19th Int. Workshop on IFSs, Burgas, 4–6 June 2015
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 2, 71–76

# An auxiliary technique for InterCriteria Analysis via a three dimensional index matrix

**Peter Vassilev, Lyudmila Todorova and Velin Andonov**

Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria
e-mails: peter.vassilev@gmail.com, lpt@biomed.bas.bg,
velin_andonov@yahoo.com

**Abstract:** In the present paper, a new auxiliary technique inspired by InterCriteria Analysis is proposed. It is also based on indexed matrices, however, since the considered estimates (again in the form of intuitionistic fuzzy pairs) are not pair-wise but triple-wise, it is three dimensional. The minimum number of required data needed to be stored, for the proposed technique, is given. The way the meaning of the obtained intuitionistic fuzzy pairs in the auxiliary method differs from that of the considered here implementation of the InterCriteria Analysis is also commented upon. Some properties of the proposed auxiliary technique are studied and possible applications are outlined.
**Keywords:** Intuitionistic fuzzy pair, InterCriteria Analysis, Index matrix.
**AMS Classification:** 03E72, 11C20.

## 1 Preliminary definitions and notions

Before outlining the main idea of the proposed auxiliary technique for InterCriteria Analysis (ICA) we shall remind some basic definitions and notions:

**Definition 1 (cf. [8]).** An Intuitionistic Fuzzy Pair (IFP) is an ordered pair of real non-negative numbers $\langle a, b\rangle$ such that: $a + b \leq 1$.

Of the relationship between the IFPs, we require the following:

**Definition 2 (cf. [8]).** We say that the IFP $u = \langle a, b\rangle$ is less or equal possibility wise to the IFP $v = \langle c, d\rangle$ and we write: $u \leq_{\Box} v$ if $a \leq c$.

**Definition 3 (cf. [2]).** An "Extended Index Matrix" (EIM) with index sets $K, L$ $(K, L \subset I^*)$ and elements from set $\mathcal{X}$ is called the object:

$$[K, L, \{a_{k_i,l_j}\}] = \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \vdots & a_{k_1,l_j} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1} & \dots & a_{k_i,l_j} & \dots & a_{k_i,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_j} & \dots & a_{k_m,l_n} \end{array}$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, and for $1 \le i \le m,\ 1 \le j \le n : a_{k_i,l_j} \in \mathcal{X}$.

**Definition 4 (cf. [3]).** A "3D-Extended Index Matrix" (3D-EIM) with index sets $K, L$ and $H$ $(K, L, H \subset I^*)$ and elements from set $\mathcal{X}$ is called the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}] = \begin{array}{c|ccccc} h_g & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \dots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \dots & a_{k_i,l_j,h_g} & \dots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \dots & a_{k_m,l_j,h_g} & \dots & a_{k_m,l_n,h_g} \end{array} \mid h_g \in H$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, $H = \{h_1, h_2, \dots, h_f\}$, and for $1 \le i \le m,\ 1 \le j \le n,\ 1 \le g \le f : a_{k_i,l_j,h_g} \in \mathcal{X}$.

Further we will only consider two types of EIMs, namely when $\mathcal{X}$ coincides with the set of all real numbers, or when $\mathcal{X}$ coincides with the set of all IFPs.

## 2 Auxilliary technique for InterCriteria Analysis

In what follows, we expand on an idea proposed in [1]. Basing our auxiliary method on [1, 4, 7] we will obtain an intuitionistic fuzzy pair as an estimation of the degrees of "agreement" and "disagreement" between three criteria applied on different objects.

Let us be given an Index Matrix (IM) with real number elements (see [5, 6]), with index sets consisting of the names of the criteria (for rows) and of the objects (for columns). In what follows we obtain a three dimensional IM with index sets consisting of the names of the criteria and elements corresponding to the "agreement" and "disagreement" between the respective criteria in the form of IFPs.

Further let $O$ denote the set of all objects $O_1, O_2, \dots, O_n$ being evaluated, and $C(O)$ be the set of values assigned by a given criteria $C$ to the objects, i.e.

$$O \stackrel{\text{def}}{=} \{O_1, O_2, O_3, \dots, O_n\}$$
$$C(O) \stackrel{\text{def}}{=} \{C(O_1), C(O_2), C(O_3), \dots, C(O_n)\}$$

Then we can define:

$$C^*(O) \stackrel{\text{def}}{=} \{\langle x, y\rangle \mid x \neq y \,\&\, \langle x, y\rangle \in C(O) \times C(O)\}$$

In order to find the agreement of different criteria we construct the vectors of all internal comparisons for each criterion, which elements fulfill exactly one of three relations $R$, $\overline{R}$ and $\tilde{R}$. In other words, we require that for a fixed criterion $C$ and any ordered pair $\langle x, y\rangle \in C^*(O)$ it is true:

$$\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in \overline{R} \tag{1}$$

$$\langle x, y\rangle \in \tilde{R} \Leftrightarrow \langle x, y\rangle \notin (R \cup \overline{R}) \tag{2}$$

$$R \cup \overline{R} \cup \tilde{R} = C^*(O). \tag{3}$$

From the above it is seen that we need only consider a subset of $C(O) \times C(O)$ for the effective calculation of the vector of internal comparisons (denoted further by $V(C)$) since from (1), (2) and (3) it follows that if the relation between $x$ and $y$ is known, then so is the relation between $y$ and $x$. Thus we are only be interested in lexicographically ordered pairs $\langle x, y\rangle$. Let us denote for brevity:

$$C_{i,j} = \langle C(O_i), C(O_j)\rangle$$

Then for a fixed criterion $C$ we construct the vector:

$$V(C) = \{C_{1,2}, C_{1,3}, \ldots, C_{1,n}, C_{2,3}, C_{2,4}, \ldots, C_{2,n}, C_{3,4}, \ldots, C_{3,n}, \ldots, C_{n-1,n}\}$$

It can be easily seen that it has exactly $\frac{n(n-1)}{2}$ elements. Let us replace the vector $V(C)$ with $\hat{V}(C)$, where for each $1 \leq k \leq \frac{n(n-1)}{2}$ the $k$-th component is:

$$\hat{V}_k(C) = \begin{cases} 1, & \text{iff } V_k(C) \in R, \\ 2, & \text{iff } V_k(C) \in \tilde{R}, \\ 3, & \text{otherwise.} \end{cases}$$

When comparing a collection of three criteria we determine their "degree of agreement" as the number of matching components of the respective vectors (divided by the length of the vector for normalization purposes). The "degree of disagreement" is the number of components completely different values (again normalized by the length). An example pseudocode for three criteria $C$, $C'$ and $C''$ is presented as **Algorithm 1**.

If we denote the respective degrees by $\mu_{C,C',C''}$ and $\nu_{C,C',C''}$, it is obvious (from the way of computation) that the order in which the criteria are taken has no effect on their value. It is also obvious that $\langle \mu_{C,C',C''}, \nu_{C,C',C''}\rangle$ is an IFP.

**Remark 1.** It is worth noting that while similar to one of the possible implementations of InterCriteria Analysis, where we obtain degrees of "agreeement" and "disagreeement" between couples of criteria $C$ and $C'$, the auxiliary method measures these for three criteria in another way. An important distinction concerns the degree of "disagreement", which confers the meaning of the inherent inconsistency presented by the three criteria rather than opposing behavior as in the case of two criteria. In order to be able to more readily compare the two methods by using the defined above vectors we present the pseudocode for the considered implementation of pairwise ICA in **Algorithm 2**.

**Algorithm 1** Calculating agreement and disagreement between three criteria

**Require:** Vectors $\hat{V}(C)$, $\hat{V}(C')$ and $\hat{V}(C'')$

**function** DEGREES OF AGREEMENT AND DISAGREEMENT($\hat{V}(C), \hat{V}(C'), \hat{V}(C'')$)
    $V \leftarrow \hat{V}(C) \odot \hat{V}(C') \odot \hat{V}(C'')$ ▷ $\odot$ denotes Hadamard (entrywise) product
    $\mu \leftarrow 0$
    $\nu \leftarrow 0$
    **for** $i \leftarrow 1$ to $\frac{n(n-1)}{2}$ **do**
        **if** $V_i \in \{1, 8, 27\}$ **then**
            $\mu \leftarrow \mu + 1$
        **else if** $(V_i) = 6$ **then**
            $\nu \leftarrow \nu + 1$
        **end if**
    **end for**
    $\mu \leftarrow \frac{2}{n(n-1)}\mu$
    $\nu \leftarrow \frac{2}{n(n-1)}\nu$
    **return** $\mu, \nu$
**end function**

Another difference between the normal and auxiliary method is the required dimension of the obtained EIM with IFPs. For the standard InterCriteria Analysis we require an IM with IFPs with index sets $K = \{C_1, C_2, \dots, C_{m-1}\}$ and $L = \{C_2, C_3, \dots, C_m\}$ where only $\frac{m(m-1)}{2}$ of the IFPs need to be stored. The result would look like the following EIM:

$$\begin{array}{c|ccccc}
 & C_2 & \dots & C_j & \dots & C_m \\
\hline
C_1 & a_{C_1,C_2} & \vdots & a_{C_1,C_j} & \dots & a_{C_1,C_m} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
C_i & \dots & \dots & a_{C_i,C_j} & \dots & a_{C_i,C_m} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
C_{m-1} & \dots & \dots & \dots & \dots & a_{C_{m-1},C_m}
\end{array},$$

where

$$a_{C_i,C_j} \stackrel{\text{def}}{=} \langle \mu_{C_i,C_j}, \nu_{C_i,C_j} \rangle.$$

On the other hand, for the proposed auxiliary method we need a 3D-EIM with $K = \{C_1, C_2, \dots, C_{m-2}\}$, $L = \{C_2, C_3, \dots, C_{m-1}\}$ and $H = \{C_3, C_4, \dots, C_m\}$ and we have to store $s$ in number IFPs, where

$$s = \sum_{t=1}^{m-2} \frac{t(t+1)}{2} = \frac{(m-2)(m-1)m}{6}$$

Since we cannot easily show how the EIM looks in a good fashion we provide the first few

**Algorithm 2** Calculating agreement and disagreement between two criteria

**Require:** Vectors $\hat{V}(C)$, $\hat{V}(C')$

**function** DEGREES OF AGREEMENT AND DISAGREEMENT($\hat{V}(C), \hat{V}(C')$)
    $V \leftarrow \hat{V}(C) \odot \hat{V}(C')$      ▷ $\odot$ denotes Hadamard (entrywise) product
    $\mu \leftarrow 0$
    $\nu \leftarrow 0$
    **for** $i \leftarrow 1$ to $\frac{n(n-1)}{2}$ **do**
        **if** $V_i \in \{1, 4, 9\}$ **then**
            $\mu \leftarrow \mu + 1$
        **else if** $(V_i) = 3$ **then**
            $\nu \leftarrow \nu + 1$
        **end if**
    **end for**
    $\mu \leftarrow \frac{2}{n(n-1)}\mu$
    $\nu \leftarrow \frac{2}{n(n-1)}\nu$
    **return** $\mu, \nu$
**end function**

and the last slice of the respective EIM:

| $C_3$ | $C_2$ |
|---|---|
| $C_1$ | $b_{C_1,C_2,C_3}$ |

,

| $C_4$ | $C_2$ | $C_3$ |
|---|---|---|
| $C_1$ | $b_{C_1,C_2,C_4}$ | $b_{C_1,C_3,C_4}$ |
| $C_2$ | $\ldots$ | $b_{C_2,C_3,C_4}$ |

,

| $C_5$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|
| $C_1$ | $b_{C_1,C_2,C_5}$ | $b_{C_1,C_3,C_5}$ | $b_{C_1,C_4,C_5}$ |
| $C_2$ | $\ldots$ | $b_{C_2,C_3,C_5}$ | $b_{C_2,C_4,C_5}$ |
| $C_3$ | $\ldots$ | $\ldots$ | $b_{C_3,C_4,C_5}$ |

,

$\vdots$

| $C_m$ | $C_2$ | $\ldots$ | $C_j$ | $\ldots$ | $C_{m-1}$ |
|---|---|---|---|---|---|
| $C_1$ | $b_{C_1,C_2,C_m}$ | $\vdots$ | $b_{C_1,C_j,C_m}$ | $\ldots$ | $b_{C_1,C_{m-1},C_m}$ |
| $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ |
| $C_i$ | $\ldots$ | $\ldots$ | $b_{C_i,C_j,C_m}$ | $\ldots$ | $b_{C_i,C_{m-1},C_m}$ |
| $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ |
| $C_{m-2}$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $b_{C_{m-2},C_{m-1},C_m}$ |

,

where

$$b_{C_i,C_j,C_k} \stackrel{\text{def}}{=} \langle \mu_{C_i,C_j,C_k}, \nu_{C_i,C_j,C_k} \rangle$$

Based on the way the values are obtained, we may state the following:

**Theorem 1.** *For every $i, j, k$ such that $1 \leq i < j < k \leq m$ it is fulfilled:*

$$b_{C_i,C_j,C_k} \leq_\Box a_{C_i,C_j}; b_{C_i,C_j,C_k} \leq_\Box a_{C_i,C_k}; b_{C_i,C_j,C_k} \leq_\Box a_{C_j,C_k}.$$

Hence, it is clear that the first component of the IFP $b_{C_i,C_j,C_k}$ provides a lower bound for the first component of each of the three IFPs above. The same however is not the case for the second components since they are arrived at in different manner. Despite that fact a high value of $\nu_{C_i,C_j,C_k}$ should suggest an inherent incompatibility between the considered criteria. This is why we believe that the proposed auxiliary method is a promising exploratory tool that can help provide better insight in seeking intercriterial relationships.

# 3 Conclusion

We have proposed a new auxiliary method for InterCriteria Analysis. In it, the "agreement' and "disagreement"' is sought not between two but between three criteria. We hope that this method will provide more information and assist the search for hidden relations, e.g. by filtering out incompatibilities or highlighting possible connections, between different criteria.

# Acknowledgements

The first and the second author are thankful for the support provided by the Bulgarian National Science Fund under Grant DFNI-I-02-5.

# References

[1] Atanassov, K., Mavrov, D., & Atanassova, V. (2014) Intercriteria decision making: A new approach for multicriteria decision making, based on index matrices and intuitionistic fuzzy sets. *Issues in IFSs and GNs*, 11, 1–8.

[2] Atanassov, K. (2014) *Index Matrices: Towards an Augmented Matrix Calculus*, Springer International Publishing, Cham.

[3] Atanassov, K. (2014) On index matrices. Part 5: 3-dimensional index matrices. *Advanced Studies in Contemporary Mathematics*, 24(4), 423–432.

[4] Atanassov, K. (1987) Generalized index matrices, *Comptes Rendus de l'Academie Bulgare des Sciences*, 40(11), 15–18.

[5] Atanassov, K. (2010) On index matrices, Part 1: Standard cases. *Advanced Studies in Contemporary Mathematics*, 20(2), 291–302.

[6] Atanassov, K. (2010) On index matrices, Part 2: Intuitionistic fuzzy case. *Proceedings of the Jangjeon Mathematical Society*, 13(2), 121–126.

[7] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.

[8] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013) On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, 19, 2013(3), 1–13.