Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy G-modules with respect to norms (T and S)

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Abstract: In this paper, we introduced intuitionistic fuzzy G-modules on M under norms (t-norm T and s-conorm S) and some related results like intersection, sum and direct sum of them has also been discussed. Also some of their properties have been investigated under G-module homomorphisms.
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1 Introduction

Group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science. Group theory is also central to public key cryptography. In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module.

In mathematics, given a group G, a G-module is an abelian group M on which G acts compatibly with the abelian group structure on M. The term G-module is also used for the more general notion of an R-module on which G acts linearly (i.e. as a group of R-module automorphisms). Representation theory (G-module theory) has had its origin in the 20^{th} century. In 1965, Zadeh [22] introduced the concept of fuzzy subset as a generalization of the notion of characteristic function in classical set theory. The concept of intuitionistic fuzzy set was introduced by Atanassov [2], as a generalization of the notion of fuzzy set. The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. Fernadez introduced and studied the notion of fuzzy G-modules in [4]. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [6], Alsina et al. [1] introduced the T-norm and the S-norm into fuzzy set theory and suggested that the T-norm be used for the intersection and union of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [5, 21]. In practice, Zadeh's conventional T-norm, $\bigwedge$ and $\bigvee$, have been used in almost every design for fuzzy logic controllers and even in the modelling of other decision-making processes. In previous works [8, 11–20], by using norms, we investigated some properties of fuzzy algebraic structures. Here in this paper, we define anti fuzzy G-submodules with respect to t-conorms and investigate some of their algebraic properties. Later we introduce the union and direct sum of them and finally, we prove that the union, direct sum, homomorphic images and pre images of theirs are also anti fuzzy G-submodules with respect to norms (T and S).

2 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. For details we refer readers to [3, 7, 9]. Throughout the paper, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ will always be rational, real and complex numbers, respectively.

Definition 2.1. Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,

(1) $r.(m + n) = r.m + r.n,$

(2) $r.(s.m) = (rs).m,$

(3) $(r + s).m = r.m + s.m.$

If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.

A right R-module can be defined in a similar fashion. Note that throughout this paper, R-modules will be left R-modules.

Definition 2.2. Let M be an R-module and N be a nonempty subset of M. Then N is called a submodule of M if N is a subgroup of M and for all $r \in R, a \in N$, we have $ra \in N.$

Definition 2.3. Let G be a finite group. A vector space M over a field K is called a G-module if for every $g \in G$ and $m \in M$, there exist a product (called the action of G on M) $m.g \in M$ satisfying the following axioms:

(1) $m.1_G = m, \forall m \in M$ (1_G being the identity element in G),

(2) $m.(g.h) = (m.g).h$, $\forall m \in M : g, h \in G$, and

(3) $(k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g)$ $\forall m_1, m_2 \in M : g \in G : k_1, k_2 \in K.$

Example 2.4. Let $G = \{1, -1, i, -i\}$ and $M = \mathbb{C}^n$ with $n \geq 1$. Then M is a vector space over $\mathbb{C}$ and under the usual addition and multiplication of complex numbers, we can show that M is a G-module.

Remark 2.5. The operation $(m, g) \to m.g$ defined above may be called a right-action of G on M and M may be said to be a right G-module. In a similar way, we can define left-action and left G-module. We shall consider all G-modules as left G-modules.

Definition 2.6. Let M be a G-module. A vector subspaee N of M is a G-submodule if N is also a G-module under the same action of G. Thus N is G-submodule of G-module M if and only if N is submodule of M and N be a G-module.

Example 2.7. Let $\mathbb{Q}$ be the field of rationals and $G = \{1, -1\}$ and $M = \mathbb{R}$. Then M is a G-module over $\mathbb{Q}$. Now for each $r \notin \mathbb{Q}$ we get that $N = \mathbb{Q}(r)$ is a G-submodule of M.

Definition 2.8. Let M and N be G-modules. A mapping $f : M \to M$ is a G-module homomorphism if

(1) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$

(2) $f(gm) = gf(m)$

for all $m_1, m_2m \in M$ and $k_1, k_2 \in K$ and $g \in G$.

Definition 2.9. Let X be an arbitrary set. By a fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ and the set $\mu_s = \{x \in X : \mu(x) \leq s\}$ is called a lower level of μ.

Definition 2.10. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0, 1) \sim 0$ and $U_X(x) = (1, 0) \sim 1$, respectively. We will denote the set of all IFSs in X as IFS(X).

Definition 2.11. Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A) \in$ IFS(X) and $B = (\mu_B, \nu_B) \in$ IFS(Y). For all $x \in X, y \in Y$, we define

$$\begin{aligned}\varphi(A)(y) &= (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) \\ &= \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0, 1) & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}\end{aligned}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Definition 2.12. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.13. (1) Standard intersection t-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum t-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) Algebraic product t-norm $T_p(x, y) = xy$.
(4) Drastic t-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Definition 2.14. An s-norm S is a function $S : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(1) $S(x, 0) = x$,
(2) $S(x, y) \leq S(x, z)$ if $y \leq z$,
(3) $S(x, y) = S(y, x)$,
(4) $S(x, S(y, z)) = S(S(x, y), z)$,

for all $x, y, z \in [0, 1]$.

Example 2.15. The basic s-norms are $S_m(x, y) = \max\{x, y\}, S_b(x, y) = \min\{1, x + y\}$ and $S_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$. Thus S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

We say that T and S are idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$ and $S(x, x) = x$.

Definition 2.16. Let $A = (\mu_A, \nu_A) \in \text{IFS}(X)$ and $B = (\mu_B, \nu_B) \in \text{IFS}(X)$. Define

$$A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) : X \to [0,1]$$

as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = S(\nu_A(x), \nu_B(x))$ for all $x \in X$.

Lemma 2.17 ([11]). *Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

$$S(S(x,y), S(w,z)) = S(S(x,w), S(y,z)),$$

for all $x, y, w, z \in [0,1]$.

3 Main results

Definition 3.1. Let G be a finite group and M be a G-module over K, which is a subfield of $\mathbb{C}$. Define $A = (\mu_A, \nu_A) \in \text{IFS}(M)$ an intuitionistic fuzzy G-module on M under norms (t-norm T and s-conorm S) if it satisfies the following inequalities:
(1) $\mu_A(ax + by) \geq T(\mu_A(x), \mu_A(y))$,
(2) $\mu_A(gm) \geq \mu_A(m)$,
(3) $\nu_A(ax + by) \leq S(\nu_A(x), \nu_A(y))$,
(4) $\nu_A(gm) \leq \nu_A(m)$,
for all $a, b \in K : x, y \in M : m \in M$ and $g \in G$.
Denote by IFMN(M), the set of all intuitionistic fuzzy G-modules on M under norms (t-norm T and s-conorm S).

Example 3.2. Let $G = \{1, -1\}$ and $M = \mathbb{R}^4$ is a vector space over real field $\mathbb{R}$. Then M is a G-module over $\mathbb{R}$. Define $\mu_A, \nu_A : M \to [0,1]$ by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x_i = 0, \forall i \\ 0.35, & \text{if at least one } x_i = 0 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0, & \text{if } x_i = 0, \forall i \\ 0.55, & \text{if at least one } x_i = 0 \end{cases}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_i \in \mathbb{R}$. If $T(a,b) = T_m(a,b) = \min\{a,b\}$ and $S(a,b) = S_m(a,b) = \max\{a,b\}$ for all $a, b \in [0,1]$, then $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$.

Example 3.3. Let F be a field, K be an extension field of F and $a \in K$. Let $F(a)$ be the field obtained by adjoining a to F as $F(a) = \{b_0 + b_1 a + b_2 a^2 + \cdots\}$ with $b_i \in F$. If $G = (a)$, be the cyclic group generated by a, then $M = F(a)$ will be G-module. Define $\mu_A, \nu_A : M \to [0,1]$ by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.15, & \text{if } x \in F - \{0\} \\ 0.55, & \text{if } x \in F(a) - F \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.65, & \text{if } x \in F - \{0\} \\ 0.25, & \text{if } x \in F(a) - F. \end{cases}$$

Let $T(a,b) = T_b(a,b) = \max\{0, a+b-1\}$ and $S(a,b) = S_b(a,b) = \min\{1, a+b\}$ for all $a, b \in [0,1]$ then $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$.

Example 3.4. Consider the G-module $M = \mathbb{C}$ over the field $\mathbb{R}$ where $G = \{\pm 1\}$.

Define $\mu_A, \nu_A : M \to [0,1]$ by

$$\mu_A(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0.25, & \text{if } z \in \mathbb{R} - \{0\} \\ 0.35, & \text{if } z \in \mathbb{C} - \mathbb{R}. \end{cases}$$

and

$$\nu_A(z) = \begin{cases} 0, & \text{if } z = 0 \\ 0.45, & \text{if } z \in \mathbb{R} - \{0\} \\ 0.55, & \text{if } z \in \mathbb{C} - \mathbb{R}. \end{cases}$$

Let $T(x,y) = T_p(x,y) = xy$ and $S(a,b) = S_p(a,b) = a + b - ab$ for all $a, b \in [0,1]$ then $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$.

Proposition 3.5. *Let M be a G-module over K and μ be a fuzzy set of M. Let $A = (\mu_A, \nu_A) \in$ IFMN(M) and T, S be idempotent. Then the $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ is either empty or a G-submodule of M for every $s, t \in [0,1]$.*

Proof. Let $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$ and $A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$ be not empty. Let $x, y \in U(\mu, \alpha)$ and $a, b \in K$. Then $\mu_A(x) \geq s$ and $\mu_A(y) \geq s$ and $\nu_A(x) \leq t$ and $\nu_A(y) \leq t$. Now

$$\mu_A(ax+by) \geq T(\mu_A(x), \mu_A(y)) \geq T(s,s) = s,$$

$$\nu_A(ax+by) \leq S(\nu_A(x), \nu_A(y)) \leq S(t,t) = t,$$

so $A(ax+by) \supseteq (s,t)$ then $ax+by \in A_{s,t}$. Also $\mu_A(gx) \geq \mu_A(x) \geq s$ and $\nu_A(gx) \leq \nu_A(x) \leq t$ mean that $A(gx) \supseteq (s,t)$ and then $gx \in A_{s,t}$. Thus $A_{s,t}$ will be G-submodule of M. □

Proposition 3.6. *Let M be a G-module over K. If $A = (\mu_A, \nu_A) \in$ IFMN(M) and $B = (\mu_B, \nu_B) \in$ IFMN(M), then $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in$ IFMN(M).*

Proof. Let $x, y \in M$ and $a, b \in K$ and $g \in G$.
As

$$\begin{aligned}(\mu_{A\cap B})(ax+by) &= T(\mu_A(ax+by), \mu_B(ax+by)) \\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \ \textit{(from Lemma 2.17)} \\ &= T((\mu_{A\cap B})(x), (\mu_{A\cap B})(y)),\end{aligned}$$

then $(\mu_{A\cap B})(ax+by) \geq T((\mu_{A\cap B})(x), (\mu_{A\cap B})(y))$.

Also

$$(\mu_{A\cap B})(gx) = T(\mu_A(gx), \mu_B(gx)) \geq T(\mu_A(x), \mu_B(x)) = (\mu_{A\cap B})(x).$$

Moreover

$$\begin{aligned}(\nu_{A\cap B})(ax+by) &= S(\nu_A(ax+by), \nu_B(ax+by))\\ &\leq S(S(\nu_A(x), \nu_A(y)), S(\nu_B(x), \nu_B(y)))\\ &= S(S(\nu_A(x), \nu_B(x)), S(\nu_A(y), \nu_B(y))) \text{ (\textit{from Lemma 2.17})}\\ &= S((\nu_{A\cap B})(x), (\nu_{A\cap B})(y)),\end{aligned}$$

then $(\nu_{A\cap B})(ax+by) \leq S((\nu_{A\cap B})(x), (\nu_{A\cap B})(y))$.
Further

$$(\nu_{A\cap B})(gx) = S(\nu_A(gx), \nu_B(gx)) \leq S(\nu_A(x), \nu_B(x)) = (\nu_{A\cap B})(x).$$

Therefore $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in$ IFMN(M). □

Corollary 3.7. *Let* $A_i = (\mu_{A_i}, \nu_{A_i}) \subseteq$ IFMN(M) *for* $i = 1, 2, 3, 4, \ldots, n$. *Then* $\cap_{i=1,2,3,\ldots,n} A_i \in$ IFMN(M).

Proposition 3.8. *Let* $f : M \to N$ *be a G-module epimorphism. If* $A = (\mu_A, \nu_A) \in$ IFMN(M), *then* $f(A) = (f(\mu_A), f(\nu_A)) \in$ IFMN(N).

Proof. Let $y_1, y_2 \in N$ and $a, b \in K$.
Then

$$\begin{aligned}f(\mu_A)(ay_1+by_2) &= \sup\{\mu_A(ax_1+bx_2) \mid x_1, x_2 \in M, f(ax_1) = ay_1, f(bx_2) = by_2\}\\ &= \sup\{\mu_A(ax_1+bx_2) \mid x_1, x_2 \in M, af(x_1) = ay_1, bf(x_2) = by_2\}\\ &\geq \sup\{T(\mu_A(x_1), \mu(x_2)) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\}\\ &= T(\sup\{\mu_A(x_1) \mid f(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid f(x_2) = y_2\})\\ &= T(f(\mu_A)(y_1), f(\mu_A)(y_2)),\end{aligned}$$

thus $f(\mu_A)(ay_1+by_2) \geq T(f(\mu_A)(y_1), f(\mu_A)(y_2))$.
Also

$$\begin{aligned}f(\nu_A)(ay_1+by_2) &= \inf\{\nu_A(ax_1+bx_2) \mid x_1, x_2 \in M, f(ax_1) = ay_1, f(bx_2) = by_2\}\\ &= \inf\{\nu_A(ax_1+bx_2) \mid x_1, x_2 \in M, af(x_1) = ay_1, bf(x_2) = by_2\}\\ &\leq \inf\{S(\nu_A(x_1), \nu(x_2)) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\}\\ &= S(\inf\{\nu_A(x_1) \mid f(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid f(x_2) = y_2\})\\ &= S(f(\nu_A)(y_1), f(\nu_A)(y_2)),\end{aligned}$$

then

$$f(\nu_A)(ay_1+by_2) \leq S(f(\nu_A)(y_1), f(\nu_A)(y_2)).$$

Let $y \in N$ and $g \in G$.

$$\begin{aligned} f(\mu_A)(gy) &= \sup\{\mu_A(gx) \mid x \in M, f(gx) = gy\} \\ &= \sup\{\mu_A(gx) \mid x \in M, gf(x) = gy\} \\ &\geq \sup\{\mu_A(x) \mid x \in M, f(x) = y\} \\ &= f(\mu_A)(y), \end{aligned}$$

and so

$$f(\mu_A)(gy) \geq f(\mu_A)(y).$$

Also

$$\begin{aligned} f(\nu_A)(gy) &= \inf\{\nu_A(gx) \mid x \in M, f(gx) = gy\} \\ &= \inf\{\nu_A(gx) \mid x \in M, gf(x) = gy\} \\ &\leq \inf\{\nu_A(x) \mid x \in M, f(x) = y\} \\ &= f(\nu_A)(y), \end{aligned}$$

thus

$$f(\nu_A)(gy) \leq f(\nu_A)(y).$$

Therefore $f(A) = (f(\mu_A), f(\nu_A)) \in$ IFMN(N). □

Proposition 3.9. *Let $f : M \to N$ be a G-module homomorphism. If $B = (\mu_B, \nu_B) \in$ IFMN(N), then $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)) \in$ IFMN(M).*

Proof. Let $x_1, x_2 \in M$ and $a, b \in K$. Then

$$\begin{aligned} f^{-1}(\mu_B)(ax_1 + bx_2) &= \mu_B(f(ax_1 + bx_2)) \\ &= \mu_B(f(ax_1) + f(bx_2)) \\ &= \mu_B(af(x_1) + bf(x_2)) \\ &\geq T(\mu_B(f(x_1), \mu_B(f(x_2)) \\ &= T(f^{-1}(\mu_B)(x_1), f^{-1}(\mu_B)(x_2)), \end{aligned}$$

then

$$f^{-1}(\mu_B)(ax_1 + bx_2) \geq T(f^{-1}(\mu_B)(x_1), f^{-1}(\mu_B)(x_2)).$$

Also

$$\begin{aligned} f^{-1}(\nu_B)(ax_1 + bx_2) &= \nu_B(f(ax_1 + bx_2)) \\ &= \nu_B(f(ax_1) + f(bx_2)) \\ &= \nu_B(af(x_1) + bf(x_2)) \\ &\leq S(\nu_B(f(x_1), \nu_B(f(x_2)) \\ &= S(f^{-1}(\nu_B)(x_1), f^{-1}(\nu_B)(x_2)), \end{aligned}$$

so

$$f^{-1}(\nu_B)(ax_1 + bx_2) \leq S(f^{-1}(\nu_B)(x_1), f^{-1}(\nu_B)(x_2)).$$

Let $x \in M$ and $g \in G$. Then

$$f^{-1}(\mu_B)(gx) = \mu_B(f(gx)) = \mu_B(gf(x)) \geq \mu_B(f(x)) = f^{-1}(\mu_B)(x)$$

and

$$f^{-1}(\nu_B)(gx) = \nu_B(f(gx)) = \nu_B(gf(x)) \leq \nu_B(f(x)) = f^{-1}(\nu_B)(x).$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)) \in$ IFMN(M). $\square$

Definition 3.10. Let $A = (\mu_A, \nu_A) \in$ IFMN(M) and $B = (\mu_B, \nu_B) \in$ IFMN(M). Define

$$A + B = (\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, \nu_A + \nu_B) = (\mu_{A+B}, \nu_{A+B}) : M \to [0,1] \times [0,1]$$

as

$$\mu_{A+B}(x) = \sup\{T(\mu_A(y), \mu_B(z)) \mid x = y + z \in M\}$$

and

$$\nu_{A+B}(x) = \inf\{S(\nu_A(y), \nu_B(z)) \mid x = y + z \in M\}$$

for all $x \in M$.

Proposition 3.11. *Let* $A = (\mu_A, \nu_A) \in$ IFMN(M) *and* $B = (\mu_B, \nu_B) \in$ IFMN(M). *Then* $A + B \in$ IFMN(M).

Proof. Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$ and $a, b \in K$. Then

$$\begin{aligned}
&\mu_{A+B}(ax_1 + bx_2)\\
&= \sup\{T(\mu_A(ay_1 + by_2), \mu_B(az_1 + bz_2)) \mid ax_1 + bx_2 = ay_1 + by_2 + az_1 + bz_2\}\\
&\geq \sup\{T(T(\mu_A(y_1), \mu_A(y_2)), T(\mu_B(z_1), \mu_B(z_2))) \mid ax_1 + bx_2 = ay_1 + az_1 + by_2 + bz_2\}\\
&= \sup\{T(T(\mu_A(y_1), \mu_A(y_2)), T(\mu_B(z_1), \mu_B(z_2))) \mid ax_1 = ay_1 + az_1, bx_2 = by_2 + bz_2\}\\
&= \sup\{T(T(\mu_A(y_1), \mu_A(y_2)), T(\mu_B(z_1), \mu_B(z_2))) \mid x_1 = y_1 + z_1, x_2 = y_2 + z_2\}\\
&\qquad\qquad\qquad\qquad (\textit{from Lemma 2.17})\\
&= \sup\{T(T(\mu_A(y_1), \mu_B(z_1)), T(\mu_A(y_2), \mu_B(z_2))) \mid x_1 + x_2 = y_1 + z_1 + y_2 + z_2\}\\
&= T(\sup\{T(\mu_A(y_1), \mu_B(z_1)) \mid x_1 = y_1 + z_1)\}, \sup\{T(\mu_A(y_2), \mu_B(z_2)) \mid x_2 = y_2 + z_2\})\\
&= T(\mu_{A+B}(x_1), \mu_{A+B}(x_2)).
\end{aligned}$$

Also

$$\begin{aligned}
&\nu_{A+B}(ax_1 + bx_2)\\
&= \inf\{S(\nu_A(ay_1 + by_2), \nu_B(az_1 + bz_2)) \mid ax_1 + bx_2 = ay_1 + by_2 + az_1 + bz_2\}\\
&\leq \inf\{S(S(\nu_A(y_1), \nu_A(y_2)), S(\nu_B(z_1), \nu_B(z_2))) \mid ax_1 + bx_2 = ay_1 + az_1 + by_2 + bz_2\}\\
&= \inf\{S(S(\nu_A(y_1), \nu_A(y_2)), S(\nu_B(z_1), \nu_B(z_2))) \mid ax_1 = ay_1 + az_1, bx_2 = by_2 + bz_2\}\\
&= \inf\{S(S(\nu_A(y_1), \nu_A(y_2)), S(\nu_B(z_1), \nu_B(z_2))) \mid x_1 = y_1 + z_1, x_2 = y_2 + z_2\}\\
&\qquad\qquad\qquad\qquad (\textit{from Lemma 2.17})\\
&= \inf\{S(S(\nu_A(y_1), \nu_B(z_1)), S(\nu_A(y_2), \nu_B(z_2))) \mid x_1 + x_2 = y_1 + z_1 + y_2 + z_2\}\\
&= S(\inf\{S(\nu_A(y_1), \nu_B(z_1)) \mid x_1 = y_1 + z_1)\}, \inf\{S(\nu_A(y_2), \nu_B(z_2)) \mid x_2 = y_2 + z_2\})\\
&= S(\nu_{A+B}(x_1), \nu_{A+B}(x_2)).
\end{aligned}$$

Let $x, y, z \in M$ and $g \in G$.
Further

$$\begin{aligned}\mu_{A+B}(gx) &= \sup\{T(\mu_A(gy), \mu_B(gz)) \mid gx = gy + gz\} \\ &\geq \sup\{T(\mu_A(y), \mu_B(z)) \mid x = y + z\} \\ &= \mu_{A+B}(x),\end{aligned}$$

so $\mu_{A+B}(gx) \geq \mu_{A+B}(x)$.
Moreover

$$\begin{aligned}\nu_{A+B}(gx) &= \inf\{S(\nu_A(gy), \nu_B(gz)) \mid gx = gy + gz\} \\ &\leq \inf\{S(\nu_A(y), \nu_B(z)) \mid x = y + z\} \\ &= \nu_{A+B}(x),\end{aligned}$$

then $\nu_{A+B}(gx) \leq \nu_{A+B}(x)$.
Therefore we get that

$$A + B = (\mu_{A+B}, \nu_{A+B}) \in \text{IFMN}(M).$$

□

Proposition 3.12. *Let M be a G-module and N be a subset of M such that $A = (\mu_A, \nu_A) \in$ IFS(M). Let*

$$A(x) = (\mu_A(x), \nu_A(x)) = \begin{cases} (1, 0), & \text{if } x \in N \\ (\alpha, \alpha), & \text{if } x \notin N \end{cases}$$

with $\alpha \in [0, 1)$. Then $A = (\mu_A, \nu_A) \in$ IFMN(M) if and only if N is a G-submodule of M.

Proof. Let $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$ and we prove that N is a submodule of M. Let $x, y \in N \subseteq M$ and $a, b \in K$. Now

$$\mu_A(ax + by) \geq T(\mu_A(x), \mu_A(y)) = T(1, 1) = 1$$

and

$$\nu_A(ax + by) \leq S(\nu_A(x), \nu_A(y)) = S(0, 0) = 0$$

thus $A(x) = (\mu_A(x), \nu_A(x)) = (1, 0)$ and then $ax + by \in N$.

Also let $g \in G$ and then $\mu_A(gx) \geq \mu_A(x) = 1$, and $\nu_A(gx) \leq \nu_A(x) = 0$, then $A(gx) = (\mu_A(gx), \nu_A(gx)) = (1, 0)$, thus $gx \in N$. Therefore N is a submodule of M and since N is a subset of M so N will be a G-submodule of M.

Conversely, let N be a submodule of M, we prove that $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$.
Suppose $x, y \in M$ and $a, b \in K$ and we investigate the following conditions:
(1) If $x, y \in N$, then

$$\mu_A(ax + by) = 1 \geq 1 = T(1, 1) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(ax + by) = 0 \leq 0 = S(0, 0) = S(\nu_A(x), \nu_A(y)).$$

(2) For any $x \in N$ and $y \notin N$, then $ax + by \notin N$ and so

$$\mu_A(ax + by) = \alpha \geq 0 = T(1, 0) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(ax + by) = \alpha \leq \alpha = S(0, \alpha) = S(\mu_A(x), \mu_A(y)).$$

(3) Let $x \notin N$ and $y \in N$, then $ax + by \notin N$ and then

$$\mu_A(ax + by) = \alpha \geq 0 = T(0, 1) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(ax + by) = \alpha \leq \alpha = S(0, \alpha) = S(\nu_A(x), \nu_A(y)).$$

(4) Finally, if $x, y \notin N$, so $ax + by \notin N$ and so

$$\mu_A(ax + by) = \alpha \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(ax + by) = \alpha \leq \alpha = S(\alpha, \alpha) = S(\nu_A(x), \nu_A(y)).$$

Therefore, from (1)–(4) we have that

$$\mu_A(ax + by) \geq T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(ax + by) \leq S(\nu_A(x), \nu_A(y)).$$

Now let $x \in M$ and $g \in G$. Then we have:
(5) If $x \in N$, then $gx \in N$ and then $\mu_A(gx) = 1 \geq \mu_A(x)$ and $\nu_A(gx) = 0 \leq \nu_A(x)$.
(6) If $x \notin N$, then $gx \notin N$ and so $\mu_A(gx) = 0 \geq 0 = \mu_A(x)$ and $\nu_A(gx) = \alpha \leq \alpha = \nu_A(x)$.
Therefore from (5) and (6) we have that $\mu_A(gx) \geq \mu_A(x)$ and $\nu_A(gx) \leq \nu_A(x)$.
Hence $A = (\mu_A, \nu_A) \in \text{IFMN}(M)$. □

Definition 3.13. Let M be a G-module over K and M_i be G-submodules of M such that $A_i = (\mu_{A_i}, \nu_{A_i}) \in \text{IFMN}(M_i)$ for all $i = 1, 2, 3, \ldots, n$. Define

$$\begin{aligned} A &= \oplus_{i=1}^n A_i \\ &= (\oplus_{i=1}^n \mu_{A_i}, \oplus_{i=1}^n \nu_{A_i}) \\ &= (\mu_{\oplus_{i=1}^n A_i}, \nu_{\oplus_{i=1}^n A_i}) : M \\ &= \oplus_{i=1}^n M_i \to [0, 1] \times [0, 1] \end{aligned}$$

as

$$\mu_{\oplus_{i=1}^n A_i}(m = \sum_i^n m_i) = \bigwedge \{\mu_{A_i}(m_i) : i = 1, 2, 3, ..., n\}$$

and

$$\nu_{\oplus_{i=1}^n A_i}(m = \sum_i^n m_i) = \bigvee \{\nu_{A_i}(m_i) : i = 1, 2, 3, \ldots, n\}$$

such that $\bigwedge$ denotes minimum [infimum] and $\bigvee$ denotes maximum [supremum] and

$$
\begin{aligned}
A(0) &= \oplus_{i=1}^{n} A_i(0) \\
&= (\oplus_{i=1}^{n} \mu_{A_i}(0), \oplus_{i=1}^{n} \nu_{A_i}(0)) \\
&= (\mu_{\oplus_{i=1}^{n} A_i}(0), \nu_{\oplus_{i=1}^{n} A_i}(0)) \\
&= (\mu_{A_i}(0), \nu_{A_i}(0)) \\
&= A_i(0)
\end{aligned}
$$

for all i. Thus $A = \oplus_{i=1}^{n} A_i$ is called the direct sum of A_i.

Proposition 3.14. *Let M be a G-module over K and M_i be G-submodules of M such that $A_i = (\mu_{A_i}, \nu_{A_i}) \in$ IFMN(M_i) for all $i = 1, 2, 3, ..., n$. Then $A = \oplus_{i=1}^{n} A_i \in$ IFMN$(\oplus_{i=1}^{n} M_i)$.*

Proof. Let $x = \sum_{i}^{n} m_i$ and $y = \sum_{j}^{n} m_j$ and $a, b \in K$ and $g \in G$. Thus

$$
\begin{aligned}
\mu_{\oplus_{i=1}^{n} A_i}(ax + by) &= \mu_{\oplus_{i=1}^{n} A_i}(a \sum_{i}^{n} m_i + b \sum_{j}^{n} m_j) \\
&= \mu_{\oplus_{i=1}^{n} A_i}(\sum_{i}^{n} am_i + \sum_{j}^{n} bm_j) \\
&= \bigwedge \{\mu_{A_i}(am_i + bm_j) : i, j = 1, 2, 3, ..., n\} \\
&\geq \bigwedge \{T(\mu_{A_i}(m_i, m_j) : i, j = 1, 2, 3, ..., n\} \text{ (\textit{Since} } A_i \in \text{IFMN}(M_i)) \\
&= T(\bigwedge \{\mu_{A_i}(m_i) : i = 1, 2, 3, ..., n\}, \bigwedge \{\mu_{A_i}(m_j) : j = 1, 2, 3, ..., n\}) \\
&= T(\mu_{\oplus_{i=1}^{n} A_i}(x), \mu_{\oplus_{i=1}^{n} A_i}(y)),
\end{aligned}
$$

then $\mu_{\oplus_{i=1}^{n} A_i}(ax + by) \geq T(\mu_{\oplus_{i=1}^{n} A_i}(x)\mu_{\oplus_{i=1}^{n} A_i}(y))$.
Also

$$
\begin{aligned}
\nu_{\oplus_{i=1}^{n} A_i}(ax + by) &= \nu_{\oplus_{i=1}^{n} A_i}(a \sum_{i}^{n} m_i + b \sum_{j}^{n} m_j) \\
&= \nu_{\oplus_{i=1}^{n} A_i}(\sum_{i}^{n} am_i + \sum_{j}^{n} bm_j) \\
&= \bigvee \{\nu_{A_i}(am_i + bm_j) : i, j = 1, 2, 3, ..., n\} \\
&\leq \bigvee \{S(\nu_{A_i}(m_i, m_j) : i, j = 1, 2, 3, ..., n\} \text{ (\textit{Since} } A_i \in \text{IFMN}(M_i)) \\
&= S(\bigvee \{\nu_{A_i}(m_i) : i = 1, 2, 3, ..., n\}, \bigvee \{\nu_{A_i}(m_j) : j = 1, 2, 3, ..., n\}) \\
&= S(\nu_{\oplus_{i=1}^{n} A_i}(x), \nu_{\oplus_{i=1}^{n} A_i}(y)),
\end{aligned}
$$

then $\nu_{\oplus_{i=1}^{n} A_i}(ax + by) \leq S(\nu_{\oplus_{i=1}^{n} A_i}(x)\nu_{\oplus_{i=1}^{n} A_i}(y))$.
Further

$$
\begin{aligned}
\mu_{\oplus_{i=1}^{n} A_i}(gx) &= \mu_{\oplus_{i=1}^{n} A_i}(g \sum_{i}^{n} m_i) \\
&= \mu_{\oplus_{i=1}^{n} A_i}(\sum_{i}^{n} gm_i)
\end{aligned}
$$

$$
\begin{aligned}
&= \bigwedge\{\mu_{A_i}(gm_i) : i = 1, 2, 3, ..., n\} \\
&\geq \bigwedge\{\mu_{A_i}(m_i) : i = 1, 2, 3, ..., n\} \ (\textit{Since } A_i \in \text{IFMN}(M_i)) \\
&= \bigwedge\{\mu_{A_i}(m_i) : i = 1, 2, 3, ..., n\} \\
&= \mu_{\oplus_{i=1}^n A_i}(x),
\end{aligned}
$$

so $\mu_{\oplus_{i=1}^n A_i}(gx) \geq \mu_{\oplus_{i=1}^n A_i}(x)$.
Moreover

$$
\begin{aligned}
\nu_{\oplus_{i=1}^n A_i}(gx) &= \nu_{\oplus_{i=1}^n A_i}(g\sum_i^n m_i) \\
&= \nu_{\oplus_{i=1}^n A_i}(\sum_i^n gm_i) \\
&= \bigvee\{\nu_{A_i}(gm_i) : i = 1, 2, 3, ..., n\} \\
&\leq \bigvee\{\nu_{A_i}(m_i) : i = 1, 2, 3, ..., n\} \ (\textit{Since } A_i \in \text{IFMN}(M_i)) \\
&= \bigvee\{\nu_{A_i}(m_i) : i = 1, 2, 3, ..., n\} \\
&= \nu_{\oplus_{i=1}^n A_i}(x),
\end{aligned}
$$

thus $\nu_{\oplus_{i=1}^n A_i}(gx) \leq \nu_{\oplus_{i=1}^n A_i}(x)$.
Therefore $A = \oplus_{i=1}^n A_i \in \text{IFMN}(\oplus_{i=1}^n M_i)$. □

Example 3.15. Let $G = \{\pm 1\}$ and $M = \mathbb{C}$ over $\mathbb{R}$. Then M is a G-module. We have $M = M_1 \oplus M_2$, where $M_1 = \mathbb{R}$ and $M_2 = i\mathbb{R}$. Define $A_1 = (\mu_{A_1}, \nu_{A_1}) : M_1 \to [0, 1] \times [0, 1]$ as

$$\mu_{A_1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \neq 0, \end{cases}$$

and

$$\nu_{A_1}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{7} & \text{if } x \neq 0. \end{cases}$$

Also $A_2 = (\mu_{A_2}, \nu_{A_2}) : M_2 \to [0, 1] \times [0, 1]$ as

$$\mu_{A_2}(y) = \begin{cases} 1 & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y \neq 0, \end{cases}$$

and

$$\nu_{A_2}(y) = \begin{cases} 0 & \text{if } y = 0 \\ \frac{1}{9} & \text{if } y \neq 0. \end{cases}$$

Define

$$A = A_1 \oplus A_2 = (\mu_{A_1 \oplus A_2}, \nu_{A_1 \oplus A_2}) : M = M_1 \oplus M_2 \to [0, 1] \times [0, 1]$$

as

$$\mu_{A_1 \oplus A_2}(x + iy) = \begin{cases} 1 & \text{if } x = y = 0 \\ \frac{1}{2} & \text{if } x \neq 0, y = 0 \\ \frac{1}{3} & \text{if } y \neq 0, \end{cases}$$

and

$$\nu_{A_1 \oplus A_2}(x+iy) = \begin{cases} 0 & \text{if } x=y=0 \\ \frac{1}{10} & \text{if } x \neq 0, y=0 \\ \frac{1}{11} & \text{if } y \neq 0. \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ and $S(a,b) = S_p(a,b) = a+b-ab$ for all $a, b \in [0,1]$. Then as $A_1 \in \text{IFMN}(M_1)$ and $A_2 \in \text{IFMN}(M_2)$ so $A = A_1 \oplus A_2 \in \text{IFMN}(M = M_1 \oplus M_2)$.

4 Open problem

In this paper, as using norms (T and S), intuitionistic fuzzy G-modules on M under norms and some related results like intersection, sum and direct sum of them has also been discussed. Now one can define and investigate intuitionistic fuzzy G-bimodules as we did for intuitionistic fuzzy G-modules and this can be an open problem.

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