Notes on Intuitionistic Fuzzy Sets
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A new intuitionistic fuzzy implication and the negation, conjunctions and disjunctions generated by it

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Abstract: A new – the 207-th – intuitionistic fuzzy implication is introduced over intuitionistic fuzzy sets. Over its basis, new intuitionistic fuzzy negation, conjunctions and disjunctions are constructed. For the first time, it is shown that all three conjunctions generated by the same implication coincide, whence their respective disjunctions are different. Some properties of the newly constructed operations are studied.
Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy operation, Intuitionistic fuzzy implication, Intuitionistic fuzzy conjunction, Intuitionistic fuzzy disjunction.
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1 Introduction

Following [1, 2] we will mention that by the moment there are 206 different intuitionistic fuzzy implications. In the present research, we will describe a new – the 207-th – intuitionistic fuzzy implication. Initially, we will give some preliminary definitions, following [1].

Let a set E be fixed. An Intuitionistic Fuzzy set (IFS) A in E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

For every two IFSs A and B, in [1] a lot of relations and operations are defined, but we will use only the following:

$$\begin{array}{lcl} A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \& \ \nu_A(x) \geq \nu_B(x)); \\ A \supset B & \text{iff} & B \subset A; \\ A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \& \ \nu_A(x) = \nu_B(x)); \\ \neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \end{array}$$

Let

$$\begin{aligned} O^* &= \{\langle x, 0, 1\rangle | x \in E\}, \\ U^* &= \{\langle x, 0, 0\rangle | x \in E\}, \\ E^* &= \{\langle x, 1, 0\rangle | x \in E\}. \end{aligned}$$

2 Main results

Here, we introduce the 207-th intuitionistic fuzzy implication as follows:

$$A \to_{207} B = \{\langle x, \mathrm{sg}(\max(\nu_A(x), \mu_B(x))), \overline{\mathrm{sg}}(\max(\nu_A(x), \mu_B(x)))\rangle | x \in E\}.$$

For it, we see that

$$\begin{array}{lcl} O^* \to_{207} O^* & = & \{\langle x, \mathrm{sg}(\max(1,0)), \overline{\mathrm{sg}}(\max(1,0))\rangle | x \in E\} = E^*, \\ O^* \to_{207} U^* & = & \{\langle x, \mathrm{sg}(\max(1,0)), \overline{\mathrm{sg}}(\max(1,0))\rangle | x \in E\} = E^*, \\ O^* \to_{207} E^* & = & \{\langle x, \mathrm{sg}(\max(1,1)), \overline{\mathrm{sg}}(\max(1,1))\rangle | x \in E\} = E^*, \\ U^* \to_{207} O^* & = & \{\langle x, \mathrm{sg}(\max(0,0)), \overline{\mathrm{sg}}(\max(0,0))\rangle | x \in E\} = O^*, \\ U^* \to_{207} U^* & = & \{\langle x, \mathrm{sg}(\max(0,0)), \overline{\mathrm{sg}}(\max(0,0))\rangle | x \in E\} = O^*, \\ U^* \to_{207} E^* & = & \{\langle x, \mathrm{sg}(\max(0,1)), \overline{\mathrm{sg}}(\max(0,1))\rangle | x \in E\} = E^*, \\ E^* \to_{207} O^* & = & \{\langle x, \mathrm{sg}(\max(0,0)), \overline{\mathrm{sg}}(\max(0,0))\rangle | x \in E\} = O^*, \\ E^* \to_{207} O^* & = & \{\langle x, \mathrm{sg}(\max(0,0)), \overline{\mathrm{sg}}(\max(0,0))\rangle | x \in E\} = O^*, \\ E^* \to_{207} E^* & = & \{\langle x, \mathrm{sg}(\max(0,1)), \overline{\mathrm{sg}}(\max(0,1))\rangle | x \in E\} = E^*. \end{array}$$

Therefore, the new implication has “the boundary” properties of the classical implication.

It has the following two geometrical interpretations, shown on Figures 1 and 2.

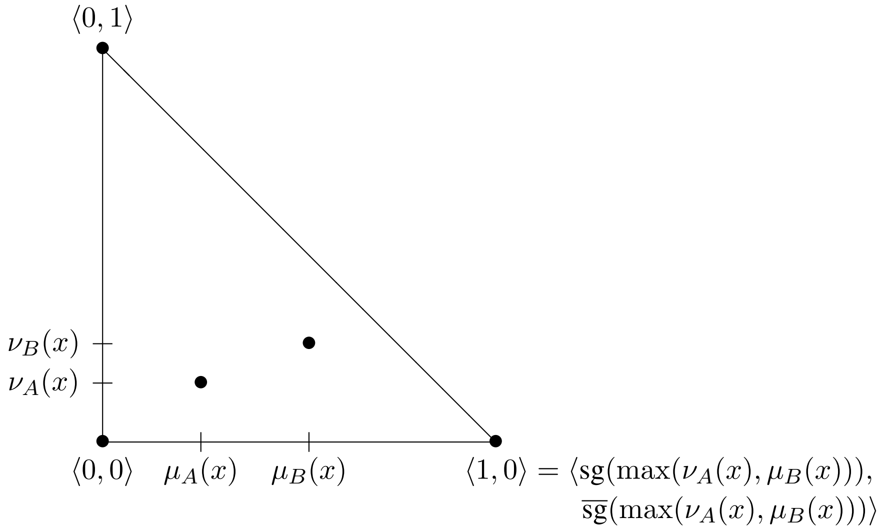

Figure 1. Geometrical interpretation of operation $\rightarrow_{207}$ – a first case

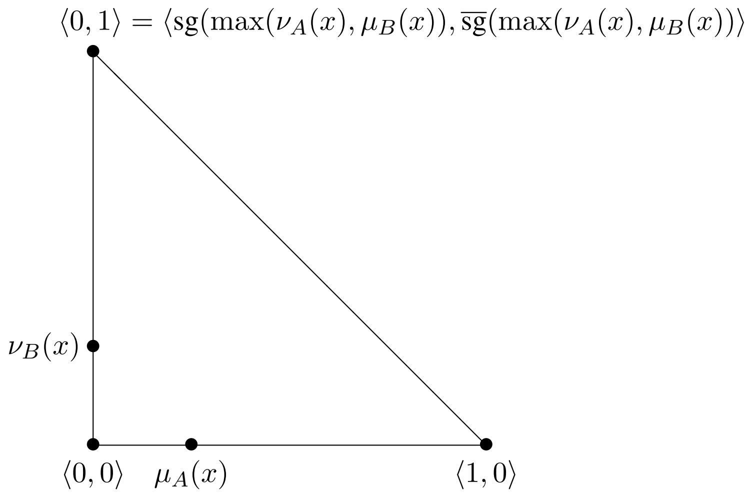

Figure 2. Geometrical interpretation of operation $\rightarrow_{207}$ – a second case

Now, we can construct a new operation intuitionistic fuzzy negation:

$$\begin{aligned}
\neg_* A \rightarrow_{207} O^* &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \rightarrow_{207} O^* \\
&= \{\langle x, \text{sg}(\max(\nu_A(x), 0)), \overline{\text{sg}}(\max(\nu_A(x), 0))\rangle | x \in E\} \\
&= \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle | x \in E\}.
\end{aligned}$$

For the new negation, we see that

$$\begin{aligned}
\neg_* O^* &= \{\langle x, \text{sg}(1), \overline{\text{sg}}(1)\rangle | x \in E\} = E^*, \\
\neg_* U^* &= \{\langle x, \text{sg}(0), \overline{\text{sg}}(0)\rangle | x \in E\} = O^*, \\
\neg_* E^* &= \{\langle x, \text{sg}(0), \overline{\text{sg}}(0)\rangle | x \in E\} = O^*.
\end{aligned}$$

Therefore, the specific equalities of the classical negation are valid, for the new negation.
The new negation has the following two geometrical interpretations, shown on Figures 3 and 4.

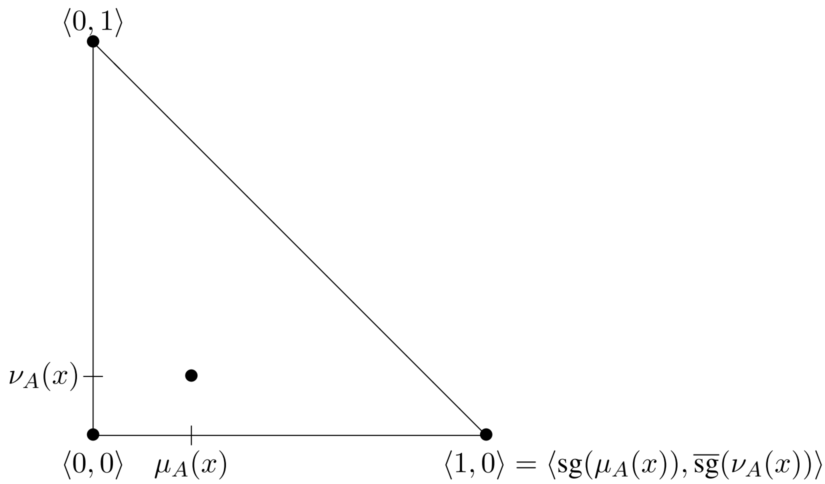

Figure 3. Geometrical interpretation of operation $\neg_*$ – a first case

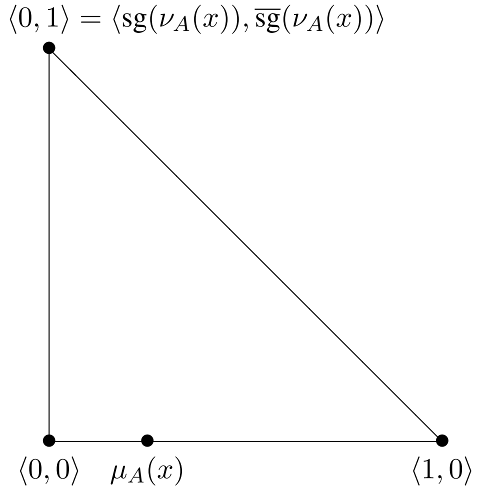

Figure 4. Geometrical interpretation of operation $\neg_*$ – a second case

The two new intuitionistic fuzzy operations generate the following three intuitionistic fuzzy conjunctions and disjunctions:

$$\begin{aligned}
A \cap^1_{207} B &= \neg_*(A \to_{207} \neg_* B)\\
&= \neg_*(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \to_{207} \{\langle x, \mathrm{sg}(\nu_B(x)), \overline{\mathrm{sg}}(\nu_B(x))\rangle | x \in E\})\\
&= \neg_*\{\langle x, \mathrm{sg}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x))), \overline{\mathrm{sg}}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x)))\rangle | x \in E\}\\
&= \{\langle x, \mathrm{sg}(\overline{\mathrm{sg}}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x))))), \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x)))))\rangle | x \in E\}\\
&= \{\langle x, \overline{\mathrm{sg}}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x)))), \mathrm{sg}(\max(\nu_A(x), \mathrm{sg}(\nu_B(x))))\rangle | x \in E\},
\end{aligned}$$

because for each real number a:

$$\mathrm{sg}(\overline{\mathrm{sg}}(a)) = \begin{cases} \text{if } a \leq 0 : 1 \\ \text{if } a > 0 : 0 \end{cases} = \overline{\mathrm{sg}}(a);$$

$$\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)) = \begin{cases} \text{if } a \leq 0 : 0 \\ \text{if } a > 0 : 1 \end{cases} = \mathrm{sg}(a).$$

$$
\begin{aligned}
A \cap_{207}^{2} B &= \neg_*(\neg_*\neg_* A \rightarrow_{207} \neg_* B)\\
&= \neg_*(\{\langle x, \mathrm{sg}(\overline{\mathrm{sg}}(\nu_A(x))), \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\nu_A(x)))\rangle | x \in E\}\\
&\quad \rightarrow_{207} \{\langle x, \mathrm{sg}(\nu_B(x)), \overline{\mathrm{sg}}(\nu_B(x))\rangle | x \in E\})\\
&= \neg_*(\{\langle x, \overline{\mathrm{sg}}(\nu_A(x)), \mathrm{sg}(\nu_A(x))\rangle | x \in E\} \rightarrow_{207} \{\langle x, \mathrm{sg}(\nu_B(x)), \overline{\mathrm{sg}}(\nu_B(x))\rangle | x \in E\})\\
&= \neg_*\{\langle x, \mathrm{sg}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x)))), \overline{\mathrm{sg}}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x))))\rangle | x \in E\}\\
&= \{\langle x, \mathrm{sg}(\overline{\mathrm{sg}}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x))))),\\
&\quad \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x)))))\rangle | x \in E\}\\
&= \{\langle x, \overline{\mathrm{sg}}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x)))), \mathrm{sg}(\max(\mathrm{sg}(\nu_A(x)), \mathrm{sg}(\nu_B(x))))\rangle | x \in E\};
\end{aligned}
$$

$$
\begin{aligned}
A \cap_{207}^{3} B &= \neg(A \rightarrow_{207} \neg B)\\
&= \neg\{\langle x, \mathrm{sg}(\max(\nu_A(x), \nu_B(x))), \overline{\mathrm{sg}}(\max(\nu_A(x), \nu_B(x)))\rangle | x \in E\};\\
&= \{\langle x, \overline{\mathrm{sg}}(\max(\nu_A(x), \nu_B(x))), \mathrm{sg}(\max(\nu_A(x), \nu_B(x)))\rangle | x \in E\}.
\end{aligned}
$$

Theorem 1. *For every two IFSs A and B*

$$A \cap_{207}^{1} B = A \cap_{207}^{2} B = A \cap_{207}^{3} B.$$

Proof. The validity of the Theorem follows from the following facts.

Let $a, b \in [0,1]$. Then

$$\overline{\mathrm{sg}}(\max(a,b)) - \overline{\mathrm{sg}}(\max(a, \mathrm{sg}(b)) = \begin{cases} \text{if } a = b = 0, & 1 - 1 = 0\\ \text{if } a > 0 \text{ or } b > 0, & 0 - 0 = 0 \end{cases};$$

and

$$\overline{\mathrm{sg}}(\max(a,b)) - \overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(b)) = \begin{cases} \text{if } a = b = 0, & 1 - 1 = 0,\\ \text{if } a > 0 \text{ or } b > 0, & 0 - 0 = 0. \end{cases} \qquad \square$$

Therefore, below we can denote the 207-th intuitionistic fuzzy conjunction by $\cap_{207}$. It has the following geometrical interpretations, shown on Figures 5 and 6.

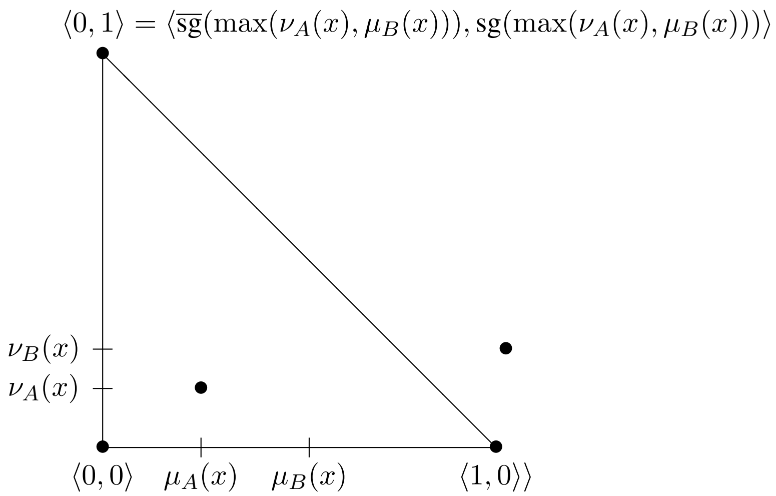

Figure 5. Geometrical interpretation of operation $\cap_{207}$ – a first case

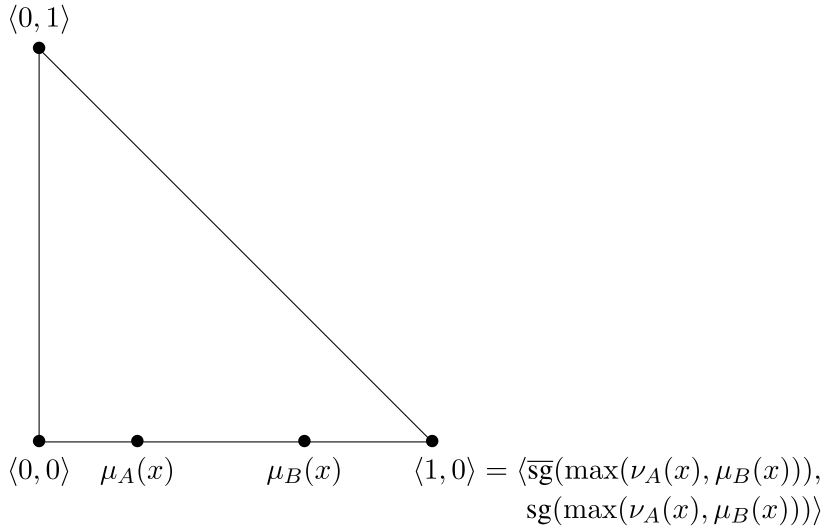

Figure 6. Geometrical interpretation of operation $\cap_{207}$ – a second case

On the other hand,

$$\begin{aligned}
A\cup^1_{207} B &= \neg_* A \rightarrow_{207} B\\
&= \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle | x\in E\} \rightarrow_{207} \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\})\\
&= \{\langle x, \text{sg}(\max(\overline{\text{sg}}(\nu_A(x)), \mu_B(x))), \overline{\text{sg}}(\max(\overline{\text{sg}}(\nu_A(x)), \mu_B(x)))\rangle | x\in E\}
\end{aligned}$$

$$\begin{aligned}
A\cup^2_{207} B &= \neg_* A \rightarrow_{207} \neg_*\neg_* B)\\
&= \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle | x\in E\} \rightarrow_{207} \{\langle x, \overline{\text{sg}}(\nu_B(x)), \text{sg}(\nu_B(x))\rangle | x \in E\})\\
&= \{\langle x, \text{sg}(\max(\overline{\text{sg}}(\nu_A(x)), \overline{\text{sg}}(\nu_B(x)))), \overline{\text{sg}}(\max(\overline{\text{sg}}(\nu_A(x)), \overline{\text{sg}}(\nu_B(x))))\rangle | x\in E\}\\
&= \{\langle x, \text{sg}(\max(\overline{\text{sg}}(\nu_A(x))), \overline{\text{sg}}(\nu_B(x)))), \overline{\text{sg}}(\max(\overline{\text{sg}}(\nu_A(x))), \overline{\text{sg}}(\nu_B(x))))\rangle | x\in E\}
\end{aligned}$$

$$\begin{aligned}
A\cup^3_{207} B &= A \rightarrow_{207} \neg B\\
&= \langle x, \text{sg}(\max(\nu_A(x), \text{sg}(\nu_B(x)))), \overline{\text{sg}}(\max(\nu_A(x), \text{sg}(\nu_B(x))))\rangle | x\in E\}.
\end{aligned}$$

Similarly to the proof of Theorem 1 we can see, that the results of the expressions $A\cup^1_{207} B$, $A\cup^2_{207} B$ and $A\cup^3_{207} B$ do not coincide, while the following assertion is valid as checked as above.

Theorem 2. *For every two IFSs A and B*

$$\neg_*\neg_* A\cup^1_{207} \neg_*\neg_* B = A\cup^2_{207} B = \neg\neg A\cup^3_{207}\neg\neg B.$$

The geometrical interpretations of the three intuitionistic fuzzy disjunctions are shown on Figures 7–12.

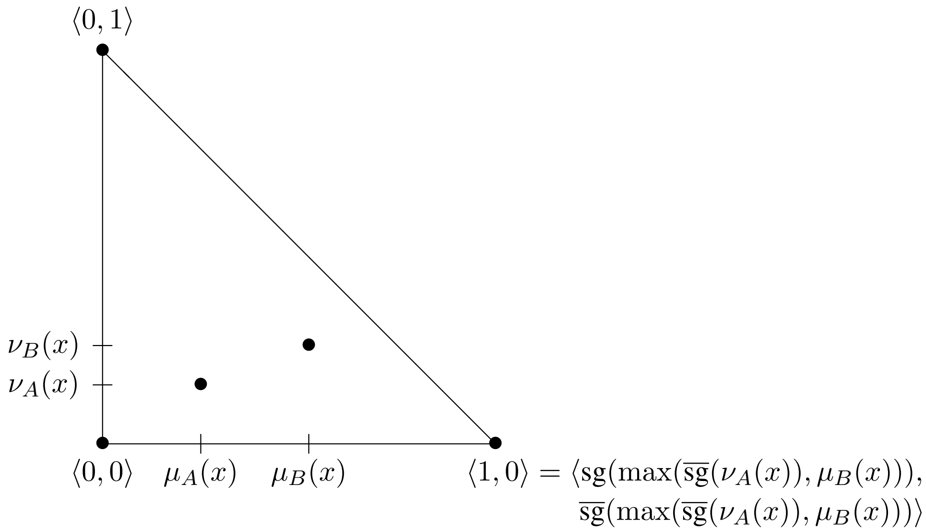

Figure 7. Geometrical interpretation of operation $\cup_{207}$ – a first case

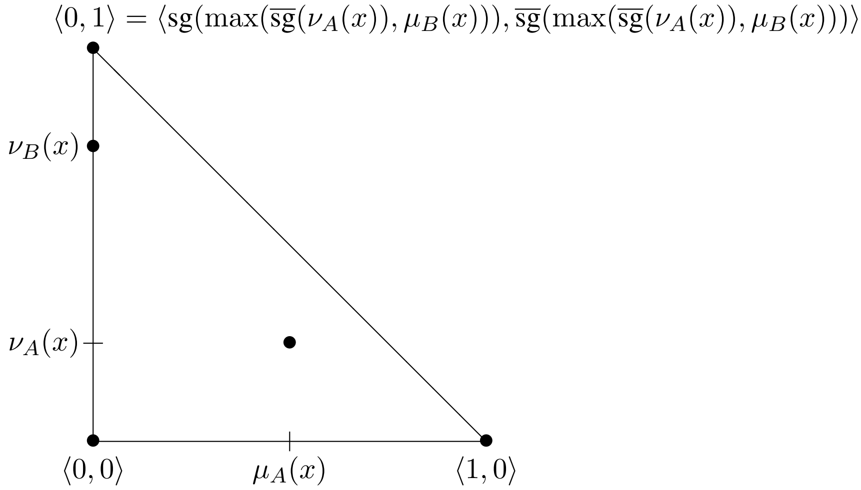

Figure 8. Geometrical interpretation of operation $\cup_{207}$ – a second case

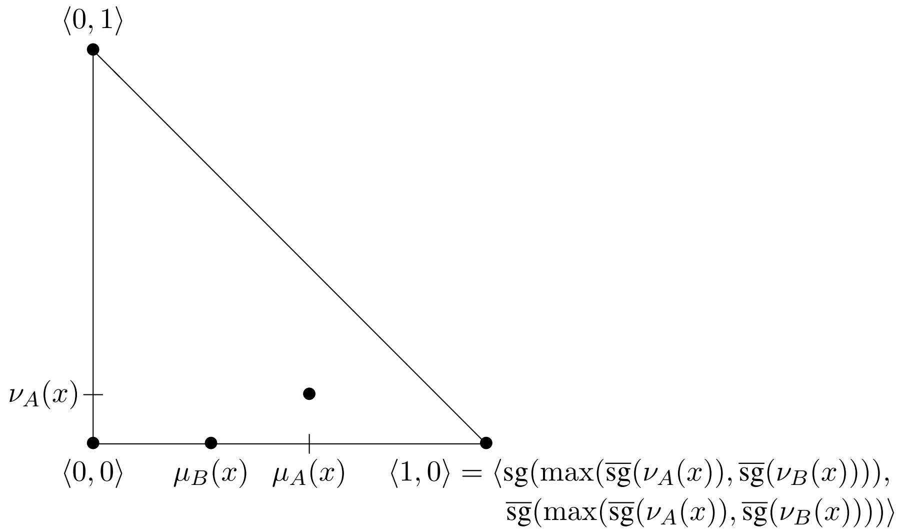

Figure 9. Geometrical interpretation of operation $\cup^2_{207}$ – a first case

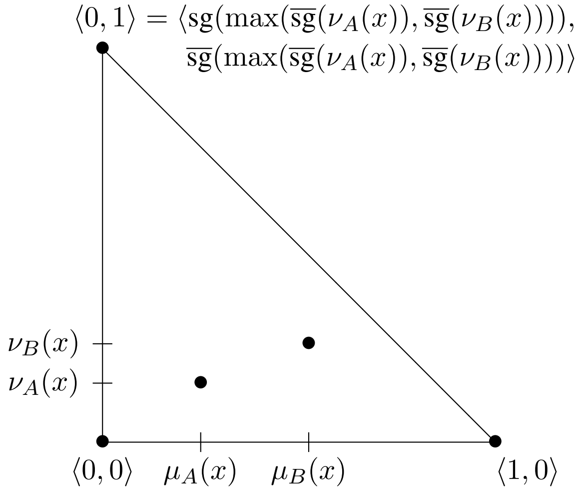

Figure 10. Geometrical interpretation of operation $\cup_{207}^{2}$ – a first case

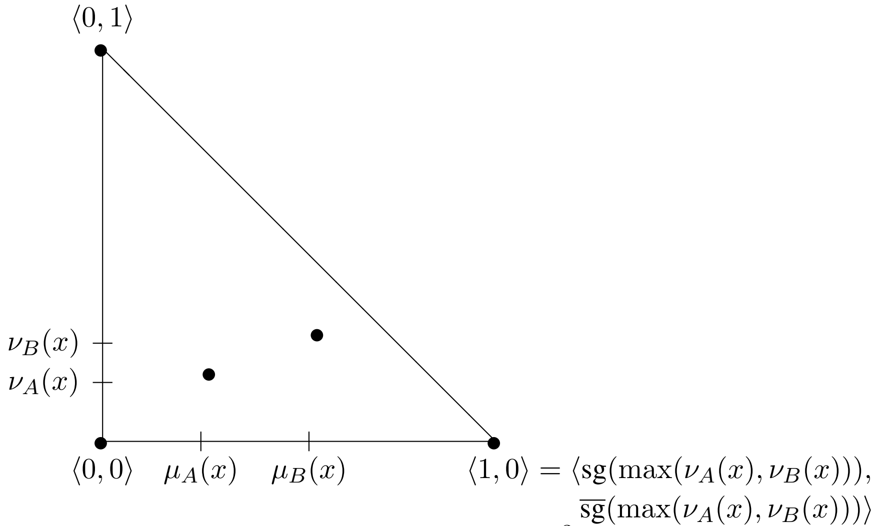

Figure 11. Geometrical interpretation of operation $\cup_{207}^{3}$ – a first case

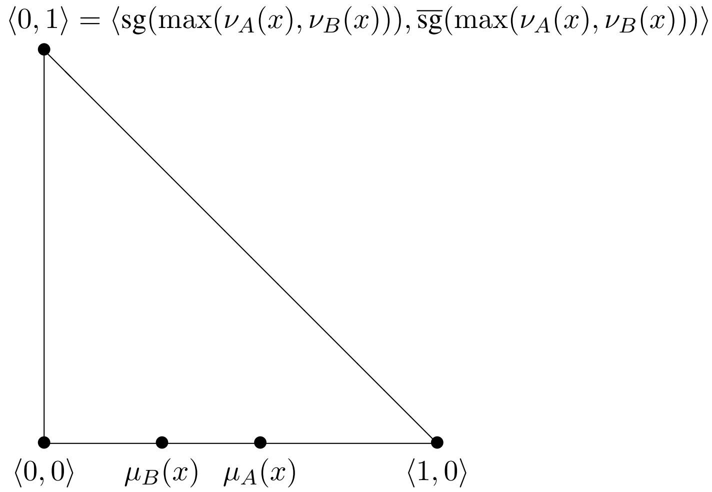

Figure 12. Geometrical interpretation of operation $\cup_{207}^{3}$ – a second case

Theorem 3. *For every two IFSs A and B and for $i = 1, 2$:*

(a) $\neg_*(\neg_* A \cup^i_{207} \neg_* B) = \neg_*\neg_* A \cap^i_{207} \neg_*\neg_* B$,

(b) $\neg(\neg A \cup^3_{207} \neg B) = \neg_*\neg_* A \cap^3_{207} \neg_*\neg_* B$,

(c) $\neg_*(\neg_* A \cap^i_{207} \neg_* B) = \neg_*\neg_* A \cup^i_{207} \neg_*\neg_* B$,

(d) $\neg(\neg A \cap^3_{207} \neg B) = \neg_*\neg_* A \cup^3_{207} \neg_*\neg_* B$.

Proof. Let A and B be given. Then

$$\begin{aligned}
\neg_*(\neg_* A \cup^1_{207} \neg_* B) &= \neg_*(\{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle | x \in E\} \\
&\quad \cup^1_{207}\{\langle x, \text{sg}(\nu_B(x)), \overline{\text{sg}}(\nu_B(x))\rangle | x \in E\}) \\
&= \neg_*(\{\langle x, \text{sg}(\max(\overline{\text{sg}}(\overline{\text{sg}}(\nu_A(x))), \text{sg}(\nu_B(x)))), \\
&\qquad \overline{\text{sg}}(\max(\overline{\text{sg}}(\overline{\text{sg}}(\nu_A(x))), \text{sg}(\nu_B(x))))\rangle | x \in E\}) \\
&= \neg_*(\{\langle x, \text{sg}(\max(\text{sg}(\nu_A(x)), \text{sg}(\nu_B(x)))), \\
&\qquad \overline{\text{sg}}(\max(\text{sg}(\nu_A(x)), \text{sg}(\nu_B(x))))\rangle | x \in E\}) \\
&= \{\langle x, \text{sg}(\overline{\text{sg}}(\max(\text{sg}(\nu_A(x)), \text{sg}(\nu_B(x))))), \\
&\qquad \overline{\text{sg}}(\overline{\text{sg}}(\max(\text{sg}(\nu_A(x)), \text{sg}(\nu_B(x)))))\rangle | x \in E\} \\
&= \{\langle x, \overline{\text{sg}}(\max(\text{sg}(\nu_A(x)), \text{sg}(\text{sg}(\nu_B(x))))), \\
&\qquad \text{sg}(\max(\text{sg}(\nu_A(x)), \text{sg}(\text{sg}(\nu_B(x)))))\rangle | x \in E\}, \\
&= \{\langle x, \overline{\text{sg}}(\nu_A(x)), \text{sg}(\nu_A(x))\rangle | x \in E\} \\
&\quad \cap^1\{\langle x, \overline{\text{sg}}(\nu_B(x)), \text{sg}(\nu_B(x))\rangle | x \in E\}) \\
&= \neg_*\neg_* A \cap^1 \neg_*\neg_* B.
\end{aligned}$$

The proof of the other equalities is similar. □

In the same manner we check the validity of the following theorem.

Theorem 4. *For every two IFSs A and B and for $i = 1, 2$, none of these equalities holds:*

(a) $\neg_*(\neg_* A \cup^i_{207} \neg_* B) = A \cap^i_{207} B$,

(b) $\neg(\neg A \cup^3_{207} \neg B) = A \cap^3_{207} B$,

(c) $\neg_*(\neg_* A \cap^i_{207} \neg_* B) = A \cup^i_{207} B$,

(d) $\neg(\neg A \cap^3_{207} \neg B) = A \cup^3_{207} B$

The results of both theorems are in accord with the results from [1].

It is obvious that operations $\cap^1_{207}, \cap^3_{207}, \cup^3_{207}$ and $\cup^3_{207}$ are not commutative and associative, while operators $\cap_{207}$ and $\cup^2_{207}$ are commutative and associative.

3 Conclusion

In the present paper, a new intuitionistic fuzzy implication was introduced and over its basis, a new intuitionistic fuzzy negation, three conjunctions and three disjunctions were constructed.

For the first time, it was checked that the three intuitionistic fuzzy conjunctions coincide, while this is not valid for the disjunctions.

For the first time, when such a new implication is being researched, we discover three coninciding conjunctions generated by one implication whence their respective disjunctions do not coincide, as is the usual case.

So, it will be interesting in future to be checked for which other intuitionistic fuzzy conjunctions and disjunctions similar coincidences occur.

References

[1] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[2] Angelova, N., Atanassov, K., & Atanassova, V. (2022). Research on intuitionistic fuzzy implications. Part 2. *Notes on Intuitionistic Fuzzy Sets*, 28(2), 172–192.