$, only DMU_2 is relatively efficient and all other DMUs are inefficient.

For each DMU, the interval efficiencies $[E'^l_{k,\beta}, E'^u_{k,\beta}]$, as determined by β- cuts (for $\beta = 0,\ 0.25,\ 0.5,\ 0.75,)$ are given in Table 3.

Table 3. The interval efficiencies $[E'^l_{k,\beta}, E'^u_{k,\beta}]$ for different values of β

β	0	0.25	0.5	0.75	1
DMU_1	0.899	$[0.881, 0.927]$	$[0.863, 0.948]$	$[0.845, 0.974]$	$(0.828, 1)$
DMU_2	0.95	$[0.935, 0.966]$	$[0.902, 0.974]$	$[0.905, 0.987]$	$(0.891, 1)$
DMU_3	0.83	$[0.816, 0.846]$	$[0.800, 0.859]$	$[0.788, 0.874]$	$(0.774, 0.89)$
DMU_4	0.932	$[0.916, 0.946]$	$[0.901, 0.961]$	$[0.885, 0.977]$	$(0.87, 0.992)$
DMU_5	0.692	$[0.674, 0.71]$	$[0.756, 0.727]$	$[0.638, 0.745]$	$(0.62, 0.763)$
DMU_6	0.778	$[0.766, 0.786]$	$[0.756, 0.795]$	$[0.745, 0.805]$	$(0.735, 0.816)$
DMU_7	0.814	$[0.801, 0.828]$	$[0.789, 0.842]$	$[0.777, 0.857]$	$(0.765, 0.871)$
DMU_8	0.733	$[0.722, 0.745]$	$[0.711, 0.756]$	$[0.701, 0.768]$	$(0.691, 0.781)$
DMU_9	0.773	$[0.761, 0.785]$	$[0.749, 0.797]$	$[0.735, 0.808]$	$(0.723, 0.821)$
DMU_{10}	0.811	$[0.801, 0.824]$	$[0.789, 0.837]$	$[0.779, 0.85]$	$(0.767, 0.864)$
DMU_{11}	0.876	$[0.867, 0.885]$	$[0.857, 0.894]$	$[0.848, 0.903]$	$(0.839, 0.913)$
DMU_{12}	0.834	$[0.827, 0.845]$	$[0.818, 0.856]$	$[0.811, 0.866]$	$(0.803, 0.877)$

Table 4. The values of $E^U_{k,\alpha}/\max\, E^U_{k,\alpha}$ for different values of α

$\boldsymbol{\alpha}$	**0**	**0.25**	**0.5**	**0.75**	**1**
DMU_1	1	0.99	0.979	0.967	0.957
DMU_2	1	1	1	1	1
DMU_3	0.881	0.88	0.878	0.875	0.873
DMU_4	1	0.996	0.992	0.987	0.984
DMU_5	0.756	0.751	0.746	0.741	0.735
DMU_6	0.825	0.824	0.823	0.821	0.821
DMU_7	0.873	0.869	0.865	0.862	0.859
DMU_8	0.793	0.79	0.785	0.78	0.775
DMU_9	0.858	0.836	0.826	0.823	0.82
DMU_{10}	0.877	0.872	0.866	0.861	0.856
DMU_{11}	0.928	0.929	0.929	0.928	0.927
DMU_{12}	0.897	0.893	0.89	0.885	0.882

Table 5. The ratio of $E'^{u}_{k,\beta}/\max\, E'^{u}_{k,\beta}$ for different values of β

$\boldsymbol{\beta}$	**0**	**0.25**	**0.5**	**0.75**	**1**
DMU_1	0.946	0.96	0.973	0.987	1
DMU_2	1	1	1	1	1
DMU_3	0.874	0.876	0.882	0.885	0.89
DMU_4	0.981	0.98	0.987	0.99	0.992
DMU_5	0.728	0.735	0.746	0.755	0.763
DMU_6	0.819	0.814	0.816	0.816	0.816
DMU_7	0.857	0.858	0.864	0.868	0.871
DMU_8	0.771	0.772	0.776	0.778	0.781
DMU_9	0.814	0.813	0.818	0.819	0.821
DMU_{10}	0.854	0.853	0.859	0.861	0.864
DMU_{11}	0.922	0.916	0.918	0.915	0.913
DMU_{12}	0.878	0.875	0.879	0.877	0.877

For $\beta \neq 1$, divide $E^u_{k,\beta}$ by $\max_k E^u_{k,\beta}$ to find the relatively efficient DMUs. The results are given in Table 5. Only DMU_2 is relatively the best performer for any $\beta \neq 1$, all other DMUs are performing worse than DMU_2.

Ranking of DMUs

For $\alpha = 0.25$ and $\beta = 0.5$, the calculated indices R_k and R'_k are given in Table 6. Overall index I_k is also given in Table 6 for each DMU_k, $k = 1, 2, \ldots, 12$. Index I_k is maximal for $k = 2$ which is 96.68. Hence, DMU_2 is the best performer, followed by DMU_4 and DMU_1 at ranks 2 and 3, respectively. Ranking of DMUs is given in Table 6. It is as follows:

$$\begin{aligned} &DMU_2 > DMU_4 > DMU_1 > DMU_{11} > DMU_{12} > DMU_3 \\ &\quad > DMU_7 > DMU_{10} > DMU_9 > DMU_6 > DMU_8 > DMU_5. \end{aligned}$$

Table 6. Indices $R_{k,\alpha},\ R'_{k,\beta},\ I_k$ and ranks of DMUs for $\alpha = 0.25$ and $\beta = 0.5$

DMU	$R_{k,\alpha}$	$R'_{k,\beta}$	I_k	**Rank**
DMU_1	56.25	16.596	36.423	3
DMU_2	158.667	34.692	96.679	1
DMU_3	6.648	5.675	6.161	6
DMU_4	94.1	23.102	58.601	2
DMU_5	2.664	2.769	2.716	12
DMU_6	4.337	3.687	4.012	10
DMU_7	6.014	4.993	5.504	7
DMU_8	3.451	2.913	3.182	11
DMU_9	4.627	3.689	4.158	9
DMU_{10}	6.142	4.840	5.491	8
DMU_{11}	11.532	8.084	9.808	4
DMU_{12}	7.616	5.680	6.648	5

Similarly, rankings can be obtained for other possible combinations of α and β as given in Table 7. It can be seen that the proposed method provides almost a uniform ranking for different possible combinations of α and β. Most of the DMUs have occupied the same rankings for different combinations of α and β. DMU_2 has emerged as the best performer of all the 12 DMUs considered, followed by $DMU_4, DMU_1, DMU_{11}, DMU_{12}, DMU_3$. Then, DMU_8 and DMU_5 are the worst performers with ranks 11 and 12, respectively. Finally, DMU_9 and DMU_6 share ranks 9 and 10 (i.e., DMU_6 and DMU_9 occupy ranks 9 and 10 interchangebly with respect to the values of α, β) and DMU_7 and DMU_{10} share ranks 7 and 8.

Table 7. Ranks of DMUs for different possible combinations of α, β

(α, β)	$(0.25, 0.25)$	$(0.25, 0.5)$	$(0.25, 0.75)$	$(0.5, 0.25)$	$(0.5, 0.5)$	$(0.75, 0.25)$
DMU_1	3	3	3	3	3	3
DMU_2	1	1	1	1	1	1
DMU_3	6	6	6	6	6	6
DMU_4	2	2	2	2	2	2
DMU_5	12	12	12	12	12	12
DMU_6	10	10	10	10	10	9
DMU_7	8	7	7	7	7	7
DMU_8	11	11	11	11	11	11
DMU_9	9	9	9	9	9	10
DMU_{10}	7	8	8	8	8	8
DMU_{11}	4	4	4	4	4	4
DMU_{12}	5	5	5	5	5	5

6 Case study: MGNREGA

Unemployment is a major setback in the growth of a country. India stands at the second place in total population in the world with one sixth of population residing here [29]. A large portion of its people live in rural areas where meeting daily basic needs is a challenge for many. The Mahatma Gandhi National Rural Employment Guarantee Act (MGNREGA), 2005 provides guaranteed 100 days employment to each adult member willing to work in rural and urban backward households. This social welfare scheme is implemented under the Ministry of Rural Development, Government of India and aims to provide basic needs to people below poverty line by providing them the opportunity to work with dignity. The scheme has empowered women and physically challenged people by giving them equal opportunities of work. At least one third of the scheme beneficiaries are women. The usual work taken up by MGNREGA workers includes construction and renovation of public roads, canals, wells, fisheries, work related to sanitation, irrigation and flood management. The Gram Panchayats decide the work to be done in each financial year in meetings with Gram Sabhas and Ward Sabhas. The adult members of family that reside in rural areas and are willing to do unskilled work are issued job cards by the Gram Panchayats. An applicant is given work within 15 days of application at prior fixed wage rate which is atleast minimum wage rate as decided by Central/ State government. If the authorities fail to provide employment within 15 days then he/she is entitled to daily unemployment allowance. For successfully creating employment opportunities for physically challenged people, the scheme was applauded in United Nations Disability and Development Report [27].

We determine the interval efficiencies of the scheme of MGNREGA in 29 States and Union territories of India. The description of inputs and outputs used is as follows:

1. **Input Variables**
   - (a) **Employment demanded**: It is the total number of labors who demanded employment in the financial year 2018-19.
   - (b) **Labor budget**: This is the budget of financial year 2018-19 provided to a State for labor work.
   - (c) **Work taken-up**: It refers to the new work added each year and spill over from last year.
   - (d) **Total budget**: This is the budget available for all the work, material and maintenance. It is shared by both State and Central government.
2. **Output variables**
   - (a) **Average employment provided per household**: It refers to the average number of days employment was provided to a household.
   - (b) **Total employment offered**: It refers to the number of labors who demanded and got employment within 15 days of application.
   - (c) **Total households worked**: It refers to the number of households that got employment.

(d) **Completed work**: It refers to the work completed out of the total work taken up.

(e) **Persondays generated** (in lakhs): It is equal to the total number of people worked times the number of days they worked. Employment generated is calculated in terms of persondays.

The crisp data has been collected from the official website of MGNREGA [14] and then transformed into TIFNs. The interval efficiencies $[E^l_{k,\alpha}, E^u_{k,\alpha}]$ for different values of α and β are given in Tables 8 and 9 respectively. Table 10 shows the ranking indices for different possible combinations of α and β and the rankings of States and Union Territories is given in Table 11. The results are as follows:

1. For $\alpha = 1$, the states of Assam, Chhattisgarh, Jharkhand, Madhya Pradesh, Rajasthan, Tamil Nadu, Telangana, Tripura and West Bengal have efficiency equal to 1 and hence are efficient.

2. For $\beta = 0$, the states of Jharkhand, Telangana and West Bengal have efficiency score equal to 1 and hence are efficient. All other states are inefficient.

3. The efficiencies $E^u_{k,\alpha},\ E'^u_{k,\beta}$ are all less than 1 for $\alpha \neq 0$ and for $\beta \neq 1$ due the restriction placed in Models 3.8 and 4.9. It justifies the comparison of States by comparing them over the same set of restrictions, i.e., the feasible sets of Models 4.8 and 4.9 do not change with each pair of values of α and β for each state.

4. By the rankings, the state of Telangana has outperformed other states in performance followed by the states of West Bengal, Jharkhand, Rajasthan, Tripura, Tamil Nadu, Assam, Madhya Pradesh, Chhattisgarh and Punjab.

5. The states of Karnataka, Nagaland, Goa, Bihar, Meghalaya and Orissa are worst performers.

6. The states of Uttar Pradesh, Manipur, Kerala, Uttarakhand, Jammu and Kashmir, Mizoram, Himachal Pradesh, Andaman and Nikobar have performed moderately.

7. Out of 29 states, 15 states have the same rank for all possible combinations of α and β. In the remaining 14 states, 10 states share a rank with another state and the rest four States share rank with 2 other states.

Another major achievement of the proposed method is almost uniform ranking of DMUs for different possible combinations of α and β. This lifts the load from decision maker to choose the values of α and β for comparison.

Table 8. The interval efficiencies $[E^l_{k,\alpha}, E^u_{k,\alpha}]$ for different values of α

DMU	**0**	**0.25**	**0.5**	**0.75**	**1**
Andhra Pradesh	$(0.993, 0.995)$	$[0.993, 0.994]$	$[0.993, 0.994]$	$[0.993, 0.994]$	0.993
Arunachal Pradesh	$(0.962, 1)$	$[0.967, 0.997]$	$[0.972, 0.991]$	$[0.977, 0.987]$	0.983
Assam	$(0.998, 1)$	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.999, 0.999]$	0.999
Bihar	$(0.984, 0.988)$	$[0.984, 0.988]$	$[0.984, 0.987]$	$[0.985, 0.986]$	0.985
Chhattisgarh	$(0.998, 1)$	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.999, 0.999]$	0.999
Goa	$(0.912, 1)$	$[0.926, 0.991]$	$[0.939, 0.982]$	$[0.952, 0.974]$	0.965
Gujarat	$(0.985, 0.989)$	$[0.986, 0.989]$	$[0.986, 0.988]$	$[0.987, 0.988]$	0.988
Haryana	$(0.994, 1)$	$[0.994, 0.998]$	$[0.995, 0.997]$	$[0.995, 0.996]$	0.995
Himachal Pradesh	$(0.985, 0.999)$	$[0.986, 0.997]$	$[0.988, 0.995]$	$[0.989, 0.993]$	0.990
Jammu & Kashmir	$(0.990, 1)$	$[0.991, 0.999]$	$[0.993, 0.998]$	$[0.994, 0.997]$	0.996
Jharkhand	$(0.999, 1)$	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	0.999
Karnataka	$(0.958, 0.962)$	$[0.959, 0.961]$	$[0.959, 0.961]$	$[0.960, 0.961]$	0.960
Kerala	$(0.995, 1)$	$[0.995, 0.999]$	$[0.996, 0.998]$	$[0.996, 0.997]$	0.997
Madhya Pradesh	$(0.998, 1)$	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.999, 0.999]$	0.999
Maharashtra	$(0.989, 0.992)$	$[0.989, 0.992]$	$[0.989, 0.991]$	$[0.990, 0.991]$	0.990
Manipur	$(0.995, 1)$	$[0.995, 0.999]$	$[0.996, 0.998]$	$[0.996, 0.998]$	0.990
Meghayala	$(0.982, 0.992)$	$[0.831, 0.991]$	$[0.984, 0.990]$	$[0.986, 0.988]$	0.987
Mizoram	$(0.991, 1)$	$[0.991, 0.998]$	$[0.992, 0.997]$	$[0.993, 0.996]$	0.994
Nagaland	$(0.894, 1)$	$[0.910, 0.989]$	$[0.925, 0.978]$	$[0.941, 0.967]$	0.957
Odisha	$(0.988, 0.991)$	$[0.988, 0.991]$	$[0.988, 0.990]$	$[0.989, 0.990]$	0.989
Punjab	$(0.997, 1)$	$[0.997, 0.999]$	$[0.998, 0.999]$	$[0.998, 0.998]$	0.998
Rjasthan	$(0.998, 1)$	$[0.998, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	0.999
Tamil Nadu	$(0.998, 1)$	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.999, 0.999]$	0.999
Telangana	$(0.999, 1)$	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	0.999
Tripura	$(0.998, 1)$	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.999, 0.999]$	0.999
Uttar Pradesh	$(0.996, 1)$	$[0.996, 0.999]$	$[0.997, 0.999]$	$[0.998, 0.999]$	0.998
Uttarakhand	$(0.992, 1)$	$[0.993, 0.999]$	$[0.994, 0.998]$	$[0.995, 0.997]$	0.996
West Bengal	$(0.999, 1)$	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	0.999
Andaman Nikobar	$(0.960, 1)$	$[0.965, 0.995]$	$[0.970, 0.990]$	$[0.975, 0.985]$	0.981

Table 9. The interval efficiencies $[E'^{l}_{k,\beta},\ E'^{u}_{k,\beta}]$ for different values of β

DMU α, β	0	0.25	0.5	0.75	1
Andhra Pradesh	0.986	$[0.986, 0.987]$	$[0.985, 0.988]$	$[0.985, 0.989]$	$(0.985, 0.991)$
Arunachal Pradesh	0.972	$[0.963, 0.979]$	$[0.954, 0.986]$	$[0.945, 0.993]$	$(0.936, 1)$
Assam	0.998	$[0.997, 0.998]$	$[0.997, 0.999]$	$[0.997, 0.999]$	$(0.996, 1)$
Bihar	0.97	$[0.968, 0.997]$	$[0.967, 0.972]$	$[0.966, 0.974]$	$(0.964, 0.976)$
Chhattisgarh	0.996	$[0.995, 0.997]$	$[0.994, 0.998]$	$[0.992, 0.999]$	$(0.991, 1)$
Goa	0.921	$[0.882, 0.942]$	$[0.842, 0.946]$	$[0.805, 0.98]$	$(0.768, 1)$
Gujarat	0.971	$[0.969, 0.972]$	$[0.967, 0.973]$	$[0.965, 0.974]$	$(0.963, 0.975)$
Haryana	0.991	$[0.989, 0.993]$	$[0.987, 0.995]$	$[0.986, 0.998]$	$(0.984, 1)$
Himachal Pradesh	0.972	$[0.967, 0.979]$	$[0.962, 0.986]$	$[0.965, 0.993]$	$(0.962, 1)$
Jammu & Kashmir	0.989	$[0.982, 0.992]$	$[0.976, 0.994]$	$[0.971, 0.997]$	$(0.967, 1)$
Jharkhand	0.999	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	$(0.998, 1)$
Karnataka	0.957	$[0.955, 0.958]$	$[0.954, 0.96]$	$[0.952, 0.961]$	$(0.95, 0.963)$
Kerala	0.979	$[0.977, 0.983]$	$[0.974, 0.986]$	$[0.972, 0.99]$	$(0.97, 0.994)$
Madhya Pradesh	0.998	$[0.998, 0.998]$	$[0.997, 0.999]$	$[0.997, 0.999]$	$(0.997, 1)$
Maharashtra	0.978	$[0.977, 0.98]$	$[0.975, 0.982]$	$[0.974, 0.984]$	$(0.972, 0.986)$
Manipur	0.993	$[0.992, 0.995]$	$[0.99, 0.997]$	$[0.989, 0.998]$	$(0.988, 1)$
Meghayala	0.964	$[0.959, 0.968]$	$[0.956, 0.972]$	$[0.953, 0.976]$	$(0.951, 0.98)$
Mizoram	0.989	$[0.987, 0.992]$	$[0.984, 0.995]$	$[0.982, 0.997]$	$(0.98, 1)$
Nagaland	0.901	$[0.871, 0.926]$	$[0.841, 0.951]$	$[0.815, 0.975]$	$(0.792, 1)$
Odisha	0.975	$[0.973, 0.977]$	$[0.972, 0.979]$	$[0.971, 0.981]$	$(0.97, 0.983)$
Punjab	0.996	$[0.995, 0.997]$	$[0.994, 0.998]$	$[0.993, 0.997]$	$(0.992, 1)$
Rajasthan	0.998	$[0.998, 0.999]$	$[0.998, 0.999]$	$[0.995, 0.999]$	$(0.994, 1)$
Tamil Nadu	0.997	$[0.996, 0.998]$	$[0.995, 0.999]$	$[0.994, 0.999]$	$(0.993, 1)$
Telangana	0.999	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	$(0.999, 1)$
Tripura	0.998	$[0.997, 0.998]$	$[0.997, 0.998]$	$[0.997, 0.999]$	$(0.996, 1)$
Uttar Pradesh	0.993	$[0.99, 0.994]$	$[0.988, 0.996]$	$[0.986, 0.998]$	$(0.984, 1)$
Uttarakhand	0.986	$[0.983, 0.99]$	$[0.98, 0.993]$	$[0.978, 0.997]$	$(0.975, 1)$
West Bengal	0.999	$[0.999, 0.999]$	$[0.999, 0.999]$	$[0.999, 0.999]$	$(0.998, 1)$
Andaman Nikobar	0.973	$[0.965, 0.98]$	$[0.958, 0.987]$	$[0.95, 0.993]$	$(0.942, 1)$

Table 10. Ranking indices I_k for different values of α and β

	$\mathbf{0.25, 0.25}$	$\mathbf{0.25, 0.5}$	$\mathbf{0.25, 0.75}$	$\mathbf{0.5, 0.25}$	$\mathbf{0.75, 0.25}$	$\mathbf{0.5, 0.5}$
Andhra Pradesh	133.07	133.07	137.7	124.53	120.25	128.31
Arunachal Pradesh	148.42	148.42	181.12	80.65	61.7	91.55
Assam	3184.84	3184.84	3625.23	1666.68	1209.56	1813.48
Bihar	57.88	57.88	59.4	54.82	52.87	55.7
Chhattisgarh	2865.3	2865.3	3105.9	1473.82	1036.74	1554.02
Goa	64.34	64.34	73.97	34.66	25.83	37.86
Gujarat	63.27	63.27	63.81	61.56	60.41	62.06
Haryana	601.93	601.93	707.86	319.24	236.79	354.56
Himachal Pradesh	234.26	234.26	268.36	128.22	94.62	139.41
Jammu & Kashmir	612.45	612.45	699.94	321.77	234.58	350.85
Jharkhand	6283.26	6283.26	7316.16	3314.6	2439.81	3658.9
Karnataka	24.4	24.4	24.81	23.9	23.78	24.28
Kerala	743.72	743.72	756.84	382.61	264.7	389.98
Madhya Pradesh	3005.24	3005.24	3502.08	1586.24	1168.45	1751.86
Maharashtra	91.58	91.58	94.86	84.41	81.37	87.03
Manipur	962.1	962.1	1103.77	505.5	369.05	552.72
Meghayala	73.58	73.58	76.49	64.39	58.94	66.55
Mizoram	476.8	476.8	568.35	254.32	190.33	284.84
Nagaland	50.81	50.81	58.76	27.41	20.49	30.04
Odisha	78.84	78.84	81.58	73.74	71.08	75.88
Punjab	1730.66	1730.66	1984.16	908.37	662.43	992.87
Rajasthan	3659.81	3659.81	4253.68	1929.88	1419.54	2128.81
Tamil Nadu	3343.78	3343.78	3731.25	1737.43	1245.06	1866.59
Telangana	15055.11	15055.11	17744.97	7976.58	5915.94	8873.19
Tripura	3524.52	3524.52	4013.87	1844.54	1338.92	2007.66
Uttar Pradesh	1628.49	1628.49	1760.75	837.36	588.26	881.17
Uttarakhand	616.18	616.18	686.24	320.63	229.9	343.99
West Bengal	11394.32	11394.32	12713.33	5917.72	4238.75	6357.39
Andaman Nicobar	134.8	134.8	170.1	73.87	57.48	85.63

Table 11. Ranking of Indian States and UTs

	RANK					
State	**Col1**	**Col2**	**Col3**	**Col4**	**Col5**	**Col6**
Andhra Pradesh	21	21	21	19	18	19
Arunachal Pradesh	19	19	19	21	22	20
Assam	7	7	7	7	7	7
Bihar	27	27	27	26	26	26
Chhattisgarh	9	9	9	9	9	9
Goa	25	25	25	27	27	27
Gujarat	26	26	26	25	23	25
Haryana	16	16	14	16	14	14
Himachal Pradesh	18	18	18	18	19	18
Jammu & Kashmir	15	15	15	14	15	15
Jharkhand	3	3	3	3	3	3
Karnataka	29	29	29	29	28	29
Kerala	13	13	13	13	13	13
Madhya Pradesh	8	8	8	8	8	8
Maharashtra	22	22	22	20	20	21
Manipur	12	12	12	12	12	12
Meghalaya	24	24	24	24	24	24
Mizoram	17	17	17	17	17	17
Nagaland	28	28	28	28	29	28
Odisha	23	23	23	23	21	23
Punjab	10	10	10	10	10	10
Rajasthan	4	4	4	4	4	4
Tamil Nadu	6	6	6	6	6	6
Telangana	1	1	1	1	1	1
Tripura	5	5	5	5	5	5
Uttar Pradesh	11	11	11	11	11	11
Uttarakhand	14	14	16	15	16	16
West Bengal	2	2	2	2	2	2
Andaman Nicobar	20	20	20	22	25	22

7 Conclusion

In this paper, we have determined the interval efficiencies of DMUs with IF inputs and outputs (TIFNs in particular). The traditional efficiency models are based on favorable–unfavorable approach [16]. The main drawback of this approach is the fact that the feasible region changes with each DMU and hence the comparison of DMUs based on this approach is not justified. For crisp data, an alternative to this approach was proposed by [28]. We have extended this approach to IFNs using α-, β- cuts. Using the proposed method, the interval efficiencies of each DMU is determined. After determining the interval efficiencies, a ranking method is proposed for ranking of DMUs. Finally, a numerical example with 12 DMUs, 2 inputs as TIFNs and 2 outputs as TIFNs is illustrated to test the validity of the model. As an application of the proposed method, the performance of Indian States and Union Territories working under the scheme MGNREGA 2005 has been checked. The states of Telangana, West Bengal and Jharkhand have emerged as the best performers in providing employment to needy while the states of Karnataka, Nagaland, Gujarat and Goa have performed poorly.

The proposed method defines a much more realistic IFDEA model with efficiencies as intervals and it also provides an almost uniform ranking of DMUs.

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