Fuzzy Sets and Systems 33 (1989) 37–45
North-Holland

MORE ON INTUITIONISTIC FUZZY SETS

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Received March 1987
Revised March 1988

Abstract: New results on intuitionistic fuzzy sets are introduced. Two news operators on intuitionistic fuzzy sets are defined and their basic properties are studied.

Keywords: Intuitionistic fuzzy sets; level; modal operator – necessity and possibility; topological operator – interior and closure.

We shall introduce new results on intuitionistic fuzzy sets (IFSs), which are a continuation of the results in [1]. All notations are from [1].

Let a set E be fixed. An IFS A^* in E is an object having the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A(x): E \to [0, 1]$ and $\nu_A(x): E \to [0, 1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set A, which is a subset of E (for simplicity below we shall write A instead of A^*), respectively, and for every $x \in E$:

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1.$$

Obviously, every fuzzy set has the form

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}.$$

For every two IFSs A and B the following relations, operations and operators are valid (see [1]):

$$A \subset B \text{ iff } (\forall x \in E)(\mu_A(x) \leqslant \mu_B(x) \;\&\; \nu_A(x) \geqslant \nu_B(x));$$

$$A = B \text{ iff } A \subset B \;\&\; B \subset A;$$

$$\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\};$$

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\};$$

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\};$$

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\};$$

$$A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\};$$

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\};$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\};$$

$$CA = \{\langle x, K, L\rangle \mid x \in E\}, \quad \text{where } K = \max_{x\in E} \mu_A(x),\ L = \min_{x\in E} \nu_A(A);$$

$$IA = \{\langle x, k, l\rangle \mid x \in E\}, \quad \text{where } k = \min_{x\in E} \mu_A(x),\ l = \max_{x\in E} \nu_A(A).$$

Theorem 10 of [1] can be formulated in a more strict form (given in [2]):

Theorem 1. *For every* IFS A:

(a) $\Box C\Box A = \Diamond C\Box A = \overline{\Box I\Diamond \bar{A}} = \overline{\Diamond I\Diamond \bar{A}} = \{\langle x, K, 1-K\rangle \mid x \in E\}$,

(b) $\Box C\Diamond A = \Diamond C\Diamond A = \overline{\Box I\Box \bar{A}} = \overline{\Diamond I\Box \bar{A}} = \{\langle x, 1-L, L\rangle \mid x \in E\}$,

(c) $\Box I\Box A = \Diamond I\Box A = \overline{\Box C\Diamond \bar{A}} = \overline{\Diamond C\Diamond \bar{A}} = \{\langle x, k, 1-k\rangle \mid x \in E\}$,

(d) $\Box I\Diamond A = \Diamond I\Diamond A = \overline{\Box C\Box \bar{A}} = \overline{\Diamond C\Box \bar{A}} = \{\langle x, 1-l, l\rangle \mid x \in E\}$,

(e) $\Box C\Box A = \Diamond C\Box A = \overline{\Box I\Diamond A} = \overline{\Diamond I\Diamond A} = \{\langle x, l, 1-l\rangle \mid x \in E\}$;

(f) $\Box C\Diamond A = \Diamond C\Diamond A = \overline{\Box I\Box A} = \overline{\Diamond I\Box A} = \{\langle x, 1-k, k\rangle \mid x \in E\}$,

(g) $\Box I\Box A = \Diamond I\Box A = \overline{\Box C\Diamond A} = \overline{\Diamond C\Diamond A} = \{\langle x, L, 1-L\rangle \mid x \in E\}$,

(h) $\Box I\Diamond \bar{A} = \Diamond I\Diamond \bar{A} = \overline{\Box C\Box A} = \overline{\Diamond C\Box A} = \{\langle x, 1-K, K\rangle \mid x \in E\}$.

Proof. For (a),

$$\begin{aligned}
\Box C\Box A &= \Box C\Box\{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\} \\
&= \Box C\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in E\} \\
&= \Box\{\langle x, K, \min_{x\in E}(1-\mu_A(x))\rangle \mid x \in E\} \\
&= \langle\{x, K, 1-K\rangle \mid x \in E\};
\end{aligned}$$

$$\begin{aligned}
\Diamond C\Box A &= \Diamond\{\langle x, K, \min_{x\in E}(1-\mu_A(x))\rangle \mid x \in E\} \\
&= \{\langle x, 1-\min_{x\in E}(1-\mu_A(x)), \min_{x\in E}(1-\mu_A(x))\rangle \mid x \in E\} \\
&= \{\langle x, \max_{x\in E}\mu_A(x), 1-\max_{x\in E}\mu_A(x)\rangle \mid x \in E\} \\
&= \{\langle x, K, 1-K\rangle \mid x \in E\};
\end{aligned}$$

$$\begin{aligned}
\overline{\Box I\Diamond \bar{A}} &= \overline{\Box I\Diamond\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\}} \\
&= \overline{\Box I\{\langle x, 1-\mu_A(x), \mu_A(x)\rangle \mid x \in E\}} \\
&= \overline{\Box\{\langle x, \min_{x\in E}(1-\mu_A(x)), \max_{x\in E}\mu_A(x)\rangle \mid x \in E\}} \\
&= \overline{\{\langle x, 1-K, K\rangle \mid x \in E\}} \\
&= \{\langle x, K, 1-K\rangle \mid x \in E\};
\end{aligned}$$

$$\overline{\Diamond I \Diamond \bar{A}} = \Diamond\{\langle x, \min_{x\in E}(1-\mu_A(x)), \max_{x \in E} \mu_A(x)\rangle \mid x \in E\}$$
$$= \overline{\{\langle x, 1-K, K\rangle \mid x \in E\}}$$
$$= \{\langle x, K, 1-K\rangle \mid x \in E\}.$$

(b)–(d) are proved analogically.
For (e),

$$\Box C \Box \bar{A} = \overline{\overline{\Box C \Box A}}$$
$$= \overline{\{\langle x, 1-l, l\rangle \mid x \in E\}}$$
$$= \{\langle x, l, 1-l\rangle \mid x \in E\},$$

and so forth.
(f)–(h) are proved analogically.

Let for a fixed IFA A:

$$S(A) = \{\Box C \Box A, \Diamond C \Box A, \overline{\Box I \Diamond \bar{A}}, \overline{\Diamond I \Diamond \bar{A}}\},$$
$$T(A) = \{\Box C \Diamond A, \Diamond C \Diamond A, \overline{\Box I \Box \bar{A}}, \overline{\Diamond I \Box \bar{A}}\},$$
$$U(A) = \{\Box I \Box A, \Diamond I \Box A, \overline{\Box I \Diamond \bar{A}}, \overline{\Diamond I \Diamond \bar{A}}\},$$
$$V(A) = \{\Box I \Diamond A, \Diamond I \Diamond A, \overline{\Box C \Box \bar{A}}, \overline{\Diamond C \Box \bar{A}}\},$$
$$W(A) = \{\Box C \Box \bar{A}, \Diamond C \Box \bar{A}, \overline{\Box I \Diamond A}, \overline{\Diamond I \Diamond A}\},$$
$$X(A) = \{\Box C \Diamond \bar{A}, \Diamond C \Diamond \bar{A}, \overline{\Box I \Box A}, \overline{\Diamond I \Box A}\},$$
$$Y(A) = \{\Box I \Box \bar{A}, \Diamond I \Box \bar{A}, \overline{\Box C \Diamond A}, \overline{\Diamond C \Diamond A}\},$$
$$Z(A) = \{\Box I \Diamond \bar{A}, \Diamond I \Diamond \bar{A}, \overline{\Box C \Box A}, \overline{\Diamond C \Box A}\}.$$

Theorem 2 (cf. Theorem 10 of [1]). *For every two* IFSs *P and Q*:
(a) *if $P \in S(A)$ and $Q \in T(A)$, then $P \subset CA \subset Q$*;
(b) *if $P \in U(A)$ and $Q \in V(A)$, then $P \subset IA \subset Q$*;
(c) *if $P \in W(A)$ and $Q \in X(A)$, then $P \subset \overline{IA} \subset Q$*;
(d) *if $P \in Y(A)$ and $Q \in Z(A)$, then $P \subset \overline{CA} \subset Q$*.

Proof. (a) Let $P \in S(A)$ and $Q \in T(A)$. Then

$$P = \{\langle x, K, 1-K\rangle \mid x \in E\} \subset CA \subset \{\langle x, 1-L, L\rangle \mid x \in E\} = Q.$$

(b)–(d) are proved analogically.

Following the idea of a fuzzy set from α-level (e.g. [5]), in [3, 4] the definition of a set from (α, β)-level, generated by the IFS A, where $\alpha, \beta \in [0, 1]$ are fixed numbers for which $\alpha + \beta \leq 1$, is introduced. Formally this set has the form:

$$N_{\alpha,\beta}(A) = \{\langle x, \mu_A(A), \nu_A(x)\rangle \mid x \in E \;\&\; \mu_A(x) \geq \alpha \;\&\; \nu_A(x) \leq \beta\}.$$

From the above definition directly follows the validity of:

Theorem 3. *For every IFS A and for every $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:*
(a) $N_{\alpha,\beta}(A)$ *is an IFS;*
(b) $N_{\alpha,\beta}(A) \subset A$, *where the relation $\subset$ is a relation in the set-theory sense.*

In [3, 4] it is proved that the class $E_{\alpha,\beta}$ of all IFSs from (α, β)-level is a filter (in the sense of [6]). Here we shall introduce two new notations, related to above mentioned one ($\alpha \in [0, 1]$ is a fixed number):
(a) we call the set

$$N_\alpha(A) = \{\langle x, \mu_A(A), \nu_A(x)\rangle \mid x \in E \,\&\, \mu_A(x) \geq \alpha\}.$$

a set of level of membership α, generated by A;
(b) we call the set

$$N^\alpha(A) = \{\langle x, \mu_A(A), \nu_A(x)\rangle \mid x \in E \,\&\, \nu_A(x) \leq \alpha\}.$$

a set of level of nonmembership α, generated by A.
From these definitions follows directly:

Theorem 4. $N_\alpha(A)$ *and* $N^\alpha(A)$ *are* IFSs *for every* IFS A *and for every* $\alpha \in [0, 1]$.

Theorem 5. *For every IFS A and for every $\alpha, \beta \in [0, 1]$:*

$$N_{\alpha,\beta}(A) = N_\alpha(A) \cap N^\beta(A).$$

Let

$$E_1 = \{N_{\alpha,\beta}(A) \mid A \subset E \,\&\, \alpha, \beta \in [0, 1] \,\&\, \alpha + \beta \leq 1\},$$
$$E_2 = \{N_\alpha(A) \mid A \subset E \,\&\, \alpha \in [0, 1]\},$$
$$E_3 = \{N^\alpha(A) \mid A \subset E \,\&\, \alpha \in [0, 1]\}.$$

Theorem 6. *The sets E_1, E_2 and E_3 are filters relating to the operation $\cap$ and relation $\subset$ (in the sense of* [6]).

Proof. For E_1 we shall check the validity of the following assertions:
(1) if $B \in E_1$ and $B \subset C$, then $C \in E_1$,
(2) if $B, C \in E_1$, then $B \cap C \in E_1$.
The first assertion is valid because from $B \in E_1$ and $B \subset C$ follows

$$(\forall x \in E)(\alpha \leq \mu_B(x) \leq \mu_C(x) \,\&\, \nu_C(x) \leq \nu_B(x) \leq \beta),$$

i.e. $C \in E_1$. From the inequality

$$(\forall x \in E)(\mu_B(x) \geq \alpha \,\&\, \mu_C(x) \geq \alpha \,\&\, \nu_B(x) \leq \beta \,\&\, \nu_C(x) \leq \beta)$$

follows

$$(\forall x \in E)(\min(\mu_B(x), \mu_C(x)) \geq \alpha \,\&\, \max(\nu_B(x), \nu_C(x)) \leq \beta),$$

i.e. $B \cap C \in E_1$. Therefore E_1 is a filter.
For E_2 and E_3 the assertions are proved analogically.

Theorem 7. $E_1 = E_2 \cap E_3$.

Proof. Let $B \in E_1$. Then there is a set $A \subset E$ and there are $\alpha, \beta \in [0, 1]$ for which $B = N_{\alpha,\beta}(A)$. From Theorem 5 it follows that

$$B = N_\alpha(A) \cap N^\beta(A),$$

i.e. $B \subset N_\alpha(A)$ and $B \subset N^\beta(A)$. Hence for every $x \in E$, $\mu_B(x) \geqslant \alpha$, i.e. $B \in E_2$ and analogically $B \in E_3$, i.e. $B \in E_2 \cap E_3$.

On the other hand, if $B \in E_2 \cap E_3$, then it can likewise be established that $B \in E_1$.

Let $\alpha \in [0, 1]$ be a fixed number. For the IFS A we shall define the operator D_α through

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + (1 - \alpha) \cdot \pi_A(x)\rangle \mid x \in E\},$$

where (see [1])

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

From this definition it follows that $D_\alpha(A)$ is a fuzzy set, because

$$\mu_A(x) + \alpha \cdot \pi_A(x) + \nu_A(x) + (1 - \alpha) \cdot \pi_A(x) = \mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

Here we shall give the basic properties of this operator.

Theorem 8. *For every* IFS A *and for every* $\alpha, \beta \in [0, 1]$:

(a) *if* $\alpha \leqslant \beta$, *then* $D_\alpha(A) \subset D_\beta(A)$;

(b) $D_\alpha(D_\beta(A)) = D_\beta(A)$.

Proof. (a) follows from the above definition.

For (b),

$$\begin{aligned} D_\alpha(D_\beta(A)) &= D_\alpha(\{\langle x, \mu_A(x) + \beta \cdot \pi_A(x), \nu_A(x) + (1 - \beta) \cdot \pi_A(x)\rangle \mid x \in E\}) \\ &= \{\langle x, \mu_A(x) + \beta \cdot \pi_A(x) + \alpha \cdot (1 - \mu_A(x) - \beta \cdot \pi_A(x) \\ &\qquad - \nu_A(x) - (1 - \beta) \cdot \pi_A(x)), \\ &\qquad \nu_A(x) + (1 - \beta) \cdot \pi_A(x) + (1 - \alpha) \cdot (1 - \mu_A(x) \\ &\qquad - \beta \cdot \pi_A(x) - \nu_A(x) - (1 - \beta) \cdot \pi_A(x))\rangle \mid x \in E\} \\ &= \{\langle x, \mu_A(x) + \beta \cdot \pi_A(x), \nu_A(x) + (1 - \beta) \cdot \pi_A(x)\rangle \mid x \in E\} \\ &= D_\beta(A). \end{aligned}$$

Theorem 9. *For every* IFS A *and for every* $\alpha \in [0, 1]$:

(a) $D_0(A) = \Box A$;

(b) $D_1(A) = \Diamond A$;

(c) $\overline{D_\alpha(\bar{A})} = D_{1-\alpha}(A)$.

Proof. For (a),

$$\begin{aligned} D_0(A) &= \{\langle x, \mu_A(x) + 0 \cdot \pi_A(x), \nu_A(x) + (1 - 0) \cdot \pi_A(x)\rangle \mid x \in E\} \\ &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\} \\ &= \Box A. \end{aligned}$$

(b) is proved analogically.

For (c),

$$\overline{D_\alpha(\overline{A})} = \overline{D_\alpha(\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\})}$$
$$= \overline{\{\langle x, \nu_A(x) + \alpha \cdot \pi_A(x), \mu_A(x) + (1-\alpha)\cdot \pi_A(x)\rangle \mid x \in E\}}$$
$$= \{\langle x, \mu_A(x) + (1-\alpha)\cdot \pi_A(x), \nu_A(x) + \alpha \cdot \pi_A(x)\rangle \mid x \in E\}$$
$$= D_{1-\alpha}(A).$$

Theorem 10. *For every* IFS A *and for every* $\alpha \in [0, 1]$:

(a) $\square D_\alpha(A) = D_\alpha(A)$,
(b) $D_\alpha(\square A) = \square A$,
(c) $\diamond D_\alpha(A) = D_\alpha(A)$,
(d) $D_\alpha(\diamond A) = \diamond A$,
(e) $C(D_\alpha(A)) \subset D_\alpha(CA)$,
(f) $I(D_\alpha(A)) \supset D_\alpha(IA)$.

Proof. The validity of (a)–(d) follows from the definition of D_α and from Theorem 8(b) and Theorem 9(a), (b).

For (e),

$$CD_\alpha(A) = C\{\langle x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + (1-\alpha)\cdot \pi_A(x)\rangle \mid x \in E\}$$
$$= \{\langle x, K_1, L_1\rangle \mid x \in E\},$$

where

$$K_1 = \max_{x\in E}(\mu_A(x) + \alpha \cdot \pi_A(x)), \qquad L_1 = \min_{x \in E}(\nu_A(x) + (1-\alpha)\cdot \pi_A(x)),$$

and

$$D_\alpha(CA) = D_\alpha(\{\langle x, K, L\rangle \mid x \in E\})$$
$$= \{\langle x, K + \alpha\cdot(1-K-L), L + (1-\alpha)\cdot(1-K-L)\rangle \mid x \in E\},$$

where K and L are as above. From

$$K + \alpha\cdot(1-K-L) - K_1 = \max_{x\in E}\mu_A(x) + \alpha \cdot \left(1 - \max_{x\in E}\mu_A(x) - \min_{x\in E}\nu_A(x)\right)$$
$$- \max_{x\in E}(\mu_A(x) + \alpha\cdot(1-\mu_A(x) - \nu_A(x)))$$
$$\geqslant \max_{x\in E}\mu_A(x) - \alpha\cdot\max_{x\in E}\mu_A(x) - \alpha\cdot\min_{x\in E}\nu_A(x)$$
$$-(1-\alpha)\cdot\max_{x\in E}\mu_A(x) + \alpha\cdot\min_{x\in E}\nu_A(x)$$
$$= 0$$

it follows that $CD_\alpha(A) \subset D_\alpha(CA)$.

(f) is proved analogically.

From the properties of the operator D_α it is seen that it is an extension of the operators $\Box$ and $\Diamond$. But it can be further extended too.

Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. We define the operator $F_{\alpha,\beta}$ for the IFS A through

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + \beta \cdot \nu_A(x)\rangle \mid x \in E\}.$$

Theorem 11. *For every* IFS A *and for every* $\alpha, \beta \in [0, 1]$ *such that* $0 \leq \alpha + \beta \leq 1$:

(a) $F_{\alpha,\beta}(A)$ *is an* IFS;
(b) *if* $0 \leq \gamma \leq \alpha$, *then* $F_{\gamma,\beta}(A) \subset F_{\alpha,\beta}(A)$;
(c) *if* $0 \leq \gamma \leq \beta$, *then* $F_{\alpha,\beta}(A) \subset F_{\alpha,\gamma}(A)$;
(d) $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$;
(e) $\Box A = F_{0,1}(A)$;
(f) $\Diamond A = F_{1,0}(A)$;
(g) $\overline{F_{\alpha,\beta}(\bar{A})} = F_{\beta,\alpha}(A)$.

Theorem 12. *For every* IFS A *and for every* $\alpha, \beta \in [0, 1]$ *such that* $0 \leq \alpha + \beta \leq 1$:

(a) $CF_{\alpha,\beta}(A) \subset F_{\alpha,\beta}(CA)$,
(b) $IF_{\alpha,\beta}(A) \supset F_{\alpha,\beta}(IA)$.

These assertions are proved as respective above.

Theorem 13. *For every two* IFSs A *and* B *and for every* $\alpha, \beta \in [0, 1]$ *such that* $0 \leq \alpha + \beta \leq 1$:

(a) $F_{\alpha,\beta}(A \cap B) \subset F_{\alpha,\beta}(A) \cap F_{\alpha,\beta}(B)$;
(b) $F_{\alpha,\beta}(A \cup B) \supset F_{\alpha,\beta}(A) \cup F_{\alpha,\beta}(B)$;
(c) $F_{\alpha,\beta}(A + B) \subset F_{\alpha,\beta}(A) + F_{\alpha,\beta}(B)$;
(d) $F_{\alpha,\beta}(A \cdot B) \supset F_{\alpha,\beta}(A) \cdot F_{\alpha,\beta}(B)$;

Proof. For (a),

$$\begin{aligned}
&F_{\alpha,\beta}(A \cap B)\\
&= \{\langle x, \min(\mu_A(x), \mu_B(x)) + \alpha \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x))),\\
&\quad \max(\nu_A(x), \nu_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x)))\rangle \mid x \in E\},\\
&F_{\alpha,\beta}(A) \cap F_{\alpha,\beta}(B)\\
&= \{\langle x, \min(\mu_A(x) + \alpha \cdot (1 - \mu_A(x) - \nu_A(x)), \mu_B(x) + \alpha \cdot (1 - \mu_B(x) - \nu_B(x))),\\
&\quad \max(\nu_A(x) + \beta \cdot (1 - \mu_A(x) - \nu_A(x)), \nu_B(x) + \beta \cdot (1 - \mu_B(x) - \nu_B(x)))\rangle \mid x \in E\}.
\end{aligned}$$

From

$$\begin{aligned}
&\min(\mu_A(x), \mu_B(x)) + \alpha \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x)))\\
&\quad = (1 - \alpha) \cdot \min(\mu_A(x), \mu_B(x)) + \alpha \cdot \min(1 - \nu_A(x), 1 - \nu_B(x))\\
&\quad \leq \min((1 - \alpha) \cdot \mu_A(x) + \alpha \cdot (1 - \nu_A(x)), (1 - \alpha) \cdot \mu_B(x) + \alpha \cdot (1 - \nu_B(x)))\\
&\quad = \min(\mu_A(x) + \alpha \cdot (1 - \mu_A(x) - \nu_A(x)), \mu_B(x) + \alpha \cdot (1 - \mu_B(x) - \nu_B(x)))
\end{aligned}$$

and from

$$\begin{aligned}
&\max(\nu_A(x), \nu_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x)))\\
&\quad = (1-\beta) \cdot \max(\nu_A(x), \nu_B(x)) + \beta \cdot \max(1 - \mu_A(x), 1 - \mu_B(x))\\
&\quad \geqslant \max((1-b) \cdot \nu_A(x) + \beta \cdot (1 - \mu_A(x)), (1-b) \cdot \nu_B(x) + \beta \cdot (1 - \mu_B(x)))\\
&\quad = \max(\nu_A(x) + \beta \cdot (1 - \mu_A(x) - \nu_A(x)), \nu_B(x) + \beta \cdot (1 - \mu_B(x) - \nu_B(x)))
\end{aligned}$$

the validity of (a) follows.

(b)–(d) are proved analogically.

This result can also be obtained concerning the D_α operator.

Theorem 14. *For every IFS A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$:*

(a) *if $\beta + \gamma \leqslant 1$, then $D_\alpha(F_{\beta,\gamma}(A)) = D_{\alpha+\beta-\alpha\cdot\beta-\alpha\cdot\gamma}(A)$,*

(b) *if $\alpha + \beta \leqslant 1$, then $F_{\alpha,\beta}(D_\gamma(A)) = D_\gamma(A)$,*

(c) *if $\alpha + \beta \leqslant 1$ and $\gamma + \delta \leqslant 1$, then*

$$F_{\alpha,\beta}(F_{\gamma,\delta}(A)) = F_{\alpha+\gamma-\alpha\cdot\gamma-\alpha\cdot\delta,\beta+\delta-\beta\cdot\gamma-\beta\cdot\delta}(A).$$

Proof. (a) For $\alpha, \beta, \gamma \in [0, 1]$, let $\beta + \gamma \leqslant 1$. Then

$$\begin{aligned}
D_\alpha(F_{\beta,\gamma}(A)) &= D_\alpha(\{\langle x, \mu_A(x) + \beta \cdot \pi_A(x), \nu_A(x) + \gamma \cdot \nu_A(x)\rangle \mid x \in E\})\\
&= \{\langle x, \mu_A(x) + \beta \cdot \pi_A(x)\\
&\qquad + \alpha \cdot (1 - \mu_A(x) - \beta \cdot \pi_A(x) - \nu_A(x) - \gamma \cdot \pi_A(x)),\\
&\qquad \nu_A(x) + \gamma \cdot \pi_A(x)\\
&\qquad + (1-\alpha) \cdot (1 - \mu_A(x) - \beta \cdot \pi_A(x) - \nu_A(x) - \gamma \cdot \pi_A(x))\rangle \mid x \in E\}\\
&= \{\langle x, \mu_A(x) + (\alpha + \beta - \alpha \cdot \beta - \alpha \cdot \gamma) \cdot \pi_A(x),\\
&\qquad \nu_A(x) + (1 - \alpha - \beta + \alpha \cdot \beta + \alpha \cdot \gamma) \cdot \pi_A(x)\rangle \mid x \in E\}\\
&= D_{\alpha+\beta-\alpha\cdot\beta-\alpha\cdot\gamma}(A).
\end{aligned}$$

(b) For $\alpha, \beta, \gamma \in [0, 1]$, let $\alpha + \beta \leqslant 1$. Then

$$\begin{aligned}
F_{\alpha,\beta}(D_\gamma(A)) &= F_{\alpha,\beta}(\{\langle x, \mu_A(x) + \gamma \cdot \pi_A(x), \nu_A(x) + (1-\gamma) \cdot \pi_A(x)\rangle \mid x \in E\})\\
&= \{\langle x, \mu_A(x) + \gamma \cdot \pi_A(x), \nu_A(x) + (1-\gamma) \cdot \pi_A(x)\rangle \mid x \in E\}\\
&= D_\gamma(x).
\end{aligned}$$

(c) For $\alpha, \beta, \gamma, \delta \in [0, 1]$, let $\alpha + \beta \leqslant 1$ and $r + \delta \leqslant 1$. Then

$$\begin{aligned}
F_{\alpha,\beta}(F_{\gamma,\delta}(A)) &= F_{\alpha,\beta}(\{\langle x, \mu_A(x) + \gamma \cdot \pi_A(x), \nu_A(x) + \delta \cdot \nu_A(x)\rangle \mid x \in E\})\\
&= \{\langle x, \mu_A(x) + \gamma \cdot \pi_A(x)\\
&\qquad + \alpha \cdot (1 - \mu_A(x) - \gamma \cdot \pi_A(x) - \nu_A(x) - \delta \cdot \pi_A(x)),\\
&\qquad \nu_A(x) + \delta \cdot \pi_A(x)\\
&\qquad + \beta \cdot (1 - \mu_A(x) - \gamma \cdot \pi_A(x) - \nu_A(x) - \delta \cdot \pi_A(x))\rangle \mid x \in E\}\\
&= \{\langle x, \mu_A(x) + (\alpha + \gamma - \alpha \cdot \gamma - \alpha \cdot \delta) \cdot \pi_A(x),\\
&\qquad \nu_A(x) + (\beta + \delta - \beta \cdot \gamma - \beta \cdot \delta) \cdot \pi_A(x)\rangle \mid x \in E\}\\
&= F_{\alpha+\gamma-\alpha\cdot\gamma-\alpha\cdot\delta,\beta+\delta-\beta\cdot\gamma-\beta\cdot\delta}(A).
\end{aligned}$$

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