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Existence and uniqueness of intuitionistic fuzzy solution for semilinear intuitionistic fuzzy integro-differential equations with nonlocal conditions

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Abstract: In this paper, we study the existence and uniqueness of an intuitionistic fuzzy solution for semi-linear intuitionistic fuzzy integro-differential equations with non-local conditions using the Banach fixed point theorem. Theorem on the existence and uniqueness of intuitionistic fuzzy solution for these problems with nonlocal conditions are presented under certain assumptions. Finally, an example is established to illustrate the effectiveness of this theorem.

Keywords: Intuitionistic fuzzy number, Intuitionistic fuzzy integro-differential equation, Intuitionistic fuzzy solution, Banach fixed point theorem.
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1 Introduction

The concept of intuitionistic fuzzy integro-differential equations is very rare, the basic notions and arithmetic of intuitionistic fuzzy sets were first introduced by Atanassov in 1983 [1, 2], he presented the notion of intuitionistic fuzzy sets (IFS) as an extension of regular fuzzy sets, this approach is a generalization of the theory of fuzzy sets introduced by L. Zadeh [18], The topic of intuitionistic fuzzy differential equations has been investigated by several writers, it has a very interesting and is one of the most important field of the intuitionistic fuzzy theory, S. Melliani and L. S. Chadli are the first authors who defined the notions of intuitionistic fuzzy differential equations [3], in [4] the authors make a new contribution to the development of the theory of intuitionistic fuzzy metrics because of the undeniable importance of the Hausdorff distance in various fields of mathematics and computer science, fixed-point theorems for complete intuitionistic fuzzy metric spaces are given in [5, 6], The important results on the intuitionistic fuzzy topological spaces are published in [12]. The authors in [11] discusses the multi-variable extension principe of IFS and presented the IF Rieman Integral.

The existence and uniqueness of a solution for integrodifferential equations, in the cases: semilinear with nonlocal conditions, semilinear parabolic, semilinear with nonlocal conditions with fuzzy data and nonlinear with fuzzy data are studied respectively in [7, 8, 9, 10], in the present work, we study the existence and uniqueness of an intuitionistic fuzzy solution for semilinear intuitionistic fuzzy integro-differential equations with nonlocal conditions based on the Banach fixed point theorem.

After discussing the motivation of this research in the introductory section then the second section aims at giving some basic notions of intuitionistic fuzzy sets, intuitionistic fuzzy numbers, derivative in the sense of hukuhara, intuitionistic fuzzy integral as well as a other basic definitions that we will use during this research, The third Section is dedicated to the study of the existence and uniqueness of an intuitionistic fuzzy solution for semilinear intuitionistic fuzzy integro-differential equations with nonlocal conditions. Finally, we illustrate our theorem by considering an example.

2 Preliminaries

An intuitionistic fuzzy set $\Upsilon \in Y$ is given by

$$\Upsilon = \{(x, \mu_\Upsilon(x), \nu_\Upsilon(x)) \quad x \in Y\},$$

where the function $\mu_\Upsilon(x), \nu_\Upsilon(x) : Y \to [0, 1]$ define respectively the degree of membership and degree of non-membership of the element $x \in Y$ to the set Υ, which is a subset of Y, and for every $x \in Y$,

$$0 \leq \mu_\Upsilon(x) + \nu_\Upsilon(x) \leq 1.$$

Obviously, every fuzzy set has the form

$$\{(x, \mu_\Upsilon(x), \nu_{\Upsilon^c}(x)) | x \in Y\}.$$

Let $I = [a, b] \subset \mathbb{R}^n$ be a compact interval. We denote by

$$\mathbb{IF}_n = \{\langle \mu, \nu \rangle : \mathbb{R}^n \to [0,1]^2, \quad \forall x \in \mathbb{R}^n, \ 0 \leq \mu(x) + \nu(x) \leq 1\}$$

the set of all intuitionistic fuzzy numbers.

An element $\langle \mu, \nu \rangle$ of $\mathbb{IF}_n$ is said an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle \mu, \nu \rangle$ is normal, i.e., there exists $x_0, x_1 \in \mathbb{R}^n$ such that $\mu(x_0) = 1$ and $\nu(x_1) = 1$.

(ii) The membership function μ is fuzzy convex, i.e., $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2))$.

(iii) The nonmembership function ν is fuzzy concave, i.e., $v(\lambda x_1 + (1–\lambda)x_2) \leq \max(\nu(x_1), \nu(x_2))$.

(iv) μ is upper semi-continuous and ν is lower semi-continuous

(v) $Supp\langle \mu, \nu \rangle = cl\{x \in \mathbb{R}^n : |\nu(x)| < 1\}$ is bounded.

Definition 1. *An intuitionistic fuzzy number in parametric form is a pair of functions*

$$\langle \mu, \nu \rangle = ((\underline{\langle \mu, \nu \rangle}^+, \overline{\langle \mu, \nu \rangle}^{\,+}), (\underline{\langle \mu, \nu \rangle}^-, \overline{\langle \mu, \nu \rangle}^{\,-}))$$

which satisfy the following requirements:

(i) $\underline{\langle \mu, \nu \rangle}^+(\alpha)$ *is a bounded monotonic increasing continuous function,*

(ii) $\overline{\langle \mu, \nu \rangle}^{\,+}(\alpha)$ *is a bounded monotonic decreasing continuous function,*

(iii) $\underline{\langle \mu, \nu \rangle}^-(\alpha)$ *is a bounded monotonic increasing continuous function,*

(iv) $\overline{\langle \mu, \nu \rangle}^{\,-}(\alpha)$ *is a bounded monotonic decreasing continuous function,*

(v) $\underline{\langle \mu, \nu \rangle}^-(\alpha) \leq \overline{\langle \mu, \nu \rangle}^{\,-}(\alpha)$ *and* $\underline{\langle \mu, \nu \rangle}^+(\alpha) \leq \overline{\langle \mu, \nu \rangle}^{\,+}(\alpha)\ \forall \alpha \in [0,1]$.

For $\alpha \in [0,1]$ and $\langle \mu, \nu \rangle \in \mathbb{F}_n$, the upper and lower α-cuts of $\langle \mu, \nu \rangle$ are defined as:

$$[\langle \mu, \nu \rangle]^\alpha = \{x \in \mathbb{R}^n : \nu(x) \leq 1 - \alpha\} \text{ and } [\langle \mu, \nu \rangle]_\alpha = \{x \in \mathbb{R}^n : \mu(x) \geq \alpha\}$$

Remark 1. *If* $\langle \mu, \nu \rangle \in \mathbb{IF}_n$*, so we can see* $[\langle \mu, \nu \rangle]_\alpha$ *as* $[\mu]^\alpha$ *and* $[\langle \mu, \nu \rangle]^\alpha$ *as* $[1-\nu]^\alpha$ *in the fuzzy case.*

We define $0_{(1,0)} \in \mathbb{IF}_n$ as:

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0 \end{cases}$$

Let $\langle \mu, \nu \rangle$, $\langle \mu', \nu' \rangle \in \mathbb{IF}_n$ and $\lambda \in \mathbb{R}^n$, we define the following operations by:

$$(\langle \mu, \nu \rangle \oplus \langle \mu', \nu' \rangle)(z) = (\sup_{z=x+y} \min(\phi(x), \mu'(y)), \inf_{z=x+y} \max(\nu(x), \nu'(y)))$$

$$\lambda \langle \mu, \nu \rangle = \begin{cases} \langle \lambda\mu, \lambda\nu \rangle, & \text{if } \lambda \neq 0, \\ 0_{(1,0)}, & \text{if } \lambda = 0. \end{cases}$$

For $\langle \mu, \nu \rangle$, $\langle \mu', \nu' \rangle \in \mathbb{IF}_n$ and $\lambda \in \mathbb{R}^n$, the addition and scalar-multiplication are defined as follows:

$$[\langle \mu, \nu \rangle \oplus \langle \mu', \nu' \rangle]^\alpha = [\langle \mu, \nu \rangle]^\alpha + [\langle \mu', \nu' \rangle]^\alpha, \qquad [\lambda \langle \mu', \nu' \rangle]^\alpha = \lambda [\langle \mu' \nu' \rangle]^\alpha$$
$$[\langle \mu, \nu \rangle \oplus \langle \mu', \nu' \rangle]_\alpha = [\langle \mu, \nu \rangle]_\alpha + [\langle \mu', \nu' \rangle]_\alpha, \qquad [\lambda \langle \mu', \nu' \rangle]_\alpha = \lambda [\langle \mu', \nu' \rangle]_\alpha$$

Definition 2. *Let $\langle \mu, \nu \rangle$ be an element of $\mathbb{IF}_n$ and $\alpha \in [0.1]$, then we define the following sets:*

$$[\langle \mu, \nu \rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R}^n | \mu(x) \geq \alpha\}, \qquad [\langle \mu, \nu \rangle]_r^+(\alpha) = \sup\{x \in \mathbb{R}^n | \mu(x) \geq \alpha\},$$
$$[\langle \mu, \nu \rangle]_l^-(\alpha) = \inf\{x \in \mathbb{R}^n | \nu(x) \leq 1 - \alpha\}, \quad [\langle \mu, \nu \rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R}^n | \nu(x) \leq 1 - \alpha\}.$$

Definition 3. *Let $\langle \mu, \nu \rangle \in \mathbb{IF}_n$ and $\alpha \in [0, 1]$, then we define the diameter of upper and lower α-cuts of $\langle \mu, \nu \rangle$, respectively, as follows:*

$$d([\langle \mu, \nu \rangle]^\alpha) = [\langle \mu, \nu \rangle]_r^-(\alpha) - [\langle \mu, \nu \rangle]_l^-(\alpha),$$
$$d([\langle \mu, \nu \rangle]_\alpha) = [\langle \mu, \nu \rangle]_r^+(\alpha) - [\langle \mu, \nu \rangle]_l^+(\alpha).$$

Proposition 1. *For all $\alpha, \beta \in [0, 1]$ and $\langle \mu, \nu \rangle \in \mathbb{F}_n$,*

(i) $[\langle \mu, \nu \rangle]_\alpha \subset [\langle \mu, \nu \rangle]^\alpha$,

(ii) $[\langle \mu, \nu \rangle]_\alpha$ *and* $[\langle \mu, \nu \rangle]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$,

(iii) If $\alpha \leq \beta$, then $[\langle \mu, \nu \rangle]_\beta \subset [\langle \mu, \nu \rangle]_\alpha$ and $[\langle \mu, \nu \rangle]^\beta \subset [\langle \mu, \nu \rangle]^\alpha$,

(iv) If $\alpha_n \nearrow \alpha$, then $[\langle \mu, \nu \rangle]_\alpha = \cap_n [\langle \mu, \nu \rangle]_{\alpha_n}$ and $[\langle \mu, \nu \rangle]^\alpha = \cap_n [\langle \mu, \nu \rangle]^{\alpha_n}$.

Let M be any set and $\alpha \in [0, 1]$ we denote by:

$$M_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}, M^\alpha \qquad = \{x \in \mathbb{R} : \nu(x) \leq 1 - \alpha\}.$$

Lemma 1. ([1]) *Let $\{M_\alpha, \alpha \in [0, 1]\}$ and $\{M^\alpha, \alpha \in [0, 1]\}$ two families of subsets of $\mathbb{R}^n$ satifies (i)-(iv) in Proposition 1, if u and v define by:*

$$\mu(x) = \begin{cases} 0, & \text{if } x \notin M_0 \\ \sup\{\alpha \in [0, 1] : x \in M_\alpha\}, & \text{if } x \in M_0 \end{cases}$$
$$\nu(x) = \begin{cases} 1, & \text{if } x \notin M_0 \\ 1 - \sup\{\alpha \in [0, 1] : x \in M^\alpha\}, & \text{if } x \in M_0 \end{cases}$$

then $\langle \mu, \nu \rangle \in \mathbb{F}_n$.

Remark 2. *It holds that:*

$$[\langle \mu, \nu \rangle]_\alpha = [[\langle \mu, \nu \rangle]_l^+(\alpha), [\langle \mu, \nu \rangle]_r^+(\alpha)],$$
$$[\langle \mu, \nu \rangle]^\alpha = [[\langle \mu, \nu \rangle]_l^-(\alpha), [\langle \mu, \nu \rangle]_r^-(\alpha)].$$

Definition 4. *Let $\langle \mu, \nu \rangle$, $\langle \mu', \nu' \rangle \in \mathbb{F}_1$, if there exists $\langle w, z \rangle \in \mathbb{F}_1$ such that*

$$\langle \mu, \nu \rangle = \langle \mu', \nu' \rangle + \langle w, z \rangle,$$

then $\langle w, z \rangle$ is called Hukuhara difference of $\langle \mu, \nu \rangle$ and $\langle \mu', \nu' \rangle$, denote by $\langle \mu, \nu \rangle \ominus_H \langle \mu', \nu' \rangle$.

Definition 5. *The generalized Hukuhara difference of two intuitionistic fuzzy number* $\langle \mu, \nu \rangle$ *and* $\langle \mu', \nu' \rangle \in \mathbb{F}_1$ *is as follows:*

$$\langle \mu', \nu' \rangle \ominus_{gH} \langle \mu, \nu \rangle = \langle \mu'', \nu'' \rangle \Longleftrightarrow \begin{cases} i) \;\; \langle \mu', \nu' \rangle = \langle \mu, \nu \rangle + \langle \mu'', \nu'' \rangle \\ \;or \\ ii) \;\; \langle \mu, \nu \rangle = \langle \mu', \nu' \rangle + (-1)\langle \mu'', \nu'' \rangle. \end{cases}$$

Let X and Y be two nonempty subsets of $\mathbb{R}^n$. Then the Hausdorff distance $D_h(X, Y)$ is defined as:

$$D_h(X, Y) = \max\{D_h^*(X, Y), D_h^*(Y, X), \} \tag{1}$$

where

$$D_h^*(X, Y) = \max_{x \in X} \min_{y \in Y} d(x, y),$$

where x and y are elements of sets X and Y, respectively, $d(x, y)$ is the distance between this elements.

In general $D_h^*(X, Y) \neq D_h^*(Y, X)$.

For any nonempty subsets of X, Y and W of $\mathbb{R}^n$. The Hausdorff distance (1) is a metric if its satisfies the following conditions.

1. $D_h(X, Y) \geq 0$ with $D_h(X, Y) = 0$ if and only if $\tilde{X} = \tilde{Y}$,
2. $D_h(X, Y) = D_h(Y, X)$,
3. $D_h(X, Y) \leq D_h(X, W) + D_h(W, Y)$, $\forall X, Y$ and $W \in \mathbb{R}^n$

On the space $\mathbb{IF}_n$ we will consider the following metric,

$$\begin{aligned} d_\infty^n(\langle \mu, \nu \rangle, \langle \mu', \nu' \rangle) = \; & \frac{1}{4} \sup_{0 < \alpha \leq 1} \parallel [\langle \mu, \nu \rangle]_r^+(\alpha) - [\langle \mu', \nu' \rangle]_r^+(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0 < \alpha \leq 1} \parallel [\langle \mu, \nu \rangle]_l^+(\alpha) - [\langle \mu', \nu' \rangle]_l^+(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0 < \alpha \leq 1} \parallel [\langle \mu, \nu \rangle]_r^-(\alpha) - [\langle \mu', \nu' \rangle]_r^-(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0 < \alpha \leq 1} \parallel [\langle \mu, \nu \rangle]_l^-(\alpha) - [\langle \mu', \nu' \rangle]_l^-(\alpha) \parallel, \end{aligned}$$

where $\parallel . \parallel$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 1. ([1]) *The metric space* $(\mathbb{IF}_n, d_\infty)$ *is complete.*

$C([0, a] \times [0, b], \mathbb{IF}_n)$ denotes the space of all continuous mappings on $[0, a] \times [0, b]$ into $\mathbb{IF}_n$. The supremum metric D on $C([0, a] \times [0, b], \mathbb{IF}_n)$ is defined by:

$$D(\langle \mu, \nu \rangle, \langle \mu', \nu' \rangle) = \sup_{(x,y) \in [0,a] \times [0,b]} d_\infty^n(\langle \mu, \nu \rangle(x, y), \langle \mu', \nu' \rangle(x, y))$$

Definition 6. *Let* $f : I \to \mathbb{IF}_n$ *and* $t_0 \in [a, b]$. *We say that f is generalized Hukuhara differentiable at* t_0 *if there exists* $f'(t_0) \in \mathbb{IF}_n$ *such that:*

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus_{gH} f(t_0 - h)}{h}.$$

Definition 7. ([15]) *Let $F : I \to \mathbb{IF}_n$ be strongly measurable if $\forall\alpha \in [0,1]$, the set-valued mappings $F_\alpha : I \to P_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]_\alpha$ and $F^\alpha : I \to P_K(\mathbb{R})$ defined by $F^\alpha(t) = [F(t)]^\alpha$ are Lebesgue measurable, when $P_K(\mathbb{R})$ is endowed with the topology generated the Hausdorff metric d_H.*

Definition 8. ([15]) *Let $F : I \to \mathbb{IF}_n$. We say that F is integrable on $[a,b]$ if there exists $\langle\mu,\nu\rangle \in \mathbb{IF}_n$ such that $\forall\alpha \in [0,1]$:*

$$
\begin{aligned}
&[\int_a^b F(t)dt]_\alpha = \{\int_a^b f(t)dt \ \ | f : [a,b] \to \mathbb{R} \textit{ is a measurable selection for } F_\alpha\},\\
&[\int_a^b F(t)dt]^\alpha = \{\int_a^b f(t)dt \ \ | f : [a,b] \to \mathbb{R} \textit{ is a measurable selection for } F^\alpha\},\\
&\quad [\langle\mu,\nu\rangle]^\alpha = [\int_a^b F(t)dt]^\alpha,\\
&\quad [\langle\mu,\nu\rangle]_\alpha = [\int_a^b F(t)dt]_\alpha
\end{aligned}
$$

and we write $\int_a^b F(t)dt = \langle\mu,\nu\rangle$. $\forall\alpha \in [0,1]$ Then F is called integrable on $[a,b]$.

Theorem 2. *For $x_0 \in \mathbb{R}$, the intuitionistic fuzzy differential equation $y'(x) = g(x,y)$, $y(x_0) = y_0 \in \mathbb{IF}_n$, where $g : \mathbb{R} \times \mathbb{IF}_1 \to \mathbb{IF}_1$ is supposed to be continuous, if equivalent to one of the integral equations:*

$$y(x) = y_0 \oplus \int_{x_0}^{x} g(t, y(t))dt$$

or

$$y(x) = y_0 \ominus (-1) \int_{x_0}^{x} g(t, y(t))dt, \forall x \in [x_0, x_1].$$

3 Existence and uniqueness of an intuitionistic fuzzy solution

In this section we consider the existence and uniqueness of the intuitionistic fuzzy solution for semilinear integro-differential equations with nonlocal conditions:

$$\frac{du}{dt} = A[u(t) + \int_0^t G(t-s)u(s)ds] + f(t, u(t)), \quad t \in \ J = [0,T] \tag{2}$$

$$u(0) = g(t_1, t_2, t_3, ..., t_p, u(\boldsymbol{.})) = u_0 \ \ \in \mathbb{F}_n, \tag{3}$$

where A is the generator of a strongly continuous semigroup $S(.)$ on $\mathbb{F}_n$, $f : J \times \mathbb{F}_n \to \mathbb{F}_n$ is a nonlinear continuous function, $G(t)$ is an $n \times n$ continuous matrix such that $G'(t)x$ is continuous for $x \in \mathbb{F}_n$ and $t \in J$ with norm $\parallel G(t) \parallel \leq k$, $k > 0$ and $g : J^p \times \mathbb{F}_n \to \mathbb{F}_n$ is a nonlinear continuous function. We can replace **.** with the elements of set $t_1, t_2, t_3, \dots, t_p$, $0 < t_1 < t_2 < t_3 < \cdots < t_p \leq T$, $p \in \mathbb{N}$.

We formulate the following two hyhpotheses.

(H_1): The nonlinear function $g : J^p \times \mathbb{F}_n \to \mathbb{F}_n$ and the inhomogeneous terms $f : J \times \mathbb{F}_n \to \mathbb{F}_n$ are continuous and satisfy a global Lipschitz condition, i.e.,

$$D_h([g(t_1, t_2, t_3, ..., t_p, \xi_1(.))], [g(t_1, t_2, t_3, ..., t_p, \xi_2(.))]) \leq \varsigma_1 D_h([\xi_1(.)], [\xi_2(.)])$$

and

$$D_h([f(s, \xi_1(s))], [f(s, \xi_2(s))]) \leq \varsigma_2 D_h([\xi_1(s)], [\xi_2(s)])$$

for all $\xi_1(.)$, $\xi_2(.)$, $\xi_1(s)$ and $\xi_2(s) \in \mathbb{F}_n$, $\varsigma_1\ \varsigma_2 > 0$.

(H_2): An intuitionistic fuzzy operator-valued function $R(t)$ is called resolvent of $(2, 3)$ if it satisfies the following:

(i) $R(0) = I$ the identity operator of $\mathbb{F}_n$.

(ii) For each $y \in \mathbb{F}_n$, the map $t \to R(t)y$ is continuous on J.

(iii) $\forall y \in \mathbb{F}_n$, $R'(t)y \in C^1(J, \mathbb{F}_n) \cap C(J, \mathbb{F}_n)$, the equation:

$$\begin{aligned}\frac{d}{dt}R(t)y &= A[R(t)y \oplus \int_0^t G(t-s)R(s)y\, ds] \\ &= R(t)Ay \oplus \int_0^t R(t-s)AG(s)y\, ds, \quad t \in J,\end{aligned}$$

such that $[R(t)]^\alpha = [R_l^-(t;\alpha), R_r^-(t;\alpha)]$, $[R(t)]_\alpha = [R_l^+(t;\alpha), R_r^+(t;\alpha)]$, and $R_l^-(t;\alpha)$, $R_r^-(t;\alpha)$, $R_l^+(t;\alpha)$, $R_r^+(t;\alpha)$ are continuous, That is, there exists a constant $\beta > 0$ such that $\mid R(t) \mid \leq \beta$.

Theorem 3. *Let $\Gamma > 0$, and Hypotheses H_1, H_2 are checked. Then, for every $u_0, g \in \mathbb{F}_n$, the intuitionistic fuzzy initial value problem $(2, 3)$ has a unique solution $u \in C(J, \mathbb{F}_n)$.*

Proof. For each $\xi(t) \in \mathbb{F}_n$, $t \in J$ define

$$(\chi\xi)(t) = R(t)(u_0 \ominus g(t_1, t_2, t_3, ..., t_p, \xi(.))) \oplus \int_0^t R(t-s)f(s, \xi(s))ds.$$

Thus, $(\chi\xi)(t) : J \to \mathbb{F}_n$ is continuous, and $\chi : C(J, \mathbb{F}_n) \to C(J, \mathbb{F}_n)$.
It is obvious that fixed point of χ are solution to the initial value problem $(2), (3)$. For ξ_1, $\xi_2 \in C(J, \mathbb{F}_n)$, we get:

$$\begin{aligned}&D_h([(\chi\xi_1)(t)], [(\chi\xi_2)(t)]) \\ &= D_h([R(t)(u_0 \ominus g(t_1, t_2, t_3, ..., t_p, \xi_1(.))) \oplus \int_0^t R(t-s)f(s, \xi_1(s))ds], \\ &\qquad [R(t)(u_0 \ominus g(t_1, t_2, t_3, ..., t_p, \xi_2(.))) \oplus \int_0^t R(t-s)f(s, \xi_2(s))ds]) \\ &= D_h([R(t)u_0] \ominus [R(t)g(t_1, t_2, t_3, ..., t_p, \xi_1(.))] \oplus [\int_0^t R(t-s)f(s, \xi_1(s))ds], \\ &\qquad [R(t)u_0] \ominus [R(t)g(t_1, t_2, t_3, ..., t_p, \xi_2(.))] \oplus [\int_0^t R(t-s)f(s, \xi_2(s))ds])\end{aligned}$$

$$
\begin{aligned}
&\leq D_h([R(t)g(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))],[R(t)g(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))]) \\
&\quad + D_h([\int_0^t R(t-s)f(s,\xi_1(s))ds],[\int_0^t R(t-s)f(s,\xi_2(s))ds]) \\
&\leq D_h([R_\alpha(t)g_\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot)),R^\alpha(t)g^\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))], \\
&\qquad [R_\alpha(t)g_\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot)),R^\alpha(t)g^\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))] \\
&\quad + \int_0^t D_h([R_\alpha(t-s)f_\alpha(s,\xi_1(s)),R^\alpha(t-s)f^\alpha(s,\xi_1(s))], \\
&\qquad\quad [R_\alpha(t-s)f_\alpha(s,\xi_2(s)),R^\alpha(t-s)f^\alpha(s,\xi_2(s))])ds \\
&\leq \max(|\,R_\alpha(t)[g_\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))-g_\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))]\,|, \\
&\qquad |\,R^\alpha(t)[g^\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))-g^\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))]\,|) \\
&\quad + \int_0^t \max(|\,R_\alpha(t-s)[f_\alpha(s,\xi_2(s))-f_\alpha(s,\xi_1(s))]\,|, \\
&\qquad\quad |\,R^\alpha(t-s)[f^\alpha(s,\xi_2(s))-f^\alpha(s,\xi_1(s))]\,|)ds \\
&\leq \beta\max(|\,[g_\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))-g_\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))]\,|, \\
&\qquad |\,[g^\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))-g^\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))]\,|) \\
&\quad + \int_0^t \max(|\,[f_\alpha(s,\xi_2(s))-f_\alpha(s,\xi_1(s))]\,|,+\,|\,[f^\alpha(s,\xi_2(s))-f^\alpha(s,\xi_1(s))]\,|)ds \\
&= \beta D_h([g_\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot)),g^\alpha(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))], \\
&\qquad [g_\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot)),g^\alpha(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))]) \\
&\quad + \beta\int_0^t D_h([f_\alpha(s,\xi_1(s)),f^\alpha(s,\xi_1(s))],[f_\alpha(s,\xi_2(s)),f^\alpha(s,\xi_2(s))])ds \\
&= \beta D_h([g(t_1,t_2,t_3,...,t_p,\xi_1(\centerdot))],[g(t_1,t_2,t_3,...,t_p,\xi_2(\centerdot))]) \\
&\quad + \beta\int_0^t D_h([f(s,\xi_1(s))],[f(s,\xi_2(s))])ds \\
&\leq \beta\varsigma_1 D_h([\xi_1(\centerdot),\xi_2(\centerdot)]) + \beta\varsigma_2\int_0^t D_h([\xi_1(s),\xi_2(s)])ds.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
d_\infty((\chi\xi_1)(t),(\chi\xi_2)(t)) &= \sup_{\alpha\in[0,1]} D_h([\chi\xi_1)(t;\alpha)],[\chi\xi_2)(t;\alpha)]) \\
&\leq \beta\varsigma_1 D_h([\xi_1(\centerdot),\xi_2(\centerdot)]) + \beta\varsigma_2\int_0^t D_h([\xi_1(s),\xi_2(s)])ds \\
&= \beta\varsigma_1 d_\infty(\xi_1(\centerdot),\xi_2(\centerdot)) + \beta\varsigma_2\int_0^t d_\infty(\xi_1(s),\xi_2(s))ds
\end{aligned}
$$

As a result,

$$
\begin{aligned}
D(\chi\xi_1,\chi\xi_2) &= \sup_{t\in J} d_\infty((\chi\xi_1)(t),(\chi\xi_2)(t)) \\
&\leq \beta\varsigma_1 \sup_{\centerdot\in J} d_\infty(\xi_1(\centerdot),\xi_2(\centerdot)) + \beta\varsigma_2 \sup_{t\in J}\int_0^t d_\infty(\xi_1(s),\xi_2(s))ds \\
&\leq \beta[\varsigma_1+\varsigma_2\Gamma]D(\xi_1(s),\xi_2(s)).
\end{aligned}
$$

Choose Γ such that $\Gamma < (1-\beta\varsigma_1)/(\beta\varsigma_2)$. In consequence, χ is a contraction mapping. By the Banach fixed-point theorem, the intuitionistic fuzzy integro-differential equation has unique fixed point $u \in C(J,\mathbb{F}_n)$. □

4 Illustrative example

Consider the semilinear heat equation on a connected domain $(0,1)$ for a material with memory, with the internal energy $u(t,x)$ and the external heat $f(t,u(t,x)) = \mathbf{2}tu(t,x)^2$. Then, the prototype of the equation is:

$$\begin{cases} \frac{du}{dt}(t,x) & = \mathbf{2}[u(t,x) - \int_0^t e^{-(t-s)}u(s,x)ds]_{xx} + \mathbf{2}tu(t,x)^2, \\ u(t,0) & = u(t,1) = 0, \\ u(0,x) - u(\Gamma,x) & = \mathbf{2}x^2 \ \in \mathbb{F}_n. \end{cases} \tag{4}$$

Let $G(t-s) = e^{-(t-s)}$, $A = \mathbf{2}\frac{(d^2)}{(dx)^2}$, $f(t,u(t,x)) = \mathbf{2}tu(t,x)^2$, $g(u(\Gamma,x)) = u(\Gamma,x) = u(0,x) - \Phi(x) = \mathbf{2}x^2 \in \mathbb{F}_n$, with $\Phi(x) \in \mathbb{F}_n$.

The α*-cuts* of the intuitionistic fuzzy number **2** is

$$[\mathbf{2}]_\alpha = [1+\alpha, 3-\alpha],$$
$$[\mathbf{2}]^\alpha = [2-2\alpha, 2+2\alpha].$$

Then, the α*-cuts* of intuitionistic fuzzy numbers $[f(t,u(t,x))]_\alpha$ and $[f(t,u(t,x))]^\alpha$ are

$$\begin{aligned} [f(t,u(t,x))]_\alpha &= [\mathbf{2}tu(t,x)^2]_\alpha = t[\mathbf{2}]_\alpha[u(t,x)^2]_\alpha \\ &= t[1+\alpha, 3-\alpha][(u_l^+(t,x;\alpha))^2, (u_r^+(t,x;\alpha))^2] \\ &= t[(1+\alpha)(u_l^+(t,x;\alpha))^2, (3-\alpha)(u_r^+(t,x;\alpha))^2], \\ [f(t,u(t,x))]^\alpha &= [\mathbf{2}tu(t,x)^2]^\alpha = t[\mathbf{2}]^\alpha[u(t,x)^2]^\alpha \\ &= t[2-2\alpha, 2+2\alpha][(u_l^-(t,x;\alpha))^2, (u_r^-(t,x;\alpha))^2] \\ &= t[(2-2\alpha)(u_l^-(t,x;\alpha))^2, (2+2\alpha)(u_r^-(t,x;\alpha))^2], \end{aligned}$$

moreover,

$$\begin{aligned} & D_h(\Phi(u(x;\alpha)), \Phi(v(y;\alpha))) \\ & = D_h([(1+\alpha)(u_l^+(x;\alpha))^2, (3-\alpha)(u_r^+(x;\alpha))^2], [(1+\alpha)(v_l^+(y;\alpha))^2, (3-\alpha)(v_r^+(y;\alpha))^2], \\ & \qquad [(2-2\alpha)(u_l^-(x;\alpha))^2, (2+2\alpha)(u_r^-(x;\alpha))^2], [(2-2\alpha)(v_l^-(y;\alpha))^2, (2+2\alpha)(v_r^-(y;\alpha))^2]) \\ & = \max\{(1+\alpha) \mid (u_l^+(x;\alpha))^2 - (v_l^+(y;\alpha))^2 \mid, (3-\alpha) \mid (u_r^+(x;\alpha))^2 - (v_r^+(y;\alpha))^2 \mid, \\ & \qquad (2-2\alpha) \mid (u_l^-(x;\alpha))^2 - (v_l^-(y;\alpha))^2 \mid, (2+2\alpha) \mid (u_r^-(x;\alpha))^2 - (v_r^-(y;\alpha))^2 \mid\}. \\ & \le (2+2\alpha)\max\{\mid u_l^+(x;\alpha) - v_l^+(y;\alpha) \mid\mid u_l^+(x;\alpha) + v_l^+(y;\alpha) \mid, \\ & \qquad \mid u_r^+(x;\alpha) - v_r^+(y;\alpha) \mid\mid u_r^+(x;\alpha) + v_r^+(y;\alpha) \mid, \\ & \qquad \mid u_l^-(x;\alpha) - v_l^-(y;\alpha) \mid\mid u_l^-(x;\alpha) + v_l^-(y;\alpha) \mid, \\ & \qquad \mid u_r^-(x;\alpha) - y_r^-(y;\alpha) \mid\mid u_r^-(x;\alpha) + v_r^-(y;\alpha) \mid\} \\ & \le 2 \mid u_r^-(x;\alpha) + v_r^-(y;\alpha) \mid\mid u_r^+(x;\alpha) + v_r^+(y;\alpha) \mid \max\{\mid u_l^+(x;\alpha) - v_l^+(y;\alpha) \mid, \\ & \qquad \mid u_r^+(x;\alpha) - v_r^+(y;\alpha) \mid, \mid u_l^-(x;\alpha) - v_l^-(y;\alpha) \mid\mid u_r^-(x;\alpha) - v_r^-(y;\alpha) \mid\} \\ & = \eta_1 D_h((u(x;\alpha), v(y;\alpha)) \end{aligned}$$

with $\eta_1 = 2 \mid u_r^-(x;\alpha) + v_r^-(y;\alpha) \mid\mid u_r^+(x;\alpha) + v_r^+(y;\alpha) \mid$.

In the same way we can prove that the function g satisfies the inequality in Hypothesis (H_1), and again:

$$
\begin{aligned}
& D_h([f(t,u(t,x))(\alpha)],[f(t,v(t,y))(\alpha)]) \\
& = D_h(t[(1+\alpha)(u_l^+(t,x;\alpha))^2,(3-\alpha)(u_r^+(t,x;\alpha))^2], \\
& \qquad t[(1+\alpha)(v_l^+(t,y;\alpha))^2,(3-\alpha)(v_r^+(t,y;\alpha))^2], \\
& \qquad t[(2-2\alpha)(u_l^-(t,x;\alpha))^2,(2+2\alpha)(u_r^-(t,x;\alpha))^2], \\
& \qquad t[(2-2\alpha)(v_l^-(t,y;\alpha))^2,(2+2\alpha)(v_r^-(t,y;\alpha))^2]) \\
& = t\max\{(1+\alpha) \mid (u_l^+(t,x;\alpha))^2-(v_l^+(t,y;\alpha))^2 \mid, (3-\alpha) \mid (u_r^+(t,x;\alpha))^2-(v_r^+(t,y;\alpha))^2 \mid, \\
& \qquad (2-2\alpha) \mid (u_l^-(t,x;\alpha))^2-(v_l^-(t,y;\alpha))^2 \mid, (2+2\alpha) \mid (u_r^-(t,x;\alpha))^2-(v_r^-(t,y;\alpha))^2 \mid\}. \\
& \leq \Gamma(2+2\alpha)\max\{\mid u_l^+(t,x;\alpha)-v_l^+(t,y;\alpha) \mid\mid u_l^+(t,x;\alpha)+v_l^+(t,y;\alpha) \mid, \\
& \qquad \mid u_r^+(t,x;\alpha)-v_r^+(t,y;\alpha) \mid\mid u_r^+(t,x;\alpha)+v_r^+(t,y;\alpha) \mid, \\
& \qquad \mid u_l^-(t,x;\alpha)-v_l^-(t,y;\alpha) \mid\mid u_l^-(t,x;\alpha)+v_l^-(t,y;\alpha) \mid, \\
& \qquad \mid u_r^-(t,x;\alpha)-y_r^-(t,y;\alpha) \mid\mid u_r^-(t,x;\alpha)+v_r^-(t,y;\alpha) \mid\} \\
& \leq 2\Gamma \mid u_r^-(t,x;\alpha)+v_r^-(t,y;\alpha) \mid\mid u_r^+(t,x;\alpha)+v_r^+(t,y;\alpha) \mid \max\{\mid u_l^+(t,x;\alpha)-v_l^+(t,y;\alpha) \mid, \\
& \qquad \mid u_r^+(t,x;\alpha)-v_r^+(t,y;\alpha) \mid, \mid u_l^-(t,x;\alpha)-v_l^-(t,y;\alpha) \mid\mid u_r^-(t,x;\alpha)-v_r^-(t,y;\alpha) \mid\} \\
& = \eta_2 D_h((u(t,x;\alpha),v(t,y;\alpha))
\end{aligned}
$$

with $\eta_2 = 2 \mid u_r^-(t,x;\alpha) + v_r^-(t,y;\alpha) \mid\mid u_r^+(t,x;\alpha) + v_r^+(t,y;\alpha) \mid$.

This is an abstract expression of the initial value problem (2), (3).

Therefore, f and g satisfy the global Lipschitz conditions, from Theorem 3 the intuitionistic fuzzy integro-differential equation has a unique intuitionistic fuzzy solution.

5 Conclusion

In this research, we have proved the existence and uniqueness of the intuitionistic fuzzy solution for semi-linear intuitionistic fuzzy integro-differential equations with non-local conditions by applying the Banach fixed point theorem, these results are shown by an example of the semilinear one-dimensional heat equation.

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