

On a Class of d-Fuzzy Sets and d-Intuitionistic Fuzzy Sets

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Abstract

In the paper a class of d-Fuzzy Sets (d-FS) and a class of d-Intuitionistic Fuzzy Sets (d-IFS) are introduced and studied. These classes are generated by specific kinds of norms on R^2 . The investigation follows [1]

1 Keywords

Fuzzy Set, d-Fuzzy Set (d-FS), Intuitionistic Fuzzy Set (IFS), d-Intuitionistic Fuzzy Set (d-IFS), metric, norm

2 Used Denotations

R^2 - for the standard 2-dimensional vector space, R_+^2 - for the set of all vectors with non-negative components from R^2 ; I - for the interval $[0,1]$; \times - for Cartesian product of sets; E - for the universe; $\mu: E \rightarrow [0,1]$, $\nu: E \rightarrow [0,1]$ - for the membership and non-membership functions.

3 Introduction

In [1] for the first time were introduced the so-called d-IFS sets. These sets are generated by an arbitrary metric on R^2 , and they are a continuation of the classical fuzzy sets (FS) (see [2] and of Atanassov's IFS, considered for the first time in [3], because FS and IFS are received for a particular choice of the metric d . Thus in [1] a possibility for a topological point of view on the introduction of FS and IFS has been proposed. In the present paper our aim is to introduce an important classes of d-FS and d-IFS, depending on infinitely many norms ψ_α on R^2 , which correspond to the norms φ_α that were considered in paragraphs 4 and 5 from [1]. Also, the corresponding domains characterising the above-mentioned classes of d-FS and of d-IFS, depending on the norms ψ_α are introduced and studied. The investigation is a continuation and furthering of the results in [1].

4 d_{φ_α} -FS and d_{φ_α} - IFS

Below we use some of the results in [1]:

Let $d: R^2 \times R^2 \rightarrow [0, +\infty)$ be an arbitrary metric on R^2 and $\mu: E \rightarrow I, \nu: E \rightarrow I$ be arbitrary mappings. We call the set

$$\{\mu(x), \nu(x) | x \in E\}$$

d -fuzzy set or d -FS, if it is fulfilled

$$\forall x \in E, d((\mu(x), \nu(x)), (0, 0)) = 1$$

Also we call the set

$$\{\mu(x), \nu(x) | x \in E\}$$

d -Intuitionistic Fuzzy Set or d -IFS, if it is fulfilled:

$$\forall x \in E, d((\mu(x), \nu(x)), (0, 0)) \leq 1$$

Let $\varphi: R^2 \rightarrow [0, +\infty)$ be an arbitrary norm on R^2 . Then as usual, φ represents a metric $d = d_\varphi$ on R^2 , that is given by the formula:

$$d_\varphi((\mu_1, \nu_1), (\mu_2, \nu_2)) = \varphi(\mu_1 - \mu_2, \nu_1 - \nu_2), (\forall (\mu_1, \nu_1), (\mu_2, \nu_2) \in R^2). \quad (1)$$

Therefore, every norm on R^2 generates d_φ -FS and d_φ -IFS.

Further we suppose that $\alpha \in (0, \infty)$. Then we introduce the following norms on R^2 :

$$\begin{aligned} \varphi_\alpha(\mu, \nu) &= (|\mu|^\alpha + |\nu|^\alpha)^{1/\alpha}; \\ \varphi_\infty(\mu, \nu) &= \max(|\mu|, |\nu|); \end{aligned} \quad (2)$$

These norms and their corresponding metrics d_{φ_α} and d_{φ_∞} generate d_{φ_α} -FS, d_{φ_α} -IFS, d_{φ_∞} -FS, and d_{φ_∞} -IFS, respectively. We must note the following:

Fact1: A set A is FS, respectively IFS iff A is d_{φ_1} -FS, respectively d_{φ_1} -IFS i.e. FS and IFS are generated by the well-known Hamming's norm.

Fact2: d_{φ_α} -FS are given by:

$$\{(\mu(x), \nu(x)) | x \in E, \mu: E \rightarrow I, \nu: E \rightarrow I \& ((\mu(x))^\alpha + (\nu(x))^\alpha) = 1\}$$

Fact3: d_{φ_α} -IFS are given by:

$$\{(\mu(x), \nu(x)) | x \in E, \mu: E \rightarrow I, \nu: E \rightarrow I \& ((\mu(x))^\alpha + (\nu(x))^\alpha) \leq 1\}$$

Fact4: Any d_{φ_∞} -FS or d_{φ_∞} -IFS is a limit of d_{φ_α} -FS or d_{φ_α} -IFS, respectively when $\alpha \rightarrow +\infty$ but μ and ν are the same functions.

We call d_{φ_2} -FS and d_{φ_2} -IFS, Euclidean-FS and Euclidean-IFS, respectively. The reason for this definition is that φ_2 is called Euclidean norm.

Let

$$K_\alpha := \{(\mu, \nu) | \mu, \nu \in [-1, 1] \& |\mu|^\alpha + |\nu|^\alpha = 1\}$$

$$\tilde{K}_\alpha := \{(\mu, \nu) | \mu, \nu \in [-1, 1] \& |\mu|^\alpha + |\nu|^\alpha \leq 1\}$$

Then K_α and \tilde{K}_α are the unit circle and disk, centered at the origin of R^2 (considered as a plane), with respect to the metric d_{φ_α} , generated by norm φ_α on R^2 .

We introduce the sets K_α^* and \tilde{K}_α^* by:

$$K_\alpha^* = K_\alpha \cap (I \times I);$$

$$\tilde{K}_\alpha^* = \tilde{K}_\alpha \cap (I \times I);$$

We introduce the domain \tilde{K}_∞^* as $I \times I$.

Below we shall discuss the important question about the connection between d_{φ_α} -IFS and d_{φ_β} -IFS, when $\alpha, \beta \in (0, +\infty]$ and $\alpha \neq \beta$.

The key to answering this question is hidden behind the fact that the norm φ_α is a strictly decreasing function with respect to α , on $(0, +\infty)$ and :

$$\lim_{\alpha \rightarrow +\infty} \varphi_\alpha = \varphi_\infty$$

This was proved in [1].

As a result of the above reasonings we conclude that the closed domains \tilde{K}_α^* grow as α increases in value, where $\alpha \in (0, +\infty)$, i.e.

$$\tilde{K}_\alpha^* \subset \tilde{K}_\beta^*$$

for $\alpha, \beta \in (0, +\infty), \alpha < \beta$

Moreover, there exists

$$\lim_{\alpha \rightarrow +\infty} \tilde{K}_\alpha^* = \tilde{K}_\infty^* = I \times I$$

In short we have:

$$\tilde{K}_1^* \subset \tilde{K}_2^* \subset \dots \subset \tilde{K}_\infty^* = I \times I$$

Let $\alpha, \beta \in (0, +\infty)$ and $\alpha < \beta$. From the above considerations we obtain that if A is a φ_α -IFS then A is φ_β -IFS.

It is also seen that any IFS is Euclidean-IFS, and moreover, any IFS is φ_α -IFS for $\alpha > 1$ but is not φ_α -IFS for $\alpha < 1$.

5 d_{ψ_α} - FS and d_{ψ_α} - IFS

Let $\alpha \in (-\infty, +\infty)$. We introduce

$$\psi_\alpha(\mu, \nu) = \left(\frac{|\mu|^\alpha + |\nu|^\alpha}{2} \right)^{\frac{1}{\alpha}} \quad (3)$$

Also we introduce

$$\psi_0(\mu, \nu) = \lim_{\alpha \rightarrow 0} \psi_\alpha(\mu, \nu) = \sqrt{|\mu||\nu|} \quad (4)$$

and

$$\psi_{+\infty}(\mu, \nu) = \lim_{\alpha \rightarrow +\infty} \psi_{\alpha}(\mu, \nu) \quad (5)$$

Considered for a fixed α , $\psi_{\alpha}(\mu, \nu)$ is known as a Power Mean of order α of the numbers $|\mu|, |\nu|$. More specifically $\psi_{-1}(\mu, \nu)$, $\psi_0(\mu, \nu)$, $\psi_1(\mu, \nu)$, $\psi_2(\mu, \nu)$ are known as Harmonic Mean, Geometric Mean (see (4)) and Root-Mean-Square of the numbers $|\mu|, |\nu|$, respectively (see[4]).

It is obvious that

$$\psi_{+\infty}(|\mu|, |\nu|) = \varphi_{\infty}(|\mu|, |\nu|) = \max(|\mu|, |\nu|) \quad (6)$$

(see (2) and (5)).

If we put in (1) $\psi_{\alpha}(\mu, \nu)$ (from (3)) we are able to generate $d_{\psi_{\alpha}}$ and $d_{\psi_{\infty}}$ - metrics on R^2 and as a result $d_{\psi_{\alpha}}$ -FS, $d_{\psi_{\alpha}}$ -IFS, $d_{\psi_{+\infty}}$ -FS, and $d_{\psi_{+\infty}}$ -IFS are introduced. For this purpose we must assume that $|\mu|$ and $|\nu|$ are the values of two non-complementary functions $\mu(x)$ and $\nu(x)$, such that $\mu: E \rightarrow I, \nu: E \rightarrow I$ (e.g. we may interpret these functions from classical point of view as degree of membership to a certain property and a degree non-membership to a different property).

Let $\alpha \in (0, +\infty]$. In this case we introduce the sets:

$$B_{\alpha} := \{(\mu, \nu) | \mu, \nu \in [-1, 1] \& \psi_{\alpha}(\mu + \nu) = 1\}$$

and

$$\tilde{B}_{\alpha} := \{(\mu, \nu) | \mu, \nu \in [-1, 1] \& \psi_{\alpha}(\mu + \nu) \leq 1\}$$

Then B_{α} and \tilde{B}_{α} are the unit circle and disk, centered at the origin of R^2 (considered as a plane), with respect to the metric $d_{\psi_{\alpha}}$, generated by norm ψ_{α} on R^2 . Also we introduce the sets B_{α}^* and \tilde{B}_{α}^* by:

$$B_{\alpha}^* = B_{\alpha} \cap R_+^2;$$

$$\tilde{B}_{\alpha}^* = \tilde{B}_{\alpha} \cap R_+^2;$$

Finally, because of (6), we introduce the domain \tilde{B}_{∞}^* as $I \times I$ (i.e. the domain \tilde{B}_{∞}^* coincides with the domain \tilde{K}_{∞}^* considered in [1]).

Below we shall discuss the important question about the connection between $d_{\varphi_{\alpha}}$ -IFS, $d_{\varphi_{\beta}}$ -IFS, $d_{\psi_{\alpha}}$ -IFS, $d_{\psi_{\beta}}$ -IFS when $\alpha, \beta \in [0, +\infty]$ and $\alpha \neq \beta$.

It is well-known that considered as a function of α only, $\psi_{\alpha}(\mu, \nu)$ is a strictly increasing function on the interval $(-\infty, +\infty]$ (see [5]). Therefore, $\psi_{\alpha}(\mu, \nu)$ is strictly increasing function with respect to α on the interval $[0, +\infty]$. Hence, the closed domains \tilde{B}_{α}^* diminishes as α increases in value when $\alpha \in [0, +\infty]$. Therefore, if $\alpha, \beta \in (0, +\infty)$ and $\alpha < \beta$ then it is fulfilled:

$$\tilde{B}_{\beta}^* \subset \tilde{B}_{\alpha}^*$$

Hence,

$$I \times I \equiv \tilde{B}_{+\infty}^* \subset \tilde{B}_{\beta}^* \subset \tilde{B}_{\alpha}^* \subset \tilde{B}_0^*$$

It is easy to see that the closed domain \tilde{B}_0^* coincides with the closed domain encapsulated between the coordinate rays of the first quadrant in the Cartesian coordinate system $O_{\mu\nu}$

centered at $(0,0)$ and the branch of the hyperbola given by the equation $\mu\nu = 1$ located in the first quadrant. This follows from the fact that

$$\psi_0(\mu, \nu) = \sqrt{|\mu||\nu|}$$

as it was mentioned above. Thus the following chains of inclusions is fulfilled:

$$\tilde{B}_0^* \supset \tilde{B}_1^* \supset \tilde{B}_2^* \supset \dots \supset \tilde{B}_{+\infty}^* \equiv I \times I$$

But when $0 < \alpha < \beta < +\infty$ we have:

$$\tilde{K}_\alpha^* \subset \tilde{K}_\beta^* \subset \tilde{K}_\infty^* \equiv I \times I \equiv \tilde{B}_{+\infty}^*$$

Hence for $0 < \alpha < \beta < +\infty$ it is fulfilled:

$$\tilde{K}_\alpha^* \subset \tilde{K}_\beta^* \subset \tilde{K}_\infty^* \equiv I \times I \equiv \tilde{B}_{+\infty}^* \subset \tilde{B}_\beta^* \subset \tilde{B}_\alpha^* \subset \tilde{B}_0^* \quad (7)$$

Hence the validity of the double chain of inclusions that is given below

$$\tilde{K}_1^* \subset \tilde{K}_2^* \subset \dots \subset \tilde{K}_\infty^* \equiv I \times I \equiv \tilde{B}_{+\infty}^* \subset \dots \subset \tilde{B}_2^* \subset \tilde{B}_1^* \subset \tilde{B}_0^*$$

is established because of (7). The above considerations give us the complete answer of the question stated before about the connection between the sets: d_{φ_α} -IFS, d_{φ_β} -IFS, d_{ψ_α} -IFS, d_{ψ_β} -IFS. They deliver an answer by providing a proof for the following:

Theorem. Let $0 < \alpha < \beta \leq +\infty$, then:

- i_1 : If A is d_{φ_α} -IFS, then A is d_{φ_β} -IFS, d_{φ_∞} -IFS, d_{ψ_β} -IFS, d_{ψ_α} -IFS and ψ_0 -IFS, simultaneously.
- i_2 : If A is d_{ψ_β} -IFS, then A is d_{ψ_α} -IFS and d_{ψ_0} -IFS too.
- i_3 : d_{φ_∞} -IFS sets coincide with $d_{\psi_{+\infty}}$ -IFS sets.

It is natural to call: d_{ψ_0} -IFS - Geometric Mean-IFS; d_{ψ_1} -IFS - Arithmetic Mean-IFS; d_{ψ_2} -IFS - Root-Mean-Square-IFS. It is clear that every IFS is d_{φ_α} -IFS, too, when $1 < \alpha \leq +\infty$. Also, every IFS is d_{ψ_α} -IFS when $0 \leq \alpha \leq +\infty$. In particular, every IFS is Geometric Mean-IFS, Arithmetic Mean-IFS and Root-Mean-Square-IFS.

Open problem: Propose a suitable definition of the modal operators possibility and necessity for the d_{ψ_α} -IFS, and investigate their properties.

References

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