Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy partition

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Abstract: In the paper we defined the notion of partition on family of intuitionistic fuzzy sets and we studied important properties.
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1 Introduction

In [12] authors studied fuzzy entropy, where set-theoretical partitions are replaced by fuzzy partitions. Fuzzy partitions is a collection $\mathcal{A} = \{f_1, \dots, f_n\}$ of fuzzy subsets of Ω, $f_i : \Omega \to [0,1]$ such that

$$\sum_{i=1}^{n} f_i(\omega) = 1$$

for every $\omega \in \Omega$.

There they presented a general algebraic theory using notion of a partition in set F as a finite collection $\mathcal{A} = \{a_1, \dots, a_n\} \subset F$ such that $\bigoplus_{i=1}^{n} a_i$ exists and

$$m(1_\Omega) = m\left(\bigoplus_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} m(a_i)$$

holds.

In [1, 2], K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. In this paper we use to define the notion of partition in family of intuitionistic fuzzy sets and we prove its properties.

Remark that in a whole text we use a notation “IF” for short a phrase “intuitionistic fuzzy”.

2 IF-events, IF-states and product operation

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \le 1_\Omega$.*

The family of all IF-sets will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ will be called the membership function, $\nu_A : \Omega \longrightarrow [0,1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1)) \end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \le \mathbf{B} \Longleftrightarrow \mu_A \le \mu_B, \nu_A \ge \nu_B.$$

Example 2.2. *Fuzzy set $f : \Omega \longrightarrow [0,1]$ can be regarded as IF-set, if we put*

$$\mathbf{A} = (f, 1 - f).$$

If $f = \chi_A$, then the corresponding IF-set has the form

$$\mathbf{A} = (\chi_A, 1 - \chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the product of sets and $\le$ to the set inclusion.

Next we defined the notion of a finitely additive IF-state, see [9, 10].

Definition 2.3. *Let $\mathcal{F}$ be the family of all IF-sets in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called a finitely IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, *then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$.

A finitely additive IF-state is an IF-state, if moreover

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Probably the most useful result in the IF-state theory is the following representation theorem ([6]):

Theorem 2.4. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \Big(1 - \int_\Omega \nu_A dP \Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

The product of two elements $\mathbf{A} = (\mu_A, \nu_A)$ and $\mathbf{B} = (\mu_B, \nu_B)$ of $\mathcal{F}$ is defines using components by the following rule.

Definition 2.5. **[4]** *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfying the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot \mathbf{A} = \mathbf{A}$ *for each* $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$*;*

(ii) the operation $\cdot$ is commutative and associative;

(iii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$*, then* $\mathbf{C} \cdot (\mathbf{A} \oplus \mathbf{B}) = (\mathbf{C} \cdot \mathbf{A}) \oplus (\mathbf{C} \cdot \mathbf{B})$ *and* $(\mathbf{C} \cdot \mathbf{A}) \odot (\mathbf{C} \cdot \mathbf{B}) = (0_\Omega, 1_\Omega)$ *for each* $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B), \mathbf{C} = (\mu_C, \nu_C) \in \mathcal{F}$*;*

(iv) if $\mathbf{A}_n \searrow (0_\Omega, 1_\Omega)$*,* $\mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n})$*,* $\mathbf{B}_n = (\mu_{B_n}, \nu_{B_n}) \in \mathcal{F}$*, then* $\mathbf{A}_n \cdot \mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$.

Remark that $\mathbf{A} \oplus \mathbf{B}$ exists if and only if $\mu_A + \mu_B \leq 1_\Omega$, $\nu_A + \nu_B - 1 \geq 0_\Omega$ if and only if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$.

Now we show an example of product operation on $\mathcal{F}$.

Theorem 2.6. **[4]** *The operation $\cdot$ defined by*

$$\mathbf{A} \cdot \mathbf{B} = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big) = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$$

for each $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ *is a product operation on* $\mathcal{F}$*.*

Proof. (i) Let (μ_A, ν_A) be an element of family $\mathcal{F}$. Then

$$(1_\Omega, 0_\Omega) \cdot (\mu_A, \nu_A) = (1_\Omega.\mu_A, 0_\Omega + \nu_A - 0_\Omega \cdot \nu_A) = (\mu_A, \nu_A).$$

(ii) The operation $\cdot$ is commutative and associative.
(iii) Let $(\mu_A, \nu_A) \odot (\mu_B, \nu_B) = (0_\Omega, 1_\Omega)$. Then

$$\begin{aligned} (\mu_C, \nu_C) \cdot \big((\mu_A, \nu_A) \oplus (\mu_B, \nu_B)\big) &= (\mu_C, \nu_C) \cdot \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big) = \\ &= (\mu_C, \nu_C) \cdot (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega) = \\ &= (\mu_A\mu_C + \mu_B\mu_C, \nu_A + \nu_B + 2\nu_C - 1_\Omega - \nu_A\nu_C - \nu_B\nu_C) \end{aligned}$$

and

$$\begin{aligned} ((\mu_C, \nu_C) \cdot (\mu_A, \nu_A)) &\oplus ((\mu_C, \nu_C) \cdot (\mu_B, \nu_B)) = \\ &= (\mu_C\mu_A, \nu_C + \nu_A - \nu_C\nu_A) \oplus (\mu_C\mu_B, \nu_C + \nu_B - \nu_C\nu_B) = \\ &= \big((\mu_C\mu_A + \mu_C\mu_B) \wedge 1_\Omega, (\nu_A + \nu_B + 2\nu_C - 1_\Omega - \nu_A\nu_C - \nu_B\nu_C) \vee 0_\Omega\big) = \\ &= (\mu_A\mu_C + \mu_B\mu_C, \nu_A + \nu_B + 2\nu_C - 1_\Omega - \nu_A\nu_C - \nu_B\nu_C). \end{aligned}$$

Hence

$$(\mu_C, \nu_C) \cdot \big((\mu_A, \nu_A) \oplus (\mu_B, \nu_B)\big) = \big((\mu_C, \nu_C) \cdot (\mu_A, \nu_A)\big) \oplus \big((\mu_C, \nu_C) \cdot (\mu_B, \nu_B)\big).$$

Moreover

$$\begin{aligned}\big((\mu_C,\nu_C)\cdot(\mu_A,\nu_A)\big) \;\odot\; & \big((\mu_C,\nu_C)\cdot(\mu_B,\nu_B)\big) = \\ = \;& (\mu_C\mu_A, \nu_C+\nu_A-\nu_C\nu_A)\odot(\mu_C\mu_B,\nu_C+\nu_B-\nu_C\nu_B) = \\ = \;& (0_\Omega, 1_\Omega).\end{aligned}$$

(iv) Let $(\mu_{A_n},\nu_{A_n}) \searrow (0_\Omega,1_\Omega)$, $(\mu_{B_n},\nu_{B_n}) \searrow (0_\Omega,1_\Omega)$. Since $\mu_{A_n} \searrow 0_\Omega$, $\nu_{A_n} \nearrow 1_\Omega$, $\mu_{B_n} \searrow 0_\Omega$ and $\nu_{B_n} \nearrow 1_\Omega$, then

$$(\mu_{A_n},\nu_{A_n})\cdot(\mu_{B_n},\nu_{B_n}) = (\mu_{A_n}\mu_{B_n}, \nu_{A_n}+\nu_{B_n}-\nu_{A_n}\nu_{B_n}) \searrow (0_\Omega,1_\Omega). \qquad \square$$

3 MV-algebras, states and embedding

By the Mundici theorem any MV-algebra can be defined by the help of an l-group (see [11]).

Definition 3.1. *By an ℓ-group we shall mean the structure $(G,+,\leq)$ such that the following properties are satisfied:*

(i) $(G,+)$ *is an Abelian group;*

(ii) $(G,\leq)$ *is a lattice;*

(iii) $a \leq b \Longrightarrow a+c \leq b+c$.

For each ℓ-group G, an element $u \in G$ is said to be a strong unit of G, if for all $a \in G$ there is an integer $n \geq 1$ such that $nu \geq a$ (nu is the sum $u+\ldots+u$ with n).

Example 3.2. *Consider $G = R^2$,*

$$\begin{aligned}(a,b)\hat{+}(c,d) \quad &= \quad (a+c, b+d-1),\\ (a,b)\leq(c,d) \quad &\Longleftrightarrow \quad a\leq c, b\geq d.\end{aligned}$$

Then $(R^2,\hat{+},\leq)$ is a lattice ordered group.

Evidently the operation $\hat{+}$ is commutative and associative, $(0,1)$ is the neutral element, since

$$(0,1)\hat{+}(a,b) = (a+0, b+1-1) = (a,b),$$

and $(-a, 2-b)$ is the inverse element, since

$$(a,b)\hat{+}(-a,2-b) = (0,1).$$

Further $\leq$ is a partial order with

$$\begin{aligned}(a,b)\vee(c,d) \;&=\; (\max(a,c), \min(b,d)),\\ (a,b)\wedge(c,d) \;&=\; (\min(a,c), \max(b,d)).\end{aligned}$$

Finally
$$(a,b) \le (c,d) \Longrightarrow a \le c, b \ge d,$$
hence
$$\begin{aligned} a+e &\le c+e, \\ b+f-1 &\ge d+f-1, \\ (a,b)\hat{+}(e,f) = (a+e, b+f-1) &\le (c+e, d+f-1) = (c,d)\hat{+}(e,f). \end{aligned}$$

Example 3.3. *Let* $(\Omega, \mathcal{S})$ *be a measurable space,* $\mathcal{S}$ *be a* σ*-algebra. Consider* $\mathcal{G} = \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \longrightarrow R$ are $\mathcal{S}-$ measurable functions$\}$,
$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega), \\ \mathbf{A} \le \mathbf{B} &\iff \mu_A \le \mu_B, \nu_A \ge \nu_B. \end{aligned}$$
Then $(\mathcal{G}, +, \le)$ *is an* ℓ*-group with the neutral element* $\mathbf{0} = (0_\Omega, 1_\Omega)$*,*
$$\mathbf{A} - \mathbf{B} = (\mu_A - \mu_B, \nu_A - \nu_B + 1_\Omega)$$
and the lattice operations
$$\begin{aligned} \mathbf{A} \vee \mathbf{B} &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \\ \mathbf{A} \wedge \mathbf{B} &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B). \end{aligned}$$

Definition 3.4. *An* MV*-algebra is an algebraic system* $(M, \oplus, \odot, \neg, 0, u)$*, where* $\oplus, \odot$ *are binary operations,* $\neg$ *is a unary operation,* $0, u$ *are fixed elements, which can be obtained by the following way: there exists a lattice group*
$$(G, +, \le)$$
such that
$$M = \{x \in G; 0 \le x \le u\}$$
where 0 *is the neutral element of* G*,* u *is a strong unit of* G*, and*
$$\begin{aligned} a \oplus b &= (a+b) \wedge u = \min(a+b, u), \\ a \odot b &= (a+b-u) \vee 0 = \max(a+b-u, 0), \\ \neg a &= u - a. \end{aligned}$$
Here $\vee, \wedge$ *are the lattice operations with respect to the order and* $\neg a$ *is the opposite element of the element* a *with respect to the operation of the group.*

Example 3.5. *Let* $([0,1], \oplus, \odot, \neg, 0, 1)$ *be an* MV*-algebra, where* $a \oplus b = \min(a+b, 1), a \odot b = \max(a+b-1, 0), \neg a = 1 - a$*. The corresponding group is* $(R, +, \le)$ *where* $+$ *is usual sum, and* $\le$ *is the usual ordering.*

Example 3.6. *Let* $([0,1]^2, \oplus, \odot, \neg, (0,1), (1,0))$ *be an* MV*-algebra, where*
$$\begin{aligned} (a,b) \oplus (c,d) &= (\min(a+c, 1), \max(b+d-1, 0)), \\ (a,b) \odot (c,d) &= (\max(a+c-1, 0), \min(b+d, 1)), \\ \neg(a,b) &= (1-a, 1-b). \end{aligned}$$
Here the corresponding group is $(R^2, \hat{+}, \le)$ *considered in* Example 3.2.

Remark that the set $L_\square = [0,1]^2$ (see Figure 1) with order relation $\leq_\square$

$$(x_1, y_1) \leq_\square (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \; and \; y_1 \geq y_2$$

defined for all $(x_1, y_1), (x_2, y_2) \in L_\square$ is a complete lattice. Since it is a product of the complete lattices $([0,1], \leq)$ and $([0,1], \geq)$ (see [3]).

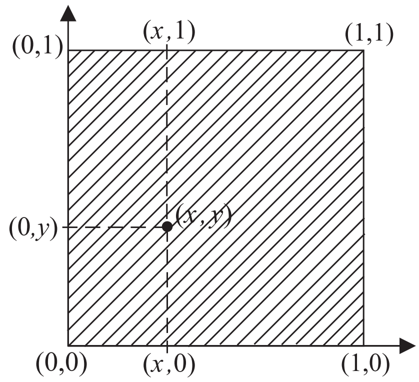

Figure 1: The shaded area constitutes the set $L_\square$

Example 3.7. *Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{M}$ the family of all pairs $\mathbf{A} = (\mu_A, \nu_A)$, where $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable functions,*

$$\begin{aligned}
\mathbf{A} \leq \mathbf{B} &\iff \mu_A \leq \mu_B, \nu_A \geq \nu_B, \\
\mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\
\mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\
\neg \mathbf{A} &= (1_\Omega - \mu_A, 1_\Omega - \nu_A).
\end{aligned}$$

Then the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ is an MV-algebra. Here the corresponding group is $(\mathcal{G}, +, \leq)$ considered in Example 3.3.

Definition 3.8. *An MV-algebra M is said to be σ-complete if its underlying lattice is σ-complete, i.e., every non-empty countable subset of M has a supremum in M.*

Every finite MV-algebra M is σ-complete - indeed, M is complete, in the sense that every non-empty subset of M has a supremum in M.

Definition 3.9. [10] *Let $(M, \oplus, \odot, \neg, 0, u)$ be an MV-algebra. By a finitely additive state on an MV-algebra M is considered each monotone mapping (i.e. $a \leq b \Rightarrow m(a) \leq m(b)$) $m : M \to [0,1]$ satisfying the following conditions:*

(i) $m(u) = 1, m(0) = 0$;

(ii) $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$.

A finitely additive state is a state, if moreover *(iii)* $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

We say that m *is faithful (also called, strictly positive) if* $m(x) \neq 0$ *whenever* $x \neq 0$, $x \in M$.

Now we presented very important result.

Theorem 3.10. [9] *Let the system* $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ *be the* MV*-algebra constructed in Example 3.7. Then* $\mathcal{F} \subset \mathcal{M}$ *and to each finitely additive IF-state* $\mathbf{m} : \mathcal{F} \to [0, 1]$ *there exists a finitely additive state* $m : \mathcal{M} \to [0, 1]$ *such that is an extension of* $\mathbf{m}$ *(i.e.* $m|\mathcal{F} = \mathbf{m}$*).*

Proof. See *Theorem 1* in paper [9]. □

Definition 3.11. [11] *An MV-algebra with product is a pair* $(M, \cdot)$*, where* M *is an MV-algebra and* $\cdot$ *is a commutative and associative binary operation on* M *satisfying the following conditions, for all* $a, b, c \in M$*:*

(i) $u \cdot a = a$*;*

(ii) if $a \odot b = 0$*, then* $c \cdot (a \oplus b) = (c \cdot a) \oplus (c \cdot b)$ *and* $(c \cdot a) \odot (c \cdot b) = 0$*;*

(iii) if $a_n \searrow 0$*,* $b_n \searrow 0$ *and* $a_n, b_n \in M$*, then* $a_n \cdot b_n \searrow 0$.

4 IF-partition

Definition 4.1. *By an IF-partition in* $\mathcal{F}$ *we shall mean a finite collection* $\xi = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\} \subset \mathcal{F}$ *such that* $\bigoplus_{i=1}^n \mathbf{A}_i$ *exists (i.e* $\bigodot_{i=1}^n \mathbf{A}_i = (0_\Omega, 1_\Omega)$*) and*

$$\mathbf{m}\left(\bigoplus_{i=1}^n \mathbf{A}_i\right) = \sum_{i=1}^n \mathbf{m}(\mathbf{A}_i) = 1$$

holds, where $\mathbf{m}$ *is a finitely additive IF-state. If* $\xi = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$*,* $\eta = \{\mathbf{B}_1, \ldots, \mathbf{B}_m\}$ *be two IF-partitions, then we define*

$$\xi \vee \eta = \{\mathbf{A}_i \cdot \mathbf{B}_j : i = 1, \ldots, n; j = 1, \ldots, m\},$$

where $\xi \neq \eta$*,* $\xi \vee \xi = \xi$*. We write* $\xi \prec \eta$ *(*η *is a refinement of* ξ*), if there exists a partition* $I(1), \ldots, I(n)$ *of the set* $\{1, \ldots, m\}$ *such that*

$$\mathbf{m}(\mathbf{A}_i) = \mathbf{m}\left(\bigoplus_{j \in I(i)} \mathbf{B}_j\right) = \sum_{j \in I(i)} \mathbf{m}(\mathbf{B}_j)$$

for every $i = 1, \ldots, n$.

Proposition 4.1. *Let* $\mathbf{A} \in \mathcal{F}$ *and* $\mathbf{m} : \mathcal{F} \to [0, 1]$ *be a finitely additive IF-state. If* $\mathbf{m}(\mathbf{A}) = 1$*, then*

$$\mathbf{m}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{m}(\mathbf{B}),$$

for each $\mathbf{B} \in \mathcal{F}$.

Proof. Let $\mathbf{m} : \mathcal{F} \to [0,1]$ be a finitely additive IF-state, $\mathbf{A}, \mathbf{B} \in \mathcal{F}$. From *Theorem 3.10* there exists finitely additive state $m : \mathcal{M} \to [0,1]$ such that $m|\mathcal{F} = \mathbf{m}$ and $\mathcal{F} \subset \mathcal{M}$. Then using (i) and (iii) property of product operation from *Definition 2.5* we have

$$\begin{aligned}\mathbf{m}(\mathbf{B}) &= m(\mathbf{B}) = m\big(\mathbf{B}\cdot(1_\Omega,0_\Omega)\big) = m\big(\mathbf{B}\cdot(\mathbf{A}\oplus\neg\mathbf{A})\big) = m(\mathbf{B}\cdot\mathbf{A}\oplus\mathbf{B}\cdot\neg\mathbf{A}) = \\ &= m(\mathbf{B}\cdot\mathbf{A}) + m(\mathbf{B}\cdot\neg\mathbf{A})\end{aligned}$$

and we use the property (ii) of *Definition 3.9*, too.

Since $m(\mathbf{A}) = \mathbf{m}(\mathbf{A}) = 1$, then

$$\begin{aligned}1 &= m(1_\Omega,0_\Omega) = m(\mathbf{A}\oplus\neg\mathbf{A}) = m(\mathbf{A}) + m(\neg\mathbf{A}) = 1 + m(\neg\mathbf{A}) \\ 0 &= m(\neg\mathbf{A}). \qquad (1)\end{aligned}$$

Moreover $\mathbf{B}\cdot\neg\mathbf{A} \le \neg\mathbf{A}$, hence

$$\begin{aligned}0 \le\ m(\mathbf{B}\cdot\neg\mathbf{A}) &\le m(\neg\mathbf{A}) \overset{(1)}{=} 0 \\ m(\mathbf{B}\cdot\neg\mathbf{A}) &= 0. \qquad (2)\end{aligned}$$

Finally

$$\begin{aligned}\mathbf{m}(\mathbf{B}) = m(\mathbf{B}) = m(\mathbf{B}\cdot\mathbf{A}) + m(\mathbf{B}\cdot\neg\mathbf{A}) \overset{(2)}{=} m(\mathbf{B}\cdot\mathbf{A}) + 0 = m(\mathbf{B}\cdot\mathbf{A}) = \\ = \mathbf{m}(\mathbf{B}\cdot\mathbf{A}).\end{aligned}$$

□

Theorem 4.2. *If ξ, η are two IF-partitions, then $\xi \vee \eta$ is an IF-partitions, too. Further, $\xi \prec \xi \vee \eta$ ($\eta \prec \xi \vee \eta$) for each IF-partitions ξ, η.*

Proof. If $\xi = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$, $\eta = \{\mathbf{B}_1, \ldots, \mathbf{B}_m\}$ be two IF-partitions, then

$$\bigoplus_{i=1}^{n} \mathbf{A}_i \text{ exists } \left(i.e \ \bigodot_{i=1}^{n} \mathbf{A}_i = (0_\Omega, 1_\Omega) \right) \text{ and } \mathbf{m}\left(\bigoplus_{i=1}^{n} \mathbf{A}_i \right) = \sum_{i=1}^{n} \mathbf{m}(\mathbf{A}_i) = 1; \qquad (3)$$

$$\bigoplus_{j=1}^{m} \mathbf{B}_j \text{ exists } \left(i.e \ \bigodot_{j=1}^{m} \mathbf{B}_j = (0_\Omega, 1_\Omega) \right) \text{ and } \mathbf{m}\left(\bigoplus_{j=1}^{m} \mathbf{B}_j \right) = \sum_{j=1}^{m} \mathbf{m}(\mathbf{B}_j) = 1. \qquad (4)$$

From *Definition 4.1* we have that

$$\xi \vee \eta = \{\mathbf{A}_i \cdot \mathbf{B}_j : i = 1, \ldots, n; j = 1, \ldots, m\}.$$

Now we prove that $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j$ exists (i.e. $\bigodot_{i=1}^{n} \bigodot_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j = (0_\Omega, 1_\Omega)$). From (iii) property of *Definition 2.5* we have $\bigodot_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j = (0_\Omega, 1_\Omega)$, hence

$$\bigodot_{i=1}^{n} \bigodot_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j = \bigodot_{i=1}^{n} \left(\bigodot_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j \right) = \bigodot_{i=1}^{n} (0_\Omega, 1_\Omega) = (0_\Omega, 1_\Omega). \qquad (5)$$

Moreover by the distributive law (iii) from *Definition 2.5* we obtain

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j = \bigoplus_{i=1}^{n} \left(\bigoplus_{j=1}^{m} \mathbf{A}_i \cdot \mathbf{B}_j \right) = \bigoplus_{i=1}^{n} \left[\mathbf{A}_i \cdot \left(\bigoplus_{j=1}^{m} \mathbf{B}_j \right) \right] = \left(\bigoplus_{i=1}^{n} \mathbf{A}_i \right) \cdot \left(\bigoplus_{j=1}^{m} \mathbf{B}_j \right). \quad (6)$$

Since $\mathbf{m}\big(\bigoplus_{j=1}^{m}\mathbf{B}_j\big) = 1$, then using *Proposition 4.1* we have

$$\mathbf{m}\Bigg(\Big(\bigoplus_{i=1}^{n}\mathbf{A}_i\Big)\cdot\Big(\bigoplus_{j=1}^{m}\mathbf{B}_j\Big)\Bigg) = \mathbf{m}\Big(\bigoplus_{i=1}^{n}\mathbf{A}_i\Big), \tag{7}$$

$$\mathbf{m}\Bigg(\mathbf{A}_i\cdot\Big(\bigoplus_{j=1}^{m}\mathbf{B}_j\Big)\Bigg) = \mathbf{m}(\mathbf{A}_i). \tag{8}$$

Finally by (6), (7) and (3) we obtain

$$\mathbf{m}\Big(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m}\mathbf{A}_i\cdot\mathbf{B}_j\Big) = \mathbf{m}\Bigg(\Big(\bigoplus_{i=1}^{n}\mathbf{A}_i\Big)\cdot\Big(\bigoplus_{j=1}^{m}\mathbf{B}_j\Big)\Bigg) = \mathbf{m}\Big(\bigoplus_{i=1}^{n}\mathbf{A}_i\Big) = 1. \tag{9}$$

Since $\bigodot_{j=1}^{m}\mathbf{A}_i\cdot\mathbf{B}_j = (0_\Omega, 1_\Omega)$, then by (ii) property of *Definition 2.3* we have

$$\sum_{j=1}^{m}\mathbf{m}(\mathbf{A}_i\cdot\mathbf{B}_j) = \mathbf{m}\Big(\bigoplus_{j=1}^{m}(\mathbf{A}_i\cdot\mathbf{B}_j)\Big). \tag{10}$$

Therefore using (10), distributive law of product operation, (8) and (3) we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{m}(\mathbf{A}_i\cdot\mathbf{B}_j) &= \sum_{i=1}^{n}\mathbf{m}\Big(\bigoplus_{j=1}^{m}(\mathbf{A}_i\cdot\mathbf{B}_j)\Big)\\ &= \sum_{i=1}^{n}\mathbf{m}\Bigg(\mathbf{A}_i\cdot\Big(\bigoplus_{j=1}^{m}\mathbf{B}_j\Big)\Bigg) = \sum_{i=1}^{n}\mathbf{m}(\mathbf{A}_i) = 1.\end{aligned} \tag{11}$$

Hence by (9) and (11) $\xi\vee\eta$ is an IF-partition.

We prove that $\xi \prec \xi\vee\eta$, yet. Let us mention that $\xi\vee\eta$ is indexed by $\{(i,j) : i = 1,\dots,n;\ j = 1,\dots,m\}$. Therefore, if we put $I(i) = \{(i,1),\dots,(i,m)\}$, then by (8), distributive law of product operation and (10) we have

$$\mathbf{m}(\mathbf{A}_i) = \mathbf{m}\Bigg(\mathbf{A}_i\cdot\Big(\bigoplus_{j=1}^{m}\mathbf{B}_j\Big)\Bigg) = \mathbf{m}\Big(\bigoplus_{j=1}^{m}(\mathbf{A}_i\cdot\mathbf{B}_j)\Big) = \sum_{j=1}^{m}\mathbf{m}(\mathbf{A}_i\cdot\mathbf{B}_j) = \sum_{(k,j)\in I(i)}\mathbf{m}(\mathbf{A}_k\cdot\mathbf{B}_j)$$

for every i. It follows $\xi \prec \xi\vee\eta$. The proof of $\eta \prec \xi\vee\eta$ is analogously. □

5 Conclusion

A notion of IF-partition is a basic notion for introduction the several types of entropy on the intuitionistic fuzzy sets. In the paper we defined the notion of IF-partition and we studied important properties.

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