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# Approximate solutions preserving parameters of intuitionistic fuzzy linear systems

**Adrian I. Ban and Lucian C. Coroianu**

Department of Mathematics and Informatics, University of Oradea,
Universităţii 1, 410087 Oradea, Romania
e-mail: aiban@uoradea.ro, lcoroianu@uoradea.ro

**Dedicated to 90-th anniversary of Prof. Lotfi Zadeh**

**Abstract.** We prove that, under some conditions, we obtain the same solution if we simplify the input data or the output data in an intuitionistic fuzzy linear system. A very recent result of approximation of intuitionistic fuzzy numbers starting from the approximation of fuzzy numbers is very useful to give few illustrative examples.

**Key words:** Fuzzy number, Intuitionistic fuzzy number, Approximation, Linear system

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## 1 Introduction

The problem of solving linear systems with fuzzy numbers or intuitionistic fuzzy numbers as data could occur whenever collections of data are processed to model a natural language abounded with imprecision and vagueness. The practitioners are interested in simplification of outputs of a modelling with fuzzy numbers or intuitionistic fuzzy numbers, in this way the results being easier interpreted and implemented.

In the present paper we prove that we can obtain the same result if we approximate the solution of an intuitionistic fuzzy linear system or we approximate the initial data of the system and then we solve the simplified linear system. The condition of linearity of the approximation operator is significant here. An important benefit is the preserving of the parameters of the exact solution by the approximate solution.

We illustrate the theoretical development by few examples. A very recent result which reduces certain approximations of intuitionistic fuzzy numbers to the approximations of fuzzy numbers is very useful.

## 2 Preliminaries

We consider the following well-known description of a fuzzy number $u$:

$$u(x) = \begin{cases} 0, & \text{if } x \le a_1, \\ l_u(x), & \text{if } a_1 \le x \le a_2, \\ 1, & \text{if } a_2 \le x \le a_3, \\ r_u(x), & \text{if } a_3 \le x \le a_4, \\ 0, & \text{if } a_4 \le x, \end{cases} \tag{1}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}, l_u : [a_1, a_2] \to [0,1]$ is a nondecreasing continuous function, $l_u(a_1) = 0, l_u(a_2) = 1$, called the left side of the fuzzy number $u$ and $r_u : [a_3, a_4] \to [0,1]$ is a nonincreasing continuous function, $r_u(a_3) = 1, r_u(a_4) = 0$, called the right side of the fuzzy number $u$. The $\alpha$-cut, $\alpha \in ]0,1]$, of a fuzzy number $u$ is the crisp set defined as

$$u_\alpha = \{x \in \mathbb{R} : u(x) \ge \alpha\}.$$

The support or 0-cut $u_0$ of a fuzzy number $u$ is defined as the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$, that is

$$u_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}.$$

Every $\alpha$-cut, $\alpha \in [0,1]$, of a fuzzy number $u$ is a closed interval

$$u_\alpha = \left[u^-(\alpha), u^+(\alpha)\right],$$

where

$$u^-(\alpha) = \inf\{x \in \mathbb{R} : u(x) \ge \alpha\},$$
$$u^+(\alpha) = \sup\{x \in \mathbb{R} : u(x) \ge \alpha\}$$

for any $\alpha \in ]0,1]$. If the sides of the fuzzy number $u$ are strictly monotone then one can see easily that $u^-$ and $u^+$ are inverse functions of $l_u$ and $r_u$, respectively.

We denote by $F(\mathbb{R})$ the set of all fuzzy numbers. A metric on the set of fuzzy numbers, which is an extension of the Euclidean distance, is defined by

$$d^2(u,v) = \int_0^1 \left(u^-(\alpha) - v^-(\alpha)\right)^2 d\alpha + \int_0^1 \left(u^+(\alpha) - v^+(\alpha)\right)^2 d\alpha. \tag{2}$$

If $\omega : \mathbb{R} \to [0,1]$ is a fuzzy set such that $1-\omega$, where $(1-\omega)(x) = 1 - \omega(x)$, for every $x \in \mathbb{R}$, is a fuzzy number and we denote

$$\omega_\alpha = \{x \in \mathbb{R} : \omega(x) \le \alpha\}, \alpha \in [0,1[$$
$$\omega_1 = \overline{\{x \in \mathbb{R} : \omega(x) < 1\}},$$

then

$$\omega_\alpha = (1-\omega)_{1-\alpha},$$

for every $\alpha \in [0,1]$. The set $\omega_\alpha$ is a closed interval $[\omega^-(\alpha), \omega^+(\alpha)]$, for every $\alpha \in [0,1]$.

Fuzzy numbers with simple membership functions are preferred in practice. The most used such fuzzy numbers are so-called trapezoidal fuzzy numbers. A trapezoidal fuzzy number $T$ is given by $T_\alpha = [T^-(\alpha), T^+(\alpha)]$,

$$T^-(\alpha) = t_1 + (t_2 - t_1)\alpha$$
$$T^+(\alpha) = t_4 - (t_4 - t_3)\alpha, \alpha \in [0,1],$$

where $t_1, t_2, t_3, t_4 \in \mathbb{R}, t_1 \leq t_2 \leq t_3 \leq t_4$. When $t_2 = t_3$ we obtain the so-called triangular fuzzy numbers, when $t_1 = t_2$ and $t_3 = t_4$ we obtain closed intervals and in the case $t_1 = t_2 = t_3 = t_4$ we obtain crisp numbers. We denote $T = (t_1, t_2, t_3, t_4)$ a trapezoidal fuzzy number as above and $F^T(\mathbb{R})$ the set of all trapezoidal fuzzy numbers.

An important kind of fuzzy number was introduced in [11] as follows. Let $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. A fuzzy number $u$ such that

$$u_\alpha = \left[u^-(\alpha), u^+(\alpha)\right] = \left[a + (b-a)\alpha^{1/r}, d - (d-c)\alpha^{1/r}\right], \alpha \in [0,1],$$

where $r > 0$, is denoted by $u = (a, b, c, d)_r$.

Let $u, v \in F(\mathbb{R}), u_\alpha = [u^-(\alpha), u^+(\alpha)], v_\alpha = [v^-(\alpha), v^+(\alpha)], \alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We consider the sum $u + v$ and the scalar multiplication $\lambda \cdot u$ by

$$(u+v)_\alpha = u_\alpha + v_\alpha = \left[u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)\right] \tag{3}$$

and

$$(\lambda \cdot u)_\alpha = \lambda \cdot u_\alpha = \begin{cases} [\lambda u^-(\alpha), \lambda u^+(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda u^+(\alpha), \lambda u^-(\alpha)], & \text{if } \lambda < 0, \end{cases} \tag{4}$$

respectively, for every $\alpha \in [0,1]$. In the case of the trapezoidal fuzzy numbers $T = (t_1, t_2, t_3, t_4)$ and $S = (s_1, s_2, s_3, s_4)$ we obtain

$$T + S = (t_1 + s_1, t_2 + s_2, t_3 + s_3, t_4 + s_4)$$

and

$$\lambda \cdot T = \begin{cases} (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4), & \text{if } \lambda \geq 0, \\ (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1), & \text{if } \lambda < 0. \end{cases}$$

If $u_1 = (a_1, b_1, c_1, d_1)_r$ and $u_2 = (a_2, b_2, c_2, d_2)_r$ then

$$u_1 + u_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)_r$$

and

$$\lambda \cdot u_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1)_r, & \text{if } \lambda \geq 0, \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1)_r, & \text{if } \lambda < 0. \end{cases}$$

Let $u \in F(\mathbb{R})$. The expected interval $EI(u)$, expected value $EV(u)$, width $w(u)$, value $Val(u)$, ambiguity $Amb(u)$, left-hand ambiguity $Amb_L(u)$, right-hand ambiguity

$Amb_R(u)$ and core $core(u)$ of $u$ were introduced ([12]-[16]) by

$$EI(u) = \left[\int_0^1 u^-(\alpha)\,d\alpha, \int_0^1 u^+(\alpha)\,d\alpha\right] \tag{5}$$

$$EV(u) = \frac{1}{2}\left(\int_0^1 u^-(\alpha)\,d\alpha + \int_0^1 u^+(\alpha)\,d\alpha\right) \tag{6}$$

$$w(u) = \int_0^1 u^+(\alpha)\,d\alpha - \int_0^1 u^-(\alpha)\,d\alpha \tag{7}$$

$$Val(u) = \int_0^1 \alpha u^+(\alpha)\,d\alpha + \int_0^1 \alpha u^-(\alpha)\,d\alpha \tag{8}$$

$$Amb(u) = \int_0^1 \alpha u^+(\alpha)\,d\alpha - \int_0^1 \alpha u^-(\alpha)\,d\alpha \tag{9}$$

$$Amb_L(u) = \int_0^1 \alpha\left(EV(u) - u^-(\alpha)\right)d\alpha \tag{10}$$

$$Amb_U(u) = \int_0^1 \alpha\left(u^+(\alpha) - EV(u)\right)d\alpha \tag{11}$$

$$core(u) = \left[u^-(1), u^+(1)\right]. \tag{12}$$

**Definition 1** *([2], [3]) Let $X \neq \emptyset$ be a given set. An intuitionistic fuzzy set in $X$ is an object $A$ given by*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in X\},$$

*where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \tag{13}$$

*for every $x \in X$.*

**Definition 2** *An intuitionistic fuzzy set $A = \{\langle x, u_A(x), v_A(x)\rangle ; x \in \mathbb{R}\}$ such that $u_A$ and $1 - v_A$ are fuzzy numbers, where*

$$(1 - v_A)(x) = 1 - v_A(x), \forall x \in \mathbb{R},$$

*is called an intuitionistic fuzzy number.*

We denote by $A = (u_A, v_A)$ an intuitionistic fuzzy number and by $IF(\mathbb{R})$ the set of all intuitionistic fuzzy numbers. It is obvious that any fuzzy number $u$ can be represented as an intuitionistic fuzzy number by $(u, 1-u)$.

With respect to the $\alpha$-cuts of the fuzzy number $1 - v_A$ the following equalities are immediate:

$$(1 - \nu_A)^-(\alpha) = \nu_A^-(1-\alpha) \tag{14}$$

and

$$(1 - \nu_A)^+(\alpha) = \nu_A^+(1-\alpha), \tag{15}$$

for every $\alpha \in [0,1]$.

We define the addition $A+B \in IF(\mathbb{R})$ of $A=(u_A,v_A), B=(u_B,v_B) \in IF(\mathbb{R})$ by

$$A+B=(u_{A+B},v_{A+B}), \tag{16}$$

where

$$u_{A+B}=u_A+u_B \tag{17}$$

and $v_{A+B}$ is given by

$$1-v_{A+B}=(1-v_A)+(1-v_B). \tag{18}$$

We define the scalar multiplication $\lambda \cdot A \in IF(\mathbb{R})$ of $A=(u_A,v_A) \in IF(\mathbb{R}), \lambda \in \mathbb{R}$ by

$$\lambda \cdot A=(u_{\lambda \cdot A},v_{\lambda \cdot A}), \tag{19}$$

where

$$u_{\lambda \cdot A}=\lambda \cdot u_A \tag{20}$$

and $v_{\lambda \cdot A}$ is given by

$$1-v_{\lambda \cdot A}=\lambda \cdot (1-v_A). \tag{21}$$

In the intuitionistic fuzzy case the distance $d$ (2) becomes (see [4])

$$\begin{aligned}\widetilde{d}^2(A,B)&=\frac{1}{2}\int_0^1 \left(u_A^-(\alpha)-u_B^-(\alpha)\right)^2 d\alpha+\frac{1}{2}\int_0^1 \left(u_A^+(\alpha)-u_B^+(\alpha)\right)^2 d\alpha\\&+\frac{1}{2}\int_0^1 \left(v_A^-(\alpha)-v_B^-(\alpha)\right)^2 d\alpha+\frac{1}{2}\int_0^1 \left(v_A^+(\alpha)-v_B^+(\alpha)\right)^2 d\alpha.\end{aligned}$$

where $A=(u_A,v_A), B=(u_B,v_B) \in IF(\mathbb{R})$.

It is immediate that

$$\widetilde{d}^2(A,B)=\frac{1}{2}d^2(u_A,u_B)+\frac{1}{2}d^2(1-v_A,1-v_B).$$

Any parameter (real number or real interval) associated with a fuzzy number can be extended in a natural way to an intuitionistic fuzzy number $A=(u_A,v_A)$ as the arithmetic mean of the same parameter applied for $u_A$ and $1-v_A$. Taking into account (14)-(15) the parameters introduced in (5)-(12) become

$$\begin{aligned}\widetilde{EI}(A)=\Bigg[&\frac{1}{2}\int_0^1 \left(u_A^-(\alpha)+v_A^-(\alpha)\right)d\alpha,\\&\frac{1}{2}\int_0^1 \left(u_A^+(\alpha)+v_A^+(\alpha)\right)d\alpha\Bigg]\end{aligned} \tag{22}$$

$$\widetilde{EV}(A)=\frac{1}{4}\int_0^1 \left(u_A^-(\alpha)+v_A^-(\alpha)+u_A^+(\alpha)+v_A^+(\alpha)\right)d\alpha \tag{23}$$

$$\widetilde{w}(A)=\frac{1}{2}\int_0^1 \left(u_A^+(\alpha)+v_A^+(\alpha)-u_A^-(\alpha)-v_A^-(\alpha)\right)d\alpha \tag{24}$$

$$\widetilde{Val}\,(A) = \frac{1}{2}\int_0^1 \alpha u_A^+(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 \alpha u_A^-(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)\,v_A^+(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)\,v_A^-(\alpha)\,d\alpha \quad (25)$$

$$\widetilde{Amb}\,(A) = \frac{1}{2}\int_0^1 \alpha u_A^+(\alpha)\,d\alpha - \frac{1}{2}\int_0^1 \alpha u_A^-(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)\,v_A^+(\alpha)\,d\alpha - \frac{1}{2}\int_0^1 (1-\alpha)\,v_A^-(\alpha)\,d\alpha \quad (26)$$

$$\widetilde{Amb}_L\,(A) = \int_0^1 \alpha\left(\widetilde{EV}\,(A) - \frac{1}{2}u_A^-(\alpha) - \frac{1}{2}\,(1-v)_A^-\,(\alpha)\right)d\alpha \quad (27)$$

$$\widetilde{Amb}_U\,(A) = \int_0^1 \alpha\left(\frac{1}{2}u_A^+(\alpha) + \frac{1}{2}\,(1-v)_A^+\,(\alpha) - \widetilde{EV}\,(A)\right)d\alpha \quad (28)$$

$$\widetilde{core}\,(A) = \left[\frac{u_A^-(1) + v_A^-(0)}{2}, \frac{u_A^+(1) + v_A^+(0)}{2}\right], \quad (29)$$

where $A = (u_A, v_A) \in IF(\mathbb{R})$.

An intuitionistic fuzzy number $\widetilde{T} = \left(u_{\widetilde{T}}, v_{\widetilde{T}}\right)$, where $u_{\widetilde{T}} = (t_1, t_2, t_3, t_4)$ and $1 - v_{\widetilde{T}} = (s_1, s_2, s_3, s_4)$ is called a trapezoidal fuzzy number. Due to (13), $\widetilde{T} = \left(u_{\widetilde{T}}, v_{\widetilde{T}}\right)$ is a trapezoidal intuitionistic fuzzy number if and only if $s_1 \leq t_1, s_2 \leq t_2, s_3 \geq t_3, s_4 \geq t_4$.

# 3 Approximations of intuitionistic fuzzy numbers

The following result is an immediate consequence of Theorem 1 in [9].

**Proposition 3** *If $A = (u_A, v_A)$ is an intuitionistic fuzzy number, then*

$$N(A) = \frac{1}{2}\cdot u_A + \frac{1}{2}\cdot(1 - v_A)$$

*is the nearest fuzzy number to $A$ with respect to the distance $\widetilde{d}$ and $N(A)$ is unique with this property.*

If $A = (u_A, v_A)$ is a trapezoidal intuitionistic fuzzy number, that is $u_A = (t_1, t_2, t_3, t_4)$ and $1 - v_A = (s_1, s_2, s_3, s_4)$ then

$$N\,(A) = \left(\frac{t_1 + s_1}{2}, \frac{t_2 + s_2}{2}, \frac{t_3 + s_3}{2}, \frac{t_4 + s_4}{2}\right)$$

is the nearest trapezoidal fuzzy number to $A$. Let us remark here that the fuzzy number $N(A)$ preserves the expected interval, value, width, value, ambiguity, left-hand ambiguity, right-hand ambiguity and core of intuitionistic fuzzy number $A$. In addition, the operator

$N : IF(\mathbb{R}) \to F(\mathbb{R})$ is linear. Indeed, according with (16)-(18) we obtain

$$\begin{aligned} N(A+B) &= \frac{1}{2} \cdot u_{A+B} + \frac{1}{2} \cdot (1 - v_{A+B}) \\ &= \frac{1}{2} \cdot (u_A + u_B) + \frac{1}{2} \cdot ((1 - v_A) + (1 - v_B)) \\ &= \frac{1}{2} \cdot u_A + \frac{1}{2} \cdot (1 - v_A) + \frac{1}{2} \cdot u_B + \frac{1}{2} \cdot (1 - v_B) \\ &= N(A) + N(B) \end{aligned}$$

and taking into account (19)-(21) we have

$$\begin{aligned} N(\lambda \cdot A) &= \frac{1}{2} \cdot u_{\lambda \cdot A} + \frac{1}{2} \cdot (1 - v_{\lambda \cdot A}) \\ &= \frac{1}{2} \cdot \lambda \cdot u_A + \frac{1}{2} \cdot \lambda \cdot (1 - v_A) \\ &= \lambda \cdot \left( \frac{1}{2} \cdot u_A + \frac{1}{2} \cdot (1 - v_A) \right) \\ &= \lambda \cdot N(A), \end{aligned}$$

for every $A = (u_A, v_A), B = (u_B, v_B) \in IF(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

Let us denote by $t_c(u)$ the nearest trapezoidal fuzzy number to $u \in F(\mathbb{R})$ (with respect to $d$), preserving the core of $u$ (see [1]). The below result is a consequence of Theorem 1 in [9] too.

**Proposition 4** *If $A = (u_A, v_A)$ is an intuitionistic fuzzy number, then*

$$T_c(A) = t_c(\frac{1}{2} \cdot u_A + \frac{1}{2} \cdot (1 - v_A)) \tag{30}$$

*is the nearest trapezoidal fuzzy number to $A$ (with respect to the distance $\widetilde{d}$), preserving the core of $A$, and $T_c(A)$ is unique with this property.*

The transfer of properties, including here scale invariance and additivity, from approximation operators over fuzzy numbers to approximation operators over intuitionistic fuzzy numbers is discussed in [9]. As a direct consequence of Theorem 10 in [9], the linearity of the operator $t_c : F(\mathbb{R}) \to F^T(\mathbb{R})$ implies the linearity of operator $T_c : IF(\mathbb{R}) \to F^T(\mathbb{R})$ given in (30).

In [4] the nearest interval (with respect to the distance $\widetilde{d}$) to an intuitionistic fuzzy number was computed as follows:

**Proposition 5** *If $A = (u_A, v_A)$ is an intuitionistic fuzzy number, $(u_A)_\alpha = \left[u_A^-(\alpha), u_A^+(\alpha)\right], (v_A)_\alpha = \left[v_A^-(\alpha), v_A^+(\alpha)\right], \alpha \in [0,1]$, then*

$$I(A) = \left[ \frac{1}{2} \int_0^1 \left(u_A^-(\alpha) + v_A^-(\alpha)\right) d\alpha, \frac{1}{2} \int_0^1 \left(u_A^+(\alpha) + v_A^+(\alpha)\right) d\alpha \right] \tag{31}$$

*is the nearest interval to $A$ with respect to the distance $\widetilde{d}$ and $I(A)$ is unique with this property.*

The linearity of the above operator $I : IF(\mathbb{R}) \to Int(\mathbb{R})$, where

$$Int(\mathbb{R}) = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$$

is immediate.

# 4 Main result

Let us consider the following system

$$\begin{cases} a_{11} \cdot x_1 + ... + a_{1n} \cdot x_n = b_1 \\ ... \\ a_{m1} \cdot x_1 + ... + a_{mn} \cdot x_n = b_m, \end{cases} \tag{32}$$

that is $A \cdot X = B$, where the elements of the coefficients matrix $A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$ are real values, the elements of the vector $B = (b_i), 1 \leq i \leq m$ and our aim is to solve (32) over $IF(\mathbb{R})$.

Let $T$ be an approximation operator on intuitionistic fuzzy numbers, which preserves the parameters $p_k, k \in \{1, ..., r\}$, that is $p_k(T(A)) = p_k(A)$, for every $A \in IF(\mathbb{R})$ and $k \in \{1, ..., r\}$.

**Theorem 6** *If $T$ is linear then the approximation of the solution of (32) with respect to $T$ is the solution of the system $A \cdot X = B^T$, where $B^T = (b_i^T), b_i^T = T(b_i), 1 \leq i \leq m$.*

**Proof.** Let us assume $(x_1^0, ..., x_n^0)$ is a solution of (32) and $(T(x_1^0), ..., T(x_n^0))$ is the approximation of its components with respect to $T$. Taking into account the linearity of $T$ we get

$$\begin{cases} a_{11} \cdot T(x_1^0) + ... + a_{1n} \cdot T(x_n^0) = T(b_1) \\ ... \\ a_{m1} \cdot T(x_1^0) + ... + a_{mn} \cdot T(x_n^0) = T(b_m), \end{cases}$$

that is $(T(x_1^0), ..., T(x_n^0))$ is the solution of the system

$$\begin{cases} a_{11} \cdot x_1 + ... + a_{1n} \cdot x_n = T(b_1) \\ ... \\ a_{m1} \cdot x_1 + ... + a_{mn} \cdot x_n = T(b_m). \end{cases}$$

■

# 5 Examples

There exist many triangular or trapezoidal approximation operators on fuzzy numbers which are not additive (see [5]-[8], [10], [17]-[19]). As an immediate consequence, the corresponding operators on intuitionistic fuzzy numbers are not additive too. Nevertheless, Theorem 6 has a fairly wide range of applicability, as the following examples prove.

**Example 7** *Let $N$ be the approximation operator of intuitionistic fuzzy numbers by fuzzy numbers, that is $N(A)$ is the nearest (with respect to $\widetilde{d}$) fuzzy number to intuitionistic fuzzy number $A$. We consider the following intuitionistic fuzzy linear system*

$$\begin{cases} 4 \cdot x_1 + x_2 = b_1 \\ x_1 + 3 \cdot x_2 = b_2, \end{cases} \tag{33}$$

*where* $u_{b_1} = (-4, 1, 1, 4), 1-v_{b_1} = (-6, -1, 3, 6), u_{b_2} = (3, 10, 10, 11), 1-v_{b_2} = (1, 5, 12, 14)$. *If $A = (u_A, v_A)$ then $N(A) = \frac{1}{2} \cdot u_A + \frac{1}{2} \cdot (1 - v_A)$ (see Proposition 3), therefore*

$$N(b_1) = (-5, 0, 2, 5),$$
$$N(b_2) = \left(2, \frac{15}{2}, 11, \frac{25}{2}\right).$$

*We obtain the system*

$$\begin{cases} 4 \cdot x_1 + x_2 = (-5, 0, 2, 5) \\ x_1 + 3 \cdot x_2 = \left(2, \frac{15}{2}, 11, \frac{25}{2}\right), \end{cases}$$

*with the solution*

$$x_1 = \left(-\frac{17}{11}, -\frac{15}{22}, -\frac{5}{11}, \frac{5}{22}\right),$$
$$x_2 = \left(\frac{13}{11}, \frac{30}{11}, \frac{42}{11}, \frac{45}{11}\right), \tag{34}$$

*which (according with Theorem 6) is the approximation of the solution of (33). It is important that the characteristics (expected interval, value, width, value, ambiguity, left-hand ambiguity, right-hand ambiguity, core) of the approximative solution (34) are identical with the characteristics of the exact solution of (33).*

**Example 8** *Let $T_c : IF(\mathbb{R}) \to F^T(\mathbb{R})$ be the trapezoidal approximation operator of intuitionistic fuzzy numbers by trapezoidal fuzzy numbers which preserves the core, that is $T_c(A)$ is the nearest (with respect to $\widetilde{d}$) trapezoidal fuzzy number to intuitionistic fuzzy number $A$ such that $core(T_c(A)) = \widetilde{core}(A)$. We consider the following intuitionistic fuzzy linear system*

$$\begin{cases} 4 \cdot x_1 + x_2 = b_1 \\ x_1 + 3 \cdot x_2 = b_2, \end{cases} \tag{35}$$

*where* $u_{b_1} = (-4, 1, 1, 4)_2, 1-v_{b_1} = (-6, -1, 3, 6)_2, u_{b_2} = (3, 10, 10, 11)_2, 1-v_{b_2} = (1, 5, 12, 14)_2$. *If $A = (u_A, v_A)$ then $T_c(A) = t_c\left(\frac{1}{2} \cdot u_A + \frac{1}{2} \cdot (1 - v_A)\right)$ (see Proposition 4), therefore (see (3.15) in [1])*

$$T_c(b_1) = t_c\left((-5, 0, 2, 5)_2\right) = \left(-\frac{7}{2}, 0, 2, \frac{41}{10}\right),$$
$$T_c(b_2) = t_c\left(\left(2, \frac{15}{2}, 11, \frac{25}{2}\right)_2\right) = \left(\frac{73}{20}, \frac{15}{2}, 11, \frac{241}{20}\right).$$

*We obtain the system*

$$\begin{cases} 4 \cdot x_1 + x_2 = \left(-\frac{7}{2}, 0, 2, \frac{41}{10}\right) \\ x_1 + 3 \cdot x_2 = \left(\frac{73}{20}, \frac{15}{2}, 11, \frac{241}{20}\right) \end{cases}$$

*with solution* $x_1 = \left(-\frac{283}{220}, -\frac{15}{22}, -\frac{5}{11}, \frac{1}{44}\right), x_2 = \left(\frac{181}{110}, \frac{30}{11}, \frac{42}{11}, \frac{441}{110}\right)$. *This solution is simple and, in addition, the core of the exact solution of (35) is equal to the core of* $x_1$ *and* $x_2$*. Indeed, the exact solution of (35) is given by*

$$x_1^0 = \left(u_{x_1^0}, v_{x_1^0}\right),$$
$$x_2^0 = \left(u_{x_2^0}, v_{x_2^0}\right),$$

*where*

$$u_{x_1^0} = \left(-\frac{15}{11}, -\frac{7}{11}, -\frac{7}{11}, \frac{1}{11}\right)_2,$$
$$1 - v_{x_1^0} = \left(-\frac{19}{11}, -\frac{8}{11}, -\frac{3}{11}, \frac{4}{11}\right)_2,$$
$$u_{x_2^0} = \left(\frac{16}{11}, \frac{39}{11}, \frac{39}{11}, \frac{40}{11}\right)_2,$$
$$1 - v_{x_2^0} = \left(\frac{10}{11}, \frac{21}{11}, \frac{45}{11}, \frac{50}{11}\right)_2$$

*and according with (29),*

$$\widetilde{core}\left(x_1^0\right) = \left[-\frac{15}{22}, -\frac{5}{11}\right],$$
$$\widetilde{core}\left(x_2^0\right) = \left[\frac{30}{11}, \frac{42}{11}\right]$$

*and* $\widetilde{core}\left(x_1^0\right) = core\left(x_1\right), \widetilde{core}\left(x_2^0\right) = core\left(x_2\right)$.

**Example 9** *Let* $I : IF(\mathbb{R}) \to Int(\mathbb{R})$ *the interval approximation operator of intuitionistic fuzzy numbers, that is* $I(A)$ *is the nearest (with respect to* $\widetilde{d}$*) real interval to intuitionistic fuzzy number* $A$*. We consider the system (35). Then (Proposition 5 is used here)*

$$I(b_1) = \left[-\frac{5}{3}, 3\right]$$

*and*

$$I(b_2) = \left[\frac{17}{3}, \frac{23}{2}\right].$$

*We obtain the system*

$$\begin{cases} 4 \cdot x_1 + x_2 = \left[-\frac{5}{3}, 3\right] \\ x_1 + 3 \cdot x_2 = \left[\frac{17}{3}, \frac{23}{2}\right] \end{cases}$$

*which has the solution* $x_1 = \left[-\frac{32}{33}, -\frac{5}{22}\right]$, $x_2 = \left[\frac{73}{33}, \frac{43}{11}\right]$. *According with (31) and taking into account the exact solution of (35) given in Example 8 we get*

$$I\left(x_1^0\right) = \left[-\frac{32}{33}, -\frac{5}{22}\right]$$

*and*

$$I\left(x_2^0\right) = \left[\frac{73}{33}, \frac{43}{11}\right]$$

*and Theorem 6 is confirmed.*

At the end of the present paper we give an example of intuitionistic fuzzy linear system without solution such that the approximate solution there exists. This aspect, together others, will be the subject of a future research.

**Example 10** *Let us consider the system*

$$\begin{cases} 2 \cdot x_1 + x_2 = b_1 \\ x_1 + 3 \cdot x_2 = b_2, \end{cases} \tag{36}$$

*where* $u_{b_1} = (-4, 1, 1, 4)$, $1 - v_{b_1} = (-6, -1, 3, 6)$, $u_{b_2} = (3, 9, 10, 11)$, $1 - v_{b_2} = (1, 5, 12, 27)$, *with the solution* $(x_1^0, x_2^0)$ *given by* $x_1^0 = \left(u_{x_1^0}, v_{x_1^0}\right)$, $x_2^0 = \left(u_{x_2^0}, v_{x_2^0}\right)$. *If* $u_{x_1^0} = (t_1, t_2, t_3, t_4)$ *and* $u_{x_2^0} = (t_1', t_2', t_3', t_4')$ *then*

$$t_3 = -\frac{7}{5} < -\frac{6}{5} = t_2,$$

*a contradiction with the quality of* $u_{x_1^0}$ *to be a trapezoidal fuzzy number . Nevertheless, (see Proposition 3)*

$$N(b_1) = (-5, 0, 2, 5),$$
$$N(b_2) = (2, 7, 11, 19)$$

*and the system*

$$\begin{cases} 2 \cdot x_1 + x_2 = (-5, 0, 2, 5) \\ x_1 + 3 \cdot x_2 = (2, 7, 11, 19), \end{cases}$$

*has the solution*

$$x_1 = \left(-\frac{17}{5}, -\frac{7}{5}, -1, -\frac{4}{5}\right),$$
$$x_2 = \left(\frac{9}{5}, \frac{14}{5}, 4, \frac{33}{5}\right).$$

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