Notes on Intuitionistic Fuzzy Sets
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Approximate solution of linear intuitionistic fuzzy Fredholm integral equations using block-pulse functions

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Abstract: This paper focuses on obtaining approximate solutions for linear intuitionistic fuzzy Fredholm integral equations (LIFFIEs) using block-pulse functions. The convergence of the proposed method is discussed, and its efficiency and accuracy are demonstrated through several numerical examples.
Keywords: Intuitionistic fuzzy set, Block-pulse function, Intuitionistic fuzzy Fredholm integral equation.
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1 Introduction

Many problems in human life, applied sciences, engineering, and medical diagnosis involve factors of uncertainty. When mathematically modeling such problems, we often encounter differential and integral equations that incorporate these uncertainty factors, necessitating solutions that account for them. These factors may be represented by fuzzy concepts [14] or their generalizations, such as intuitionistic fuzzy sets (IFSs) [3–5]. An IFS is characterized by two functions that respectively express the degree of belongingness and non-belongingness of an element to the set.

Since finding exact solutions for intuitionistic fuzzy differential and integral equations is often challenging, many researchers have proposed numerical methods to solve such equations [1, 8, 11–13].

In this paper, linear intuitionistic fuzzy Fredholm integral equations (LIFFIEs) are considered in the following form:

$$y(t) = f(t) \oplus \int_0^1 k(t,s) \odot y(s)ds, t \in [0,1], \tag{1}$$

where $y(t)$ is an unknown intuitionistic fuzzy function for every $t \in (0,1]$, while $f(t)$ and $k(t,s)$ are a known intuitionistic fuzzy function and a kernel function, respectively, defined on $D \subset \mathbb{R} \times \mathbb{R}$.

Here $y(t)$ and $f(t)$ can be expressed as $y(t) = \langle \mu_y(t), \nu_y(t) \rangle$, $f(t) = \langle \mu_f(t), \nu_f(t) \rangle$, where $\mu_y(t),\ \mu_f(t)$ represent the membership functions and $\nu_y(t),\ \nu_f(t)$ represent the non-membership functions of $y(t)$ and $f(t)$, respectively.

The structure of this paper is as follows: in Section 2, an overview of fuzzy and intuitionistic fuzzy concepts is provided. The main ideas are presented in Section 3. Section 4 investigates the uniqueness of the solution and conducts a convergence analysis. The proposed method is tested through two numerical examples in Section 5. Finally, the conclusions are presented in the last section.

2 Preliminaries

This section provides the basic definitions and concepts of fuzzy sets, as outlined in [6, 7], as well as intuitionistic fuzzy sets and intuitionistic fuzzy numbers, as described in [3].

Definition 1. *A fuzzy set $y : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy number if it satisfies the conditions:*

1. *y is upper semicontinuous.*
2. *$y(x) = 0$ outside some interval $[c,d]$.*
3. *There exist real numbers a, b such that $c \leq a \leq b \leq d$, for which:*
   - *3.1 $y(x)$ is monotonically increasing on $[c,a]$.*
   - *3.2 $y(x)$ is monotonically decreasing on $[b,d]$.*
   - *3.3 $y(x) = 1,\ a \leq x \leq b$.*

Definition 2. *A pair $y = (\underline{y}(r), \overline{y}(r))$ for $r \in [0,1]$ is called a parametric form of y if:*

1. *$\underline{y}(r)$ is a bounded left continuous monotonically increasing function on $[0,1]$,*
2. *$\overline{y}(r)$ is a bounded left continuous monotonically decreasing function on $[0,1]$,*
3. *$\forall r \in [0,1],\ \underline{y}(r) \leq \overline{y}(r)$.*

The collection of all the fuzzy numbers with addition and multiplication is denoted by E^1 and is a convex cone.

The set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is an IFS in universe set of X if μ_A and ν_A are functions from X to $[0,1]$ defined as membership and non-membership functions on X, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1,\ \forall x \in X$ [3, 13].

Definition 3. [2] *The α-cut of an IFN*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in R\}$$

is defined as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in R,\ \mu_A(x) \geq \alpha \textit{ and } \nu_A(x) \leq 1-\alpha\}, \forall \alpha \in [0,1].$$

The α-cut representation of an IFN A generates the following pair of intervals and is denoted by

$$[A]_\alpha = \{[A_1(\alpha), A_2(\alpha)], [A_3(\alpha), A_4(\alpha)].$$

Definition 4. [10] *A Triangular Intuitionistic Fuzzy Number (TIFN) A is an intuitionistic fuzzy set defined on R, characterized by the following membership function $\mu_A(x)$ and non-membership function $\nu_A(x)$:*

$$\mu_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2, \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \end{cases} \tag{2}$$

and

$$\nu_A(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1'} & a_1' \leq x \leq a_2, \\ \dfrac{x-a_2}{a_3'-a_2}, & a_2 \leq x \leq a_3', \end{cases} \tag{3}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$, and TIFN is denoted by $A = (a_1, a_2, a_3; a_1', a_2, a_3')$. For more details see [10]. The set of all intuitionistic fuzzy numbers (IFNs) is denoted by $\mathfrak{I}$.
For arbitrary intuitionistic fuzzy numbers

$$[A]_\alpha = \{[A_1(\alpha), A_2(\alpha)], [A_3(\alpha), A_4(\alpha)],$$

and

$$[B]_\alpha = \{[B_1(\alpha), B_2(\alpha)], [B_3(\alpha), B_4(\alpha)],$$

the quantity

$$\begin{aligned} D(A,B) =& \frac{1}{2} \sup_{0\leq\alpha\leq 1} \Bigg\{ \max\left[|A_1(\alpha)-B_1(\alpha)|, |A_2(\alpha)-B_2(\alpha)|\right] \\ & + \max\left[|A_3(\alpha)-B_3(\alpha)|, |A_4(\alpha)-B_4(\alpha)|\right] \Bigg\} \end{aligned} \tag{4}$$

represents the distance between A and B.

2.1 Block-pulse functions and transformation matrix

The block-pulse functions (BPFs) and their well-known properties are introduced.

$$\mathfrak{b}_i(t) = \begin{cases} 1, & \frac{(i-1)T}{m} \leq t < \frac{iT}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

for $i = 1, 2, \ldots, m$ defines a set of BPFs with the following properties:

$$\mathfrak{b}_i(t)\mathfrak{b}_j(t) = \begin{cases} \mathfrak{b}_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{6}$$

$$\int_0^T \mathfrak{b}_i(t)\mathfrak{b}_j(t)dt = \begin{cases} \frac{T}{m}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7}$$

The expansion of $f \in L[0,T)$, with respect to BPFs $\mathfrak{B}(t) = (\mathfrak{b}_1(t), \mathfrak{b}_2(t), \ldots, \mathfrak{b}_m(t))^T$ is defined as [9]:

$$f(t) \simeq (f_1, f_2, \ldots, f_m)\mathfrak{B}(t) = F^T\mathfrak{B}(t) = \mathfrak{B}^T(t)F,$$

where

$$F = \frac{1}{h}\int_0^T f(t)\mathfrak{B}(t)dt.$$

For $K(t,s)$ in $L^2([0,T] \times [0,T])$ we have

$$K(t,s) \simeq \mathfrak{B}^T(t)K\mathfrak{B}(s)$$

with

$$K = \frac{1}{h^2}\int_0^T\int_0^T \mathfrak{B}^T(t)K(t,s)\mathfrak{B}(s)dsdt,$$

and $h = \frac{T}{m}$.

Additionally, from [9], it can be shown that

$$\int_0^T \mathfrak{B}(t)\mathfrak{B}^T(t)dt = hI, \tag{8}$$

and $\int_0^t \mathfrak{B}(t)dt \simeq \mathcal{P}\mathfrak{B}(t)$ where

$$\mathcal{P} = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Therefore, we have

$$\int_0^T f(t)dt \simeq \int_0^T F^T\mathfrak{B}(t)dt \simeq F^T\mathcal{P}\mathfrak{B}(t). \tag{9}$$

3 Main idea

Consider the parametric form of LIFFIEs as follows:

$$\widetilde{y}(t) = \widetilde{f}(t) \oplus \int_0^1 k(t,s) \odot \widetilde{y}(s)ds, \tag{10}$$

where $\widetilde{y}(t)$ and $\widetilde{f}(t)$ are parametric form of $y(t)$ and $f(t)$ in Equation (1), respectively. Specifically,

$$\begin{aligned}
\widetilde{y}(t) &= ([y_1(t,r), y_2(t,r)],\ [y_3(t,r), y_4(t,r)]),\\
\widetilde{f}(t) &= ([f_1(t,r), [f_2(t,r)],\ [f_3(t,r), f_4(t,r)]),\ \ r \in [0,1].
\end{aligned}$$

The expansion of functions $\widetilde{y}_i(t), \widetilde{f}_i(t)$ for $i = 1,2,3,4$ and $k(t,s)$ can be written as follows:

$$y_i(t,r) \simeq \mathfrak{B}^T(t) Y_i \mathfrak{B}(r), \quad f_i(t,r) \simeq \mathfrak{B}^T(t) F_i \mathfrak{B}(r), \quad k(t,s) \simeq \mathfrak{B}^T(t) K \mathfrak{B}(s), \tag{11}$$

where $\mathfrak{B}(t) = (\mathfrak{b}_1(t), \mathfrak{b}_2(t), \ldots, \mathfrak{b}_m(t))^T,\ h = 1/m$ and $Y_i,\ F_i$ for $i = 1,2,3,4$ and K are $m \times m$ matrices defined as:

$$Y_i = \frac{1}{h^2}\int_0^1\int_0^1 \mathfrak{B}^T(t) y_i(t,r)\mathfrak{B}(r)drdt,\ F_i = \frac{1}{h^2}\int_0^1\int_0^1 \mathfrak{B}^T(t) f_i(t,r)\mathfrak{B}(r)drdt,$$

and

$$K = \frac{1}{h^2}\int_0^1\int_0^1 \mathfrak{B}^T(t) k(t,s)\mathfrak{B}(s)dsdt.$$

Substituting Eqs. (11) into Equation (10) for any $i = 1,2,3,4$, we obtain

$$\begin{aligned}
\mathfrak{B}^T(t) Y_i \mathfrak{B}(r) &= \mathfrak{B}^T(t) F_i \mathfrak{B}(r) \oplus \int_0^1 \mathfrak{B}^T(t) K \mathfrak{B}(s) \mathfrak{B}^T(s) Y_i \mathfrak{B}(r) ds\\
\mathfrak{B}^T(t) Y_i \mathfrak{B}(r) &= \mathfrak{B}^T(t) F_i \mathfrak{B}(r) \oplus \mathfrak{B}^T(t) hKY_i \mathfrak{B}(r). \qquad (12)
\end{aligned}$$

Hence

$$Y_i = F_i + hKY_i$$

and solving for Y_i yields:

$$Y_i = (I - hK)^{-1} F_i.$$

Then we can find an approximate solution $y_i(t,r)$ for $i = 1,2,3,4$ as follows:

$$y_i(t,r) = \mathfrak{B}^T(t) Y_i \mathfrak{B}(r).$$

4 Convergence analysis

Consider the equation

$$\widetilde{y}(t) = \widetilde{f}(t) \oplus \int_0^1 k(t,s) \odot \widetilde{y}(s)ds, \tag{13}$$

Theorem 5. *Suppose that $\widetilde{y},\ \widetilde{f} \in C([0,1], \mathfrak{I})$ and $k \in C([0,1] \times [0,1], \mathbb{R})$, if $\| k \|_\infty = K < 1$ for every $0 \le s,t \le 1$, then Equation* (13) *has a unique solution $\widetilde{y}(t)$ on* $[0,1]$.

Proof. We define the operator T on $C([0,1],\mathfrak{I})$ by

$$T\widetilde{u}(t)=\widetilde{f}(t)\oplus\int_0^1 k(t,s)\odot\widetilde{u}(s)ds.$$

Then

$$\begin{aligned}D(T\widetilde{u}(t),T\widetilde{v}(t))&=D\left(\widetilde{f}(t)\oplus\int_0^1 k(t,s)\odot\widetilde{u}(s)ds,\widetilde{f}(t)\oplus\int_0^1 k(t,s)\odot\widetilde{v}(s)ds\right)\\&=D\left(\int_0^1 k(t,s)\odot\widetilde{u}(s)ds,\int_0^1 k(t,s)\odot\widetilde{v}(s)ds\right)\\&\leq\int_0^1 KD(\widetilde{u}(s),\widetilde{v}(s))ds.\end{aligned}$$

Hence

$$\parallel D(T\widetilde{u}(t),T\widetilde{v}(t))\parallel_\infty\leq\int_0^1 K\parallel D(\widetilde{u}(s),\widetilde{v}(s))\parallel_\infty ds.$$

Finally,

$$\parallel D(T\widetilde{u}(t),T\widetilde{v}(t))\parallel_\infty<\parallel D(\widetilde{u}(s),\widetilde{v}(s))\parallel_\infty.$$

This shows that the operator T is a contraction on $C([0,1],\mathfrak{I})$. Therefore, T has a unique fixed point $\widetilde{y}\in C([0,1],\mathfrak{I})$, and consequently this $\widetilde{y}=\widetilde{y}(t)$ is the unique solution of Equation (13) on $[0,1]$. □

Theorem 6. *Suppose that $y_{im}(t,r)$ and $y_i(t,r)$ are the approximate solution obtained by the block-pulse function and the exact solution of i-th component of $\widetilde{y}(t)$ in Equation (10), respectively. If $k(t,s)$ is continuous and bounded for all $s,t\in[0,1]$, then $y_{im}(t,r)\longrightarrow y_i(t,r)$ as $m\longrightarrow\infty$, for any $i=1,2,3,4$.*

Proof.

$$\begin{aligned}&D(y_i(t,r),y_{im}(t,r))\\&\quad=D\left(\widetilde{f}(t)\oplus\int_0^1 k(t,s)\odot y_i(s,r)ds,\widetilde{f}(t)\oplus\int_0^1 k(t,s)\odot y_{im}(t,r)ds\right)\\&\quad=D\left(\int_0^1 k(t,s)\odot y_i(s,r)ds,\int_0^1 k(t,s)\odot\mathfrak{B}^T(s)Y_i\mathfrak{B}(r)ds\right)\\&\quad\leq\int_0^1 KD(y_i(s,r),\sum_{n=1}^m\sum_{j=1}^m c_{nj}^{(i)}\mathfrak{b}_n(s)\mathfrak{b}_j(r))ds.\end{aligned}$$

where

$$\max_{0\leq s,t\leq 1}\mid k(s,t)\mid=K.$$

Since

$$\lim_{m\to\infty}y_i(t,r)=\lim_{m\to\infty}\sum_{n=1}^m\sum_{j=1}^m c_{nj}^{(i)}\mathfrak{b}_n(t)\mathfrak{b}_j(r).$$

Therefore, for every $i=1,2,3,4$,

$$\lim_{m\to\infty}D\left(y_i(t,r),\sum_{n=1}^m\sum_{j=1}^m c_{nj}^{(i)}\mathfrak{b}_n(t)\mathfrak{b}_j(r)\right)=0.$$ □

5 Examples

Consider the following LIFFIEs:

$$\widetilde{y}(t) = \widetilde{f}(t) \oplus \int_0^1 k(t,s) \odot \widetilde{y}(s)ds, \quad t \in [0,1], \tag{14}$$

where $\widetilde{y}(t)$ and $\widetilde{f}(t)$ are the parametric forms of $y(t)$ and $f(t)$ in Equation (1), respectively. Specifically,

$$\widetilde{y}(t) = ([y_1(t,r), y_2(t,r)],\ [y_3(t,r), y_4(t,r)]),$$

and

$$\widetilde{f}(t) = ([f_1(t,r), [f_2(t,r)],\ [f_3(t,r), f_4(t,r)])$$

for $r \in [0,1]$.

Example 1. *Consider for $t, r \in [0,1]$*

$$\begin{aligned}
f_1(t,r) &= (1.25 + 0.25r)(t + \exp(t)),\\
f_2(t,r) &= (1.75 - 0.25r)(t + \exp(t)),\\
f_3(t,r) &= (1.5 - 0.5r)(t + \exp(t)),\\
f_4(t,r) &= (1.5 + 0.5r)(t + \exp(t)),
\end{aligned}$$

and $k(t,s) = ts$. The exact solution is as follows:

$$\begin{aligned}
y_1(t,r) &= (1.25 + 0.25r)\exp(t),\\
y_2(t,r) &= (1.5 - 0.25r)\exp(t),,\\
y_3(t,r) &= (1.5 - 0.5r)\exp(t),\\
y_4(t,r) &= (1.5 + 0.5r)\exp(t).
\end{aligned}$$

The numerical solutions and error values are given in Tables 1 to 4. Tables 1 and 2 show the results for $t = 0.0,\ m = 8$ and $t = 0.125,\ m = 8$, respectively. Tables 3 and 4 show the results for $t = 0.0,\ m = 16$ and $t = 0.125,\ m = 16$, respectively.

Table 1. Aproximate solutions and errors for $t = 0.0$ and $m = 8$ in Example 1.

$t = 0.0$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	1.56514	0.31514	2.19120	0.44120	1.87817	0.37817	1.87817	0.37817
0.12500	1.60427	0.32302	2.15207	0.43332	1.79991	0.36241	1.95643	0.39393
0.25000	1.64340	0.33090	2.11294	0.42544	1.72166	0.34666	2.03468	0.40968
0.37500	1.68253	0.33878	2.07381	0.41756	1.64340	0.33090	2.11294	0.42544
0.50000	1.72166	0.34666	2.03468	0.40968	1.56514	0.31514	2.19120	0.44120
0.62500	1.76078	0.35453	1.99555	0.40180	1.48688	0.29938	2.26945	0.45695
0.75000	1.79991	0.36241	1.95643	0.39393	1.40863	0.28363	2.34771	0.47271
0.87500	1.83904	0.37029	1.91730	0.38605	1.33037	0.26787	2.42597	0.48847
1.00000	1.87817	0.37817	1.87817	0.37817	1.25211	0.25211	2.50423	0.50423

Table 2. Aproximate solutions and errors for $t = 0.125$ and $m = 8$ in Example 1.

$t = 0.125$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	2.20974	0.79330	3.09363	1.11063	2.65169	0.95196	2.65169	0.95196
0.12500	2.26498	0.81314	3.03839	1.09079	2.54120	0.91230	2.76217	0.99163
0.25000	2.32023	0.83297	2.98315	1.07096	2.43071	0.87263	2.87266	1.03129
0.37500	2.37547	0.85280	2.92790	1.05113	2.32023	0.83297	2.98315	1.07096
0.50000	2.43071	0.87263	2.87266	1.03129	2.20974	0.79330	3.09363	1.11063
0.62500	2.48596	0.89247	2.81742	1.01146	2.09925	0.75364	3.20412	1.15029
0.75000	2.54120	0.91230	2.76217	0.99163	1.98877	0.71397	3.31461	1.18996
0.87500	2.59644	0.93213	2.70693	0.97180	1.87828	0.67431	3.42510	1.22962
1.00000	2.65169	0.95196	2.65169	0.95196	1.76779	0.63464	3.53558	1.26929

Table 3. Aproximate solutions and errors for $t = 0.0$ and $m = 16$ in Example 1.

$t = 0.0$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	1.40699	0.15699	1.96978	0.21978	1.68838	0.18838	1.68838	0.18838
0.12500	1.44216	0.16091	1.93461	0.21586	1.61803	0.18053	1.75873	0.19623
0.25000	1.47734	0.16484	1.89943	0.21193	1.54769	0.17269	1.82908	0.20408
0.37500	1.51251	0.16876	1.86426	0.20801	1.47734	0.16484	1.89943	0.21193
0.50000	1.54769	0.17269	1.82908	0.20408	1.40699	0.15699	1.96978	0.21978
0.62500	1.58286	0.17661	1.79391	0.20016	1.33664	0.14914	2.04013	0.22763
0.75000	1.61803	0.18053	1.75873	0.19623	1.26629	0.14129	2.11048	0.23548
0.87500	1.65321	0.18446	1.72356	0.19231	1.19594	0.13344	2.18083	0.24333
1.00000	1.68838	0.18838	1.68838	0.18838	1.12559	0.12559	2.25118	0.25118

Table 4. Aproximate solutions and errors for $t = 0.125$ and $m = 16$ in Example 1.

$t = 0.125$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	2.04712	0.63069	2.86597	0.88296	2.45655	0.75683	2.45655	0.75683
0.12500	2.09830	0.64646	2.81480	0.86720	2.35419	0.72529	2.55890	0.78836
0.25000	2.14948	0.66222	2.76362	0.85143	2.25184	0.69376	2.66126	0.81989
0.37500	2.20066	0.67799	2.71244	0.83566	2.14948	0.66222	2.76362	0.85143
0.50000	2.25184	0.69376	2.66126	0.81989	2.04712	0.63069	2.86597	0.88296
0.62500	2.30301	0.70952	2.61008	0.80413	1.94477	0.59915	2.96833	0.91450
0.75000	2.35419	0.72529	2.55890	0.78836	1.84241	0.56762	3.07069	0.94603
0.87500	2.40537	0.74106	2.50773	0.77259	1.74006	0.53608	3.17304	0.97757
1.00000	2.45655	0.75683	2.45655	0.75683	1.63770	0.50455	3.27540	1.00910

Example 2. *Consider for* $t, r \in [0,1]$

$$
\begin{aligned}
f_1(t,r) &= (1.5+0.5r)(\frac{5}{6}t^2+t+1),\\
f_2(t,r) &= (3-r)(\frac{5}{6}t^2+t+1),\\
f_3(t,r) &= (2-r)(\frac{5}{6}t^2+t+1),\\
f_4(t,r) &= (2+2r)(\frac{5}{6}t^2+t+1),
\end{aligned}
$$

and $k(t,s) = t^2 s$. *The exact solution is as follows:*

$$
\begin{aligned}
y_1(t,r) &= (1.5+0.5r)(t+1),\\
y_2(t,r) &= (3-r)(t+1),,\\
y_3(t,r) &= (2-r)(t+1),\\
y_4(t,r) &= (2+2r)(t+1).
\end{aligned}
$$

The numerical solutions and error values are given in Tables 5 to 8. Tables 5 and 6 show the results for $t = 0.0,\ m = 8$ and $t = 0.125,\ m = 8$, respectively. Tables7 and 8 show the results for $t = 0.0,\ m = 16$ and $t = 0.125,\ m = 16$, respectively.

Table 5. Aproximate solutions and errors for $t = 0.0$ and $m = 8$ in Example 2.

$t = 0.0$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	1.69351	0.19351	3.38701	0.38701	2.25801	0.25801	2.25801	0.25801
0.12500	1.76407	0.20157	3.24589	0.37089	2.11688	0.24188	2.54026	0.29026
0.25000	1.83463	0.20963	3.10476	0.35476	1.97576	0.22576	2.82251	0.32251
0.37500	1.90519	0.21769	2.96363	0.33863	1.83463	0.20963	3.10476	0.35476
0.50000	1.97576	0.22576	2.82251	0.32251	1.69351	0.19351	3.38701	0.38701
0.62500	2.04632	0.23382	2.68138	0.30638	1.55238	0.17738	3.66926	0.41926
0.75000	2.11688	0.24188	2.54026	0.29026	1.41125	0.16125	3.95151	0.45151
0.87500	2.18744	0.24994	2.39913	0.27413	1.27013	0.14513	4.23376	0.48376
1.00000	2.25801	0.25801	2.25801	0.25801	1.12900	0.12900	4.51601	0.51601

Table 6. Aproximate solutions and errors for $t = 0.125$ and $m = 8$ in Example 2.

$t = 0.125$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	2.10656	0.41906	4.21312	0.83812	2.80874	0.55874	2.80874	0.55874
0.12500	2.19433	0.43652	4.03757	0.80319	2.63320	0.52382	3.15984	0.62859
0.25000	2.28210	0.45398	3.86202	0.76827	2.45765	0.48890	3.51093	0.69843
0.37500	2.36988	0.47144	3.68648	0.73335	2.28210	0.45398	3.86202	0.76827
0.50000	2.45765	0.48890	3.51093	0.69843	2.10656	0.41906	4.21312	0.83812
0.62500	2.54542	0.50636	3.33538	0.66351	1.93101	0.38414	4.56421	0.90796
0.75000	2.63320	0.52382	3.15984	0.62859	1.75547	0.34922	4.91530	0.97780
0.87500	2.72097	0.54128	2.98429	0.59367	1.57992	0.31429	5.26640	1.04765
1.00000	2.80874	0.55874	2.80874	0.55874	1.40437	0.27937	5.61749	1.11749

Table 7. Aproximate solutions and errors for $t = 0.0$ and $m = 16$ in Example 2.

$t = 0.0$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	1.59531	0.09531	3.19063	0.19063	2.12709	0.12709	2.12709	0.12709
0.12500	1.66179	0.09929	3.05769	0.18269	1.99414	0.11914	2.39297	0.14297
0.25000	1.72826	0.10326	2.92474	0.17474	1.86120	0.11120	2.65886	0.15886
0.37500	1.79473	0.10723	2.79180	0.16680	1.72826	0.10326	2.92474	0.17474
0.50000	1.86120	0.11120	2.65886	0.15886	1.59531	0.09531	3.19063	0.19063
0.62500	1.92767	0.11517	2.52591	0.15091	1.46237	0.08737	3.45651	0.20651
0.75000	1.99414	0.11914	2.39297	0.14297	1.32943	0.07943	3.72240	0.22240
0.87500	2.06061	0.12311	2.26003	0.13503	1.19649	0.07149	3.98829	0.23829
1.00000	2.12709	0.12709	2.12709	0.12709	1.06354	0.06354	4.25417	0.25417

Table 8. Aproximate solutions and errors for $t = 0.125$ and $m = 16$ in Example 2.

$t = 0.125$ r	App. $y1$	Error	App. $y2$	Error	App. $y3$	Error	App. $y4$	Error
0.00000	1.99936	0.31186	3.99872	0.62372	2.66581	0.41581	2.66581	0.41581
0.12500	2.08266	0.32485	3.83210	0.59773	2.49920	0.38982	2.99904	0.46779
0.25000	2.16597	0.33785	3.66549	0.57174	2.33258	0.36383	3.33226	0.51976
0.37500	2.24928	0.35084	3.49888	0.54575	2.16597	0.33785	3.66549	0.57174
0.50000	2.33258	0.36383	3.33226	0.51976	1.99936	0.31186	3.99872	0.62372
0.62500	2.41589	0.37683	3.16565	0.49378	1.83275	0.28587	4.33194	0.67569
0.75000	2.49920	0.38982	2.99904	0.46779	1.66613	0.25988	4.66517	0.72767
0.87500	2.58250	0.40282	2.83242	0.44180	1.49952	0.23389	4.99840	0.77965
1.00000	2.66581	0.41581	2.66581	0.41581	1.33291	0.20791	5.33162	0.83162

6 Conclusion

First, the block-pulse functions and their properties are presented. Then, the operational matrices of these functions are calculated. By substituting these matrices into LIFFIEs, we arrive at a system of algebraic equations. Solving this system of linear equations yields a numerical solution to the problem. We then prove the uniqueness of the solution and the convergence of the method. Additionally, several numerical examples are provided to demonstrate the effectiveness of the proposed method. The results indicate that this method is highly effective for solving these types of equations. For future research, this method can be extended to handle such equations with nonlinear unknown functions.

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