Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy level topological structures

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Abstract: The paper is a continuation of the series of papers devoted to the concept of a Modal Topological Structure (MTS) and its modification and extension. Here we introduce two new modifications of MTSs – the Level Topological Structures (LTSs, when the level operators play the role of the modal ones) and Modal Level (Topological) Structures (ML(T)Ss, when the level operators play the role of the topological ones). In addition, three extended MTSs are discussed, too – the Modal LTSs (MLTSs, when they contain simultaneously modal, level and topological operators), Modal Level Temporal Topological Structures (ML$\mathcal{T}$TSs, when they contain simultaneously modal, level, temporal and topological operators) and multi-LTSs (mLTSs, when the number of their level operators is higher than 1). All these structures are illustrated with examples from intuitionistic fuzzy sets theory.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy modal topological structure, Modal topological structure.
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1 Introduction

The concept of a Modal Topological Structure (MTS) was introduced by the author in 2022 in [2] and it was modified and extended in about 20 subsequent publications. One of the resultant modifications are the Temporal Topological Structures ($\mathcal{T}$TSs, [6, 8]), while among the MTS extensions are the Modal Temporal Topological Structures (M$\mathcal{T}$TS, [9]), Bi-, Tri- and Multi-MTSs (see [4, 5, 7]), and others. All these structures are illustrated with examples from the area of the intuitionistic fuzzy sets (IFSs, see, e.g., [1]).

In the present paper, we will introduce two new modifications of MTSs and three extended MTSs. All these structures are again illustrated with examples from intuitionistic fuzzy sets theory.

All notations, related to IFSs, can be found, e.g., in [1]. In Section 2, we will discuss only some of the definitions, related to the present research.

The conditions for the topological operators in a given MTS are formulated in [2] and coincide with these from Section 4 (they are based on the conditions from [10, 13]), while the conditions for the logical operators in the same MTS are formulated also in [2] and coincide with these from Section 3 (they are modifications of these from [11, 12, 14]).

2 Short remarks on the intuitionistic fuzzy level operators

The following two level operators defined over the IFS

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

are (see [1]):

$$\begin{aligned} P_{\alpha,\beta}(A) &= \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\}; \\ Q_{\alpha,\beta}(A) &= \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\}, \end{aligned}$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. We directly see that

$$\begin{aligned} P_{\alpha,\beta}(A) &= A \cup \{\langle x, \alpha, \beta\rangle | x \in E\}; \\ Q_{\alpha,\beta}(A) &= A \cap \{\langle x, \alpha, \beta\rangle | x \in E\}. \end{aligned}$$

Therefore, we can say that operator $P_{\alpha,\beta}$ is from a "union ($\cup$-) type" or from closure type, while operator $Q_{\alpha,\beta}$ is from "intersection" ($\cap$-) type" or from interior type.

In this paper, the level operators will play two roles: the role of the modal operators and the role of the topological operators.

3 Intuitionistic fuzzy level topological structures

When the level operators play the role of the modal operators, they must satisfy the conditions Cm1–Cm3 and Im1–Im3, in the following forms, respectively (we will keep the notation from Section 3 for these conditions):

Cm1. $\circ(A \nabla B) = \circ A \nabla \circ B$,

Cm2. $A \subseteq \circ A$,

Cm3. $\circ \circ A = \circ A$,

Im1. $\bullet(A\Delta B) = \bullet A\Delta \bullet B$,

Im2. $\bullet A \subseteq A$,

Im3. $\bullet\bullet A = \bullet A$,

where $\circ$ and $\bullet$ are the level operators, Δ and ∇ are operations, $A, B \in \mathcal{P}(X)$. In addition, when $\mathcal{E}$ is a topological operator (from cl- or in-type) and $*$ is a level operator (from cl- or in-type), we will assume that they satisfy the equality:

$$* \,\mathcal{E}(A) = \mathcal{E}(*A) \qquad (*)$$

keeping the notation from [2].

Now, we will call the structure with the form

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta, *, \nabla\rangle$$

a Level Topological Structure (LTS).

Theorem 1. *For each universe E and for every two real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$,*

(a) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup\rangle$ *is an IFLTS.*

(b) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap\rangle$ *is an IFLTS.*

(c) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, Q_{\alpha,\beta}, \cup\rangle$ *is an IFLTS.*

(d) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, Q_{\alpha,\beta}, \cap\rangle$ *is an IFLTS.*

(e) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, Q_{\alpha,\beta}, \cup\rangle$ *is an IFLTS.*

(f) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, Q_{\alpha,\beta}, \cap\rangle$ *is an IFLTS.*

(g) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup\rangle$ *is an IFLTS.*

(h) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap\rangle$ *is an IFLTS.*

Proof. Let us prove some of the assertions formulated above.

(b) Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. The proof of the validity of the conditions Ct1–Ct4 are given in [2]. So, we will sequentially prove the validity only of the conditions Cm1–Cm3 and (*).

Cm1.

$$\begin{aligned}
P_{\alpha,\beta}(A \cap B) &= P_{\alpha,\beta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, \max(\alpha, \min(\mu_A(x), \mu_B(x))), \min(\beta, \max(\nu_A(x), \nu_B(x)))\rangle | x \in E\} \\
&= \{\langle x, \min(\max(\alpha, \mu_A(x)), \max(\alpha, \mu_B(x))), \\
&\qquad \max(\min(\beta, \nu_A(x)), \min(\beta, \nu_B(x)))\rangle | x \in E\} \\
&= \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\} \\
&\quad \cap \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\} \\
&= P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B);
\end{aligned}$$

Cm2.

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\ &\subseteq \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\} \\ &= P_{\alpha,\beta}(A); \end{aligned}$$

Cm3.

$$\begin{aligned} P_{\alpha,\beta}(P_{\gamma,\delta}(A)) &= P_{\alpha,\beta}(\{\langle x, \max(\gamma, \mu_A(x)), \min(\delta, \nu_A(x))\rangle | x \in E\}) \\ &= \{\langle x, \max(\alpha, \max(\gamma, \mu_A(x))), \min(\beta, \min(\delta, \nu_A(x)))\rangle | x \in E\} \\ &= \{\langle x, \max(\max(\alpha, \gamma), \mu_A(x)), \min(\min(\beta, \delta), \nu_A(x))\rangle | x \in E\} \\ &= P_{\max(\alpha,\gamma),\min(\beta,\delta)}(A); \end{aligned}$$

(*)

$$\begin{aligned} P_{\alpha,\beta}(\mathcal{C}(A)) &= P_{\alpha,\beta}(\{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\}) \\ &= \{\langle x, \max(\alpha, \sup_{y\in E} \mu_A(y)), \min(\beta, \inf_{y\in E} \nu_A(y))\rangle | x \in E\} \\ &= \{\langle x, \sup_{y\in E}(\max(\alpha, \mu_A(y))), \inf_{y\in E}(\min(\beta, \nu_A(y)))\rangle | x \in E\} \\ &= \mathcal{C}(\{\langle x, \max(\alpha, \mu_A(y)), \min(\beta, \nu_A(y))\rangle | x \in E\}) \\ &= \mathcal{C}(P_{\alpha,\beta}(A)). \end{aligned}$$

(c) Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. We will sequentially prove the validity of the conditions Im1–Im3 and (*).

Im1.

$$\begin{aligned} Q_{\alpha,\beta}(A \cup B) &= Q_{\alpha,\beta}(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\ &= \{\langle x, \min(\alpha, \max(\mu_A(x), \mu_B(x))), \max(\beta, \min(\nu_A(x), \nu_B(x)))\rangle | x \in E\} \\ &= \{\langle x, \max(\min(\alpha, \mu_A(x)), \min(\alpha, \mu_B(x))), \\ &\qquad \min(\max(\beta, \nu_A(x)), \max(\beta, \nu_B(x)))\rangle | x \in E\} \\ &= \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\} \\ &\quad \cup \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\} \\ &= Q_{\alpha,\beta}(A) \cup Q_{\alpha,\beta}(B); \end{aligned}$$

Im2.

$$\begin{aligned} Q_{\alpha,\beta}(A) &= \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\} \\ &\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\ &= A; \end{aligned}$$

Im3.

$$\begin{aligned} Q_{\alpha,\beta}(Q_{\gamma,\delta}(A)) &= Q_{\alpha,\beta}(\{\langle x, \min(\gamma, \mu_A(x)), \max(\delta, \nu_A(x))\rangle\} | x \in E\}) \\ &= \{\langle x, \min(\alpha, \min(\gamma, \mu_A(x))), \max(\beta, \max(\delta, \nu_A(x)))\rangle\} | x \in E\} \\ &= \{\langle x, \min(\min(\alpha, \gamma), \mu_A(x)), \max(\max(\beta, \delta), \nu_A(x))\rangle\} | x \in E\} \\ &= Q_{\min(\alpha,\gamma),\max(\beta,\delta)}(A); \end{aligned}$$

(*)

$$\begin{aligned}Q_{\alpha,\beta}(\mathcal{I}(A)) &= Q_{\alpha,\beta}(\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x\in E\})\\ &= \{\langle x, \min(\alpha, \inf_{y\in E}\mu_A(y)), \max(\beta, \sup_{y\in E}\nu_A(y))\rangle | x\in E\})\\ &= \{\langle x, \inf_{y\in E}(\min(\alpha,\mu_A(y))), \sup_{y\in E}(\max(\beta,\nu_A(y)))\rangle | x\in E\}\\ &= \mathcal{I}(\{\langle x, \min(\alpha,\mu_A(y)), \max(\beta,\nu_A(y))\rangle | x\in E\})\\ &= \mathcal{I}(Q_{\alpha,\beta}(A)).\end{aligned}$$

(e) Let the IFS $A \in \mathcal{P}(E^*)$ be given. Now, we must check only the validity of the conditions Cm1 and (*).

For the validity of condition (Cm1) we obtain:

$$\begin{aligned}Q_{\alpha,\beta}(A\cap B) &= Q_{\alpha,\beta}(\{\langle x, \min(\mu_A(x),\mu_B(x)), \max(\nu_A(x),\nu_A(x))\rangle | x\in E\}\\ &= \{\langle x, \min(\alpha, \min(\mu_A(x),\mu_B(x))), \max(\beta, \max(\nu_A(x),\nu_B(x)))\rangle | x\in E\}\\ &= \{\langle x, \min(\min(\alpha,\mu_A(x)), \min(\alpha,\mu_B(x))),\\ &\qquad \max(\max(\beta,\nu_A(x)), \max(\beta,\nu_B(x)))\rangle | x\in E\}\\ &= \{\langle x, \min(\alpha,\mu_A(x)), \max(\beta,\nu_A(x))\rangle | x\in E\}\\ &\qquad \cap\{\langle x, \min(\alpha,\mu_B(x)), \max(\beta,\nu_B(x))\rangle | x\in E\}\\ &= Q_{\alpha,\beta}(A)\cap Q_{\alpha,\beta}(B).\end{aligned}$$

For the validity of condition (*), we obtain

$$\begin{aligned}Q_{\alpha,\beta}(\mathcal{I}(A)) &= Q_{\alpha,\beta}(\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x\in E\})\\ &= \{\langle x, \min(\alpha, \inf_{y\in E}\mu_A(y)), \max(\beta, \sup_{y\in E}\nu_A(y))\rangle | x\in E\}\\ &= \{\langle x, \inf_{y\in E}\min(\alpha,\mu_A(y)), \sup_{y\in E}\max(\beta,\nu_A(y))\rangle | x\in E\}\\ &= \mathcal{I}(\{\langle x, \min(\alpha,\mu_A(x)), \max(\beta,\nu_A(x))\rangle | x\in E\})\\ &= \mathcal{I}(Q_{\alpha,\beta}(A)).\end{aligned}$$

(g) Let the IFS $A \in \mathcal{P}(E^*)$ be given. We must check only the validity of the conditions Im1 and (*).

$$\begin{aligned}P_{\alpha,\beta}(A\cup B) &= P_{\alpha,\beta}(\{\langle x, \max(\mu_A(x),\mu_B(x)), \min(\nu_A(x),\nu_A(x))\rangle | x\in E\}\\ &= \{\langle x, \max(\alpha, \max(\mu_A(x),\mu_B(x))), \min(\beta, \min(\nu_A(x),\nu_B(x)))\rangle | x\in E\}\\ &= \{\langle x, \max(\max(\alpha,\mu_A(x)), \max(\alpha,\mu_B(x))),\\ &\qquad \min(\min(\beta,\nu_A(x)), \min(\beta,\nu_B(x)))\rangle | x\in E\}\\ &= \{\langle x, \max(\alpha,\mu_A(x)), \min(\beta,\nu_A(x))\rangle | x\in E\}\\ &\qquad \cup\{\langle x, \max(\alpha,\mu_B(x)), \min(\beta,\nu_B(x))\rangle | x\in E\}\\ &= P_{\alpha,\beta}(A)\cup P_{\alpha,\beta}(B).\end{aligned}$$

For the validity of condition (*), we obtain

$$\begin{aligned}P_{\alpha,\beta}(\mathcal{I}(A)) &= P_{\alpha,\beta}(\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x\in E\})\\ &= \{\langle x, \max(\alpha, \inf_{y\in E}\mu_A(y)), \min(\beta, \sup_{y\in E}\nu_A(y))\rangle | x\in E\})\\ &= \{\langle x, \inf_{y\in E}\max(\alpha,\mu_A(y)), \sup_{y\in E}\min(\beta,\nu_A(y))\rangle | x\in E\})\\ &= \mathcal{I}(\{\langle x, \max(\alpha,\mu_A(x)), \min(\beta,\nu_A(x))\rangle | x\in E\})\\ &= \mathcal{I}(P_{\alpha,\beta}(A)).\end{aligned}$$

The remaining assertions are proved by the same manner. □

4 Intuitionistic fuzzy modal level (topological) structures

The level operators can play the role of the topological operators, because if for some IFS A:

$$K(A) = \sup_{y \in E} \mu_A(y), \qquad L(A) = \inf_{y \in E} \nu_A(y),$$

$$k(A) = \inf_{y \in E} \mu_A(y), \qquad l(A) = \sup_{y \in E} \nu_A(y),$$

then we cal see immediately that:

$$\mathcal{C}(A) = P_{K,L}(A),$$

$$\mathcal{I}(A) = Q_{k,l}(A).$$

Now, for every $A, B \in \mathcal{P}(X)$ and for every $\alpha, \beta \in [0, 1]$, so that $\alpha + \beta \in [0, 1]$, the level operators must satisfy the conditions Ct1–Ct4 and It1–It4. Unfortunately, as we will show below, both operators do not satisfy the respective fourth condition. So, the structures that we will describe below are feeble ones (cf. [3]). They have the following forms, respectively (we will keep the notation from Section 4 for these conditions):

Ct1. $P_{\alpha,\beta}(A \Delta B) = P_{\alpha,\beta}(A) \Delta P_{\alpha,\beta}(B)$,

Ct2. $A \subseteq P_{\alpha,\beta}(A)$,

Ct3. $P_{\alpha,\beta}(P_{\alpha,\beta}(A)) = P_{\alpha,\beta}(A)$,

Ct4. $P_{\alpha,\beta}(O) \supseteq O$,

and

It1. $Q_{\alpha,\beta}(A \nabla B) = Q_{\alpha,\beta}(A) \nabla Q_{\alpha,\beta}(B)$,

It2. $Q_{\alpha,\beta}(A) \subseteq A$,

It3. $Q_{\alpha,\beta}(Q_{\alpha,\beta}(A)) = Q_{\alpha,\beta}(A)$,

It4. $Q_{\alpha,\beta}(X) \subseteq X$.

In this section, we will use the notations "FM($cl, 4 \supseteq$)-L(T)TS" and "FM($in, 4 \subseteq$)-L(T)TS". When X is a universe E of IFSs, the notations will be IFFM($cl, 4 \supseteq$)-L(T)TS and "IFFM($in, 4 \subseteq$)-L(T)TS".

Theorem 2. *For each universe E and for every two real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$,*

(a) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cup, \diamondsuit, \cup \rangle$ *is an IFFM($cl, 4 \supseteq$)-L(T)TS.*

(b) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cup, \diamondsuit, \cap \rangle$ *is an IFFM($cl, 4 \supseteq$)-L(T)TS.*

(c) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cup, \square, \cup \rangle$ *is an IFFM($cl, 4 \supseteq$)-L(T)TS.*

(d) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cup, \square, \cap \rangle$ *is an IFFM($cl, 4 \supseteq$)-L(T)TS.*

(e) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cap, \diamondsuit, \cup \rangle$ *is an IFFM*$(cl, 4 \supseteq)$*-L(T)TS.*

(f) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cap, \diamondsuit, \cap \rangle$ *is an IFFM*$(cl, 4 \supseteq)$*-L(T)TS.*

(g) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFFM*$(cl, 4 \supseteq)$*-L(T)TS.*

(h) $\langle \mathcal{P}(E^*), P_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFFM*$(cl, 4 \supseteq)$*-L(T)TS.*

(i) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cup, \diamondsuit, \cup \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(j) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cup, \diamondsuit, \cap \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(k) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cup, \Box, \cup \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(l) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cup, \Box, \cap \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(m) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cap, \diamondsuit, \cup \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(n) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cap, \diamondsuit, \cap \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(o) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

(p) $\langle \mathcal{P}(E^*), Q_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFFM*$(in, 4 \subseteq)$*-L(T)TS.*

Proof. Almost all checks are already done. So, we must check only the conditions Ct4 and It4, that are:

$$\begin{aligned} P_{\alpha,\beta}(O^*) &= P_{\alpha,\beta}(\{\langle x, 0, 1\rangle | x \in E\}) \\ &= \{\langle x, \max(\alpha, 0), \min(\beta, 1)\rangle | x \in E\} \\ &= (\{\langle x, \alpha, \beta\rangle | x \in E\}) \\ &\supseteq O^*; \end{aligned}$$

and

$$\begin{aligned} Q_{\alpha,\beta}(E^*) &= \{\langle x, \min(\alpha, 1), \max(\beta, 0)\rangle | x \in E\} \\ &= (\{\langle x, \alpha, \beta\rangle | x \in E\}) \\ &\subseteq E^* \end{aligned}$$

that proves the theorem. □

5 Other intuitionistic fuzzy level topological structures

Here, the level operators will play the role of modal ones, i.e., for them the conditions from the Ct4- and It4-type will be not necessary. Therefore, we can introduce some new structures.

First, the structure

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta, \mathcal{R}, \nabla_1, *, \nabla_2 \rangle$$

will be named Modal Level Topological Structure (MLTS), when $\mathcal{E}$ is a topological operator, $\mathcal{R}$ is a level operator, * is a modal operator,

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta, *, \nabla_2 \rangle$$

is a MTS and

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta, \mathcal{R}, \nabla_1 \rangle$$

is a LTS.

When X is a universe E of IFSs, the abbreviation will be "IFMLTS", because below we will not discuss the feeble cases.

Theorem 3. *For each universe E and for every two real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$,*

(a) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Diamond, \cup \rangle$ *is an IFMLTS.*

(b) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Diamond, \cap \rangle$ *is an IFMLTS.*

(c) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Box, \cup \rangle$ *is an IFMLTS.*

(d) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Box, \cap \rangle$ *is an IFMLTS.*

(e) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Diamond, \cup \rangle$ *is an IFMLTS.*

(f) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Diamond, \cap \rangle$ *is an IFMLTS.*

(g) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFMLTS.*

(h) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFMLTS.*

(i) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Diamond, \cup \rangle$ *is an IFMLTS.*

(j) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Diamond, \cap \rangle$ *is an IFMLTS.*

(k) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Box, \cup \rangle$ *is an IFMLTS.*

(l) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cup, \Box, \cap \rangle$ *is an IFMLTS.*

(m) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Diamond, \cup \rangle$ *is an IFMLTS.*

(n) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Diamond, \cap \rangle$ *is an IFMLTS.*

(o) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFMLTS.*

(p) $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, P_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFMLTS.*

Theorem 4. *For each universe E and for every two real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$,*

(a) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Diamond, \cup \rangle$ *is an IFMLTS.*

(b) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Diamond, \cap \rangle$ *is an IFMLTS.*

(c) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Box, \cup \rangle$ *is an IFMLTS.*

(d) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Box, \cap \rangle$ *is an IFMLTS.*

(e) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Diamond, \cup \rangle$ *is an IFMLTS.*

(f) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Diamond, \cap \rangle$ *is an IFMLTS.*

(g) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFMLTS.*

(h) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFMLTS.*

(i) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Diamond, \cup \rangle$ *is an IFMLTS.*

(j) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Diamond, \cap \rangle$ *is an IFMLTS.*

(k) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Box, \cup \rangle$ *is an IFMLTS.*

(l) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cup, \Box, \cap \rangle$ *is an IFMLTS.*

(m) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Diamond, \cup \rangle$ *is an IFMLTS.*

(n) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Diamond, \cap \rangle$ *is an IFMLTS.*

(o) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Box, \cup \rangle$ *is an IFMLTS.*

(p) $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, P_{\alpha,\beta}, \cap, \Box, \cap \rangle$ *is an IFMLTS.*

Second, the structure

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta_1, \mathcal{T}, \Delta_2, \mathcal{R}, \nabla_1, *, \nabla_2 \rangle$$

will be named Modal Level Temporal Topological Structure (ML$\mathcal{T}$TS), when $\mathcal{E}$ is a topological operator, $\mathcal{T}$ – a temporal operator, $\mathcal{R}$ – a level operator, * – a modal operator,

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta_1, \mathcal{T}, \Delta_2, *, \nabla_2 \rangle$$

is a M$\mathcal{T}$TS and

$$\langle \mathcal{P}(X), \mathcal{E}, \Delta_1, \mathcal{R}, \nabla_1, *, \nabla_2 \rangle$$

is a MLTS.

In general, we can construct 64 different ML$\mathcal{T}$TSs and 192 feeble ML$\mathcal{T}$TSs. So, we can prove the following assertion in the above manner.

Theorem 5. *For each universe E and for every two real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$,*

$$\langle \mathcal{P}(E^*), \mathcal{E}, \Delta_1, \mathcal{T}, \Delta_2, \mathcal{R}, \nabla_1, *, \nabla_2 \rangle$$

is a ML$\mathcal{T}$TS, where:

- *when $\mathcal{E}$ is the topological operator $\mathcal{C}$, then Δ_1 is operation $\cup$,*
- *when $\mathcal{E}$ is the topological operator $\mathcal{I}$, then Δ_1 is operation $\cap$,*
- *when $\mathcal{T}$ is the temporal operator $\mathcal{C}^*$, then Δ_2 is operation $\cup$,*
- *when $\mathcal{T}$ is the temporal operator $\mathcal{I}^*$, then Δ_2 is operation $\cap$,*
- *when $\mathcal{R}$ is the level operator $P_{\alpha,\beta}$ or $Q_{\alpha,\beta}$, then ∇_1 is operation $\cup$ or $\cap$,*
- *when $\mathcal{R}$ is the modal operator $\Diamond$ or $\Box$, then ∇_2 is operation $\cup$ or $\cap$.*

Third, let us have two increasing sequences $\{\alpha_i\}_{i=1}^t$ and $\{\gamma_j\}_{j=1}^w$, and two decreasing sequences $\{\beta_i\}_{i=1}^t$, $\{\delta_j\}_{j=1}^w$ so that for each i $(1 \leq i \leq t)$, for each j $(1 \leq j \leq w)$:

$$\alpha_i, \beta_i, \alpha_i + \beta_i \in [0, 1]$$

and

$$\gamma_j, \delta_j, \gamma_j + \delta_j \in [0, 1].$$

Then for each $A \in \mathcal{P}(E^*)$:

$$Q_{\gamma_1,\delta_1}(A) \subseteq \cdots \subseteq Q_{\gamma_w,\delta_w}(A) \subseteq A \subseteq P_{\alpha_1,\beta_1}(A) \subseteq \cdots \subseteq P_{\alpha_t,\beta_t}(A).$$

Therefore, we can construct an object with the form

$$\langle \mathcal{P}(E^*), \mathcal{E}, \Delta, P_{\alpha_1,\beta_1}(A), \ldots, P_{\alpha_t,\beta_t}(A), \nabla_1, Q_{\gamma_1,\delta_1}(A), \ldots, Q_{\gamma_w,\delta_w}(A), \nabla_2 \rangle$$

for which, as above, we can prove that it is an IF multi-LTS.

Fourth, by analogy with the previous three structures, we can construct a structure with u cl-topological and v in-topological operators, t cl-temporal and w in-temporal operators, and n cl-modal and m in-modal operators.

6 Conclusion

The present research shows the possibility the level operators defined over IFSs to play as the role of modal operators, as well as the role of topological operators in topological structures. By the moment, the level operators do not have analogues in topology, but as we saw above, they can be components of some types of IFTSs. It will be interesting, if they can obtain other interpretations.

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