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Research on intuitionistic fuzzy implications. Part 5

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Abstract: Continuing the research from four previous papers of the authors, here we check the validity of the two tautologies from the two tautologies of propositional logic

$$((A \to B)\&(A \to C)) \to (A \to (B\&C)),$$
$$((B \vee C) \to A) \to ((B \to A) \vee (C \to A))$$

and give the list of the intuitionistic fuzzy implications that satisfy these two formulas.

Keywords: Intuitionistic fuzzy implication, Intuitionistic fuzzy pair.
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1 Introduction

In the series of four articles [1–4] the authors study some properties of intuitionistic fuzzy implication and determine those of them that satisfy the largest number of tautologies (in particular - axioms) of logic. In the present study, we will determine which of these intuitionistic fuzzy implication satisfy the two obviously true tautologies of the propositional logic

$$((A \to B)\&(A \to C)) \to (A \to (B\&C)), \tag{1}$$

$$((B \vee C) \to A) \to ((B \to A) \vee (C \to A)) \tag{2}$$

discussed in [9].

All necessary definitions and notations are given in Part 1 [1] and in [8], but below, we will repeat a part of them, wherever this is appropriate.

2 Main results

Let everywhere below $a, b, c, d, e, f \in [0, 1]$, $a + b \leq 1, c + d \leq 1, e + f \leq 1$ and the truth-values of the logical variables A, B and C (evaluated by the function V that determine their degrees of truth and false) be:

$$V(A) = \langle a, b\rangle,$$
$$V(B) = \langle c, d\rangle,$$
$$V(C) = \langle e, f\rangle.$$

Furthermore, we will use the following two sign-type functions:

$$\mathrm{sg}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0, \end{cases} \qquad \overline{\mathrm{sg}}(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$

2.1 First case (standard IF operations)

First, we will discuss the case, when operations intuitionistic fuzzy negation ($\neg$), conjunction ($\&$) and disjunction ($\vee$) are the standard ones:

$$\neg\langle a, b\rangle = \langle b, a\rangle,$$
$$\langle a, b\rangle\&\langle c, d\rangle = \langle \min(a, c), \max(b, d)\rangle,$$
$$\langle a, b\rangle \vee \langle c, d\rangle = \langle \max(a, c), \min(b, d)\rangle.$$

Theorem 1. *For* $i = 1–9, 11–14, 17, 18, 20–25, 27–38, 40, 42–46, 48–53, 55, 57, 61–64, 66, 68–69, 71, 72, 74–77, 79–85, 88–91, 93, 94, 97, 100–106, 109–122, 124–137, 139, 141, 146–170, 177–189,$ *and* 191 *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Proof. First, let us check the validity of formula (1) for $i = 4$. We have

$$\begin{aligned}
&((A \to_4 B) \,\&\, (A \to_4 C)) \to_4 (A \to_4 (B \,\&\, C)) \\
&= (((\langle a, b\rangle \to_4 \langle c, d\rangle) \,\&\, (\langle a, b\rangle \to_4 \langle e, f\rangle)) \to_4 (\langle a, b\rangle \to_4 (\langle c, d\rangle \,\&\, \langle e, f\rangle))) \\
&= (\langle \max(b, c), \min(a, d)\rangle \,\&\, \langle \max(b, e), \min(a, f)\rangle) \\
&\qquad\quad \to_4 (\langle a, b\rangle \to_4 \langle \min(c, e), \max(d, f)\rangle) \\
&= \langle \max(b, \min(c, e)), \min(a, \max(d, f))\rangle \\
&\qquad\quad \to_4 \langle \max(b, \min(c, e)), \min(a, \max(d, f))\rangle \\
&= \Big\langle \max\big(\min(a, \max(d, f)), \max(b, \min(c, e))\big), \\
&\qquad \min\big(\max(b, \min(c, e)), a, \max(d, f)\big)\Big\rangle.
\end{aligned}$$

Let

$$X \equiv \max\big(\min(a, \max(d, f)), \max(b, \min(c, e))\big) - \min\big(\max(b, \min(c, e)), a, \max(d, f)\big).$$

Then

$$X \geq \max(b, \min(c, e)) - \min\big(\max(b, \min(c, e)), a, \max(d, f)\big) \geq 0.$$

Therefore, formula (1) is an IFT for the intuitionistic fuzzy implication $\to_4$.

Second, let us check formula (1) for $i = 16$. We obtain

$$\begin{aligned}
&((A \to_{16} B) \,\&\, (A \to_{16} C)) \to_{16} (A \to_{16} (B \,\&\, C)) \\
&= (\langle \max(\overline{\mathrm{sg}}(a), c), \min(\mathrm{sg}(a), d)\rangle \,\&\, \langle \max(\overline{\mathrm{sg}}(a), e), \min(\mathrm{sg}(a), f)\rangle) \\
&\qquad\quad \to_{16} \langle \max(\overline{\mathrm{sg}}(a), \min(c, e)), \min(\mathrm{sg}(a), \max(d, f))\rangle \\
&= \langle \max(\overline{\mathrm{sg}}(a), \min(c, e)), \min(\mathrm{sg}(a), \max(d, f))\rangle \\
&\qquad\quad \to_{16} \langle \max(\overline{\mathrm{sg}}(a), \min(c, e)), \min(\mathrm{sg}(a), \max(d, f))\rangle \\
&= \Big\langle \max\big(\overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(a), \min(c, e))), \max(\overline{\mathrm{sg}}(a), \min(c, e))\big), \\
&\qquad \min\big(\mathrm{sg}(\max(\overline{\mathrm{sg}}(a), \min(c, e))), \min(\mathrm{sg}(a), \max(d, f))\big)\Big\rangle.
\end{aligned}$$

Let

$$Y \equiv \max\big(\overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(a), \min(c, e))), \max(\overline{\mathrm{sg}}(a), \min(c, e))\big).$$

We distinguish two main cases.

1. If $a = 0$, then $\overline{\mathrm{sg}}(a) = 1$, and hence

$$Y = \max\big(\overline{\mathrm{sg}}(\max(1, \min(c, e))), \max(1, \min(c, e))\big) = \max(0, 1) = 1.$$

2. Let $a > 0$. Then $\overline{\mathrm{sg}}(a) = 0$, and therefore

$$Y = \max\big(\overline{\mathrm{sg}}(\min(c, e)), \min(c, e)\big).$$

There are three subcases:

(a) if $\min(c, e) = 0$, then
$$Y = \max(\overline{\mathrm{sg}}(0), 0) = 1;$$

(b) if $\min(c, e) = 1$, then
$$Y = \max(\overline{\mathrm{sg}}(1), 1) = 1;$$

(c) if $0 < \min(c, e) < 1$, then
$$Y = \max(0, \min(c, e)) = \min(c, e) < 1.$$

Thus, in the last subcase formula (1) cannot be a tautology.

Now let
$$Z \equiv \min\big(\mathrm{sg}(\max(\overline{\mathrm{sg}}(a), \min(c, e))), \min(\mathrm{sg}(a), \max(d, f))\big).$$

Again, we distinguish two main cases.

1. If $a = 0$, then $\overline{\mathrm{sg}}(a) = 1$ and $\mathrm{sg}(a) = 0$, so
$$Z = \min\big(\mathrm{sg}(\max(1, \min(c, e))), \min(0, \max(d, f))\big) = 0.$$

2. Let $a > 0$. Then $\overline{\mathrm{sg}}(a) = 0$ and $\mathrm{sg}(a) = 1$, so
$$Z = \min\big(\mathrm{sg}(\min(c, e)), \max(d, f)\big).$$

There are three subcases:

(a) if $\min(c, e) = 0$, then
$$Z = \min(\mathrm{sg}(0), \max(d, f)) = 0;$$

(b) if $\min(c, e) = 1$, then
$$Z = \min(\mathrm{sg}(1), \max(d, f)) = \max(d, f);$$

(c) if $0 < \min(c, e) < 1$, then
$$Z = \min(\mathrm{sg}(\min(c, e)), \max(d, f)) = \max(d, f).$$

Therefore, formula (1) is an IFT precisely when $Y \geq Z$. In particular, if $a > 0$ and $0 < \min(c, e) < 1$, then
$$Y - Z = \min(c, e) - \max(d, f). \tag{3}$$

This difference need not be nonnegative. For example, if $c = e = 0.3,\ d = f = 0.6$, then
$$Y - Z = 0.3 - 0.6 = -0.3 < 0,$$
while
$$c + d = e + f = 0.9.$$
Hence formula (1) is not an IFT for $\rightarrow_{16}$ and, consequently, cannot be a tautology for $\rightarrow_{16}$.

The remaining intuitionistic fuzzy implications can be checked in the same manner. $\square$

Theorem 2. *For* $i = 2, 3, 8, 11, 14, 20, 23, 31, 32, 34, 37, 40, 42, 43, 45, 48, 49, 52, 55, 57, 62, 63, 69, 74, 77, 83, 84, 88, 90, 93, 97, 153, 177 - 186$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

Proof. First, we will check the validity of (1) when $i = 2$:

$$((A \to_2 B)\&(A \to_2 C)) \to_2 (A \to_2 (B\&C))$$
$$= (((\langle a, b\rangle \to_2 \langle c, d\rangle)\&(\langle a, b\rangle \to_2 \langle e, f\rangle)) \to_2 (\langle a, b\rangle \to_2 (\langle c, d\rangle\&\langle e, f\rangle))$$
$$= (\langle \overline{\text{sg}}(a - c), d\,\text{sg}(a - c)\rangle\&\langle \overline{\text{sg}}(a - e), f\,\text{sg}(a - e)\rangle) \to_2 (\langle a, b\rangle \to_2 \langle\min(c, e), \max(d, f)\rangle)$$
$$= \langle\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)), \max(d\,\text{sg}(a - c), f\,\text{sg}(a - e))\rangle$$
$$\to_2 \langle \overline{\text{sg}}(a - \min(c, e)), \max(d, f)\,\text{sg}(a - \max(d, f))\rangle$$
$$= \langle \overline{\text{sg}}(\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e))),$$
$$\max(d, f)\,\text{sg}(a - \max(d, f))\,\text{sg}(\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e)))\rangle$$

Let
$$Y \equiv \overline{\text{sg}}\big(\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e))\big).$$

We distinguish the following cases.

1. If $a > c$, then $\overline{\text{sg}}(a - c) = 0$, and hence
$$Y = \overline{\text{sg}}\big(\min(0, \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e))\big) = 1.$$

2. Let $0 < a \le c$. Then $\overline{\text{sg}}(a - c) = 1$, so
$$Y = \overline{\text{sg}}\big(\overline{\text{sg}}(a - e) - \overline{\text{sg}}(a - \min(c, e))\big).$$

   Now:

   (a) if $a > e$, then $\overline{\text{sg}}(a - e) = 0$, and therefore $Y = 1$;

   (b) if $a \le e$, then $a \le \min(c, e)$, and consequently $Y = \overline{\text{sg}}(1 - 1) = 1$.

3. If $a = 0$, then
$$Y = \overline{\text{sg}}\big(\min(1, \overline{\text{sg}}(-e)) - 1\big) = \overline{\text{sg}}(1 - 1) = 1.$$

Thus, in all cases, $Y = 1$.
Now, let

$$Z \equiv \max(d, f)\,\text{sg}(a - \max(d, f))\,\text{sg}\big(\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e))\big).$$

We again consider two cases.

1. If $a \le \max(d, f)$, then $\text{sg}(a - \max(d, f)) = 0$, and therefore $Z = 0$.

2. If $a > \max(d, f)$, then $\text{sg}(a - \max(d, f)) = 1$, and hence
$$Z = \max(d, f)\,\text{sg}\big(\min(\overline{\text{sg}}(a - c), \overline{\text{sg}}(a - e)) - \overline{\text{sg}}(a - \min(c, e))\big).$$

There are two subcases.

(a) If $a \le \min(c, e)$, then $\mathrm{sg}(a - \min(c, e)) = 1$, so

$$Z = \max(d, f)\ \mathrm{sg}\big(\min(\mathrm{sg}(a-c), \mathrm{sg}(a-e)) - 1\big) = 0.$$

(b) If $a > \min(c, e)$, then $\mathrm{sg}(a - \min(c, e)) = 0$. Moreover, $a > c$ or $a > e$, and hence

$$\mathrm{sg}(a-c) = 0 \quad \text{or} \quad \mathrm{sg}(a-e) = 0.$$

Therefore, $\min(\mathrm{sg}(a-c), \mathrm{sg}(a-e)) = 0$, and consequently

$$Z = \max(d, f)\,\mathrm{sg}(0) = 0.$$

Thus, in all cases,

$$Z = 0.$$

Therefore, the implication $\rightarrow_2$ satisfies (1) as a tautology.

As we showed in the proof of Theorem 1, intuitionistic fuzzy implication $\rightarrow_{16}$ does not satisfy (1) as a tautology.

In the same manner, we can check the validity of formulas (1) with remaining intuitionistic fuzzy implications as a tautology. □

Theorem 3. *For* $i = 1-9, 11-15, 17, 18, 20-25, 27-38, 40, 42-46, 48-53, 55, 57, 61-66, 68, 69, 71, 72, 74–77, 79–85, 88–91, 93, 94, 97, 100–107, 109–122, 124–137, 139, 141, 146–170, 176-191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\rightarrow_i$ *as an IFT for every three IFPs* A, B *and* C.

Proof. First, let us check the validity of formula (2) for $i = 1$. We have

$$\begin{aligned}
&((B \vee C) \rightarrow_1 A) \rightarrow_1 ((B \rightarrow_1 A) \vee (C \rightarrow_1 A))\\
&= ((\langle c, d\rangle \vee \langle e, f\rangle) \rightarrow_1 \langle a, b\rangle)\\
&\qquad \rightarrow_1 \big((\langle c, d\rangle \rightarrow_1 \langle a, b\rangle) \vee (\langle e, f\rangle \rightarrow_1 \langle a, b\rangle)\big)\\
&= (\langle \max(c, e), \min(d, f)\rangle \rightarrow_1 \langle a, b\rangle)\\
&\qquad \rightarrow_1 \big(\langle \max(d, \min(c, a)), \min(c, b)\rangle \vee \langle \max(f, \min(e, a)), \min(f, b)\rangle\big)\\
&= \langle \max(\min(d, f), \min(\max(c, e), a)), \min(\max(c, e), b)\rangle\\
&\qquad \rightarrow_1 \langle \max(d, \min(c, a), f, \min(e, a)), \min(c, b, f)\rangle\\
&= \Big\langle \max\Big(\min(\max(c, e), b),\\
&\qquad\qquad \min\big(\max(\min(d, f), \min(\max(c, e), a)), \max(d, \min(c, a), f, \min(e, a))\big)\Big),\\
&\qquad \min\big(\max(\min(d, f), \min(\max(c, e), a)), c, b, f\big)\Big\rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X \equiv\ &\max\Big(\min(\max(c, e), b),\\
&\qquad \min\big(\max(\min(d, f), \min(\max(c, e), a)), \max(d, \min(c, a), f, \min(e, a))\big)\Big)\\
&- \min\big(\max(\min(d, f), \min(\max(c, e), a)), c, b, f\big).
\end{aligned}$$

We distinguish two cases.

1. If $b \le \max(c, e)$, then

$$\begin{aligned} X = {} & \max\Big(b, \min\big(\max(\min(d,f), \min(\max(c,e),a)), \max(d, \min(c,a), f, \min(e,a))\big)\Big) \\ & - \min\big(\max(\min(d,f), \min(\max(c,e),a)), c, b, f\big) \\ \ge {} & b - \min\big(\max(\min(d,f), \min(\max(c,e),a)), c, b, f\big) \ge 0. \end{aligned}$$

2. If $b > \max(c, e)$, then

$$\begin{aligned} X = {} & \max\Big(\max(c,e), \min\big(\max(\min(d,f), \min(\max(c,e),a)), \max(d, \min(c,a), f, \min(e,a))\big)\Big) \\ & - \min\big(\max(\min(d,f), \min(\max(c,e),a)), c, b, f\big) \\ = {} & \max\Big(c, e, \min\big(\max(\min(d,f), \min(\max(c,e),a)), \max(d, \min(c,a), f, \min(e,a))\big)\Big) \\ & - \min\big(\max(\min(d,f), \min(\max(c,e),a)), c, b, f\big) \\ \ge {} & c - \min\big(\max(\min(d,f), \min(\max(c,e),a)), c, b, f\big) \ge 0. \end{aligned}$$

Therefore, formula (2) is an IFT for the intuitionistic fuzzy implication $\to_1$.

Notice, however, that it is not a tautology for $\to_1$. For example, if $a = b = c = d = e = f$, the value of formula (2) is $\langle a, a\rangle$, which is not necessarily $\langle 1, 0\rangle$. This case will be considered again in the proof of Theorem 4.

Second, let us show that formula (2) is not an IFT for the intuitionistic fuzzy implication $\to_{59}$. Recall that

$$\langle x, y\rangle \to_{59} \langle u, v\rangle = \langle \max(\overline{\text{sg}}(1-y), u), 1 - \max(y, u)\rangle.$$

We obtain

$$\begin{aligned} & ((B \vee C) \to_{59} A) \to_{59} ((B \to_{59} A) \vee (C \to_{59} A)) \\ & = \big\langle \max\big(\overline{\text{sg}}(1 - \min(d,f)), a\big), 1 - \max(\min(d,f), a)\big\rangle \\ & \qquad \to_{59} \big\langle \max\big(\overline{\text{sg}}(1-d), \overline{\text{sg}}(1-f), a\big), 1 - \max(d, f, a)\big\rangle \\ & = \Big\langle \max\big(\overline{\text{sg}}(\max(\min(d,f), a)), \overline{\text{sg}}(1-d), \overline{\text{sg}}(1-f), a\big), \\ & \qquad\quad 1 - \max\big(1 - \max(\min(d,f),a), \overline{\text{sg}}(1-d), \overline{\text{sg}}(1-f), a\big)\Big\rangle. \end{aligned}$$

Now let, for example, $a = b = c = e = 0.2,\ d = f = 0.3$. Then

$$\overline{\text{sg}}(1-d) = \overline{\text{sg}}(1-f) = \overline{\text{sg}}(0.7) = 0$$

and

$$\overline{\text{sg}}(\max(\min(d,f), a)) = \overline{\text{sg}}(0.3) = 0.$$

Hence

$$\begin{aligned} & ((B \vee C) \to_{59} A) \to_{59} ((B \to_{59} A) \vee (C \to_{59} A)) \\ & = \langle \max(0, 0, 0, 0.2), 1 - \max(0.7, 0, 0, 0.2)\rangle \\ & = \langle 0.2, 0.3\rangle. \end{aligned}$$

Since

$$0.2 - 0.3 = -0.1 < 0,$$

formula (2) is not an IFT for the intuitionistic fuzzy implication $\to_{59}$.

The remaining intuitionistic fuzzy implications can be checked in the same manner. $\square$

Theorem 4. *For* $i = 2, 3, 8, 11, 14, 15, 20, 23, 24, 31, 32, 34, 37, 40, 42q43, 45, 48, 49, 52, 55, 57, 62, 63, 65, 68, 69, 74, 77, 83, 84, 88, 90, 93, 97, 153, 176–186, 190, 191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

Proof. Let us check the validity of formula (2) for $i = 62$. We have

$$\begin{aligned}
&((B \vee C) \to_{62} A) \to_{62} ((B \to_{62} A) \vee (C \to_{62} A)) \\
&= (((\langle c, d\rangle \vee \langle e, f\rangle) \to_{62} \langle a, b\rangle) \to_{62} \big(((\langle c, d\rangle \to_{62} \langle a, b\rangle) \vee (\langle e, f\rangle \to_{62} \langle a, b\rangle)\big) \\
&= (\langle \max(c, e), \min(d, f)\rangle \to_{62} \langle a, b\rangle) \\
&\qquad \to_{62} \big(\langle \overline{\mathrm{sg}}(b - d), c\,\mathrm{sg}(b - d)\rangle \vee \langle \overline{\mathrm{sg}}(b - f), e\,\mathrm{sg}(b - f)\rangle\big) \\
&= \langle \overline{\mathrm{sg}}(b - \min(d, f)), \max(c, e)\,\mathrm{sg}(b - \min(d, f))\rangle \\
&\qquad \to_{62} \langle \max(\overline{\mathrm{sg}}(b - d), \overline{\mathrm{sg}}(b - f)), \min(c\,\mathrm{sg}(b - d), e\,\mathrm{sg}(b - f))\rangle \\
&= \Big\langle \overline{\mathrm{sg}}(D),\ \overline{\mathrm{sg}}(b - \min(d, f))\ \mathrm{sg}(D)\Big\rangle,
\end{aligned}$$

where

$$D \equiv \min(c\,\mathrm{sg}(b - d), e\,\mathrm{sg}(b - f)) - \max(c, e)\,\mathrm{sg}(b - \min(d, f)).$$

It remains to show that $D \leq 0$. We distinguish two cases.

1. If $b > d$, then $\mathrm{sg}(b - d) = 1$ and, since $\min(d, f) \leq d < b$,

   $$\mathrm{sg}(b - \min(d, f)) = 1.$$

   Therefore,

   $$D = \min(c, e\,\mathrm{sg}(b - f)) - \max(c, e) \leq c - \max(c, e) \leq 0.$$

2. If $b \leq d$, then $\mathrm{sg}(b - d) = 0$, and hence

   $$\begin{aligned} D &= \min(0, e\,\mathrm{sg}(b - f)) - \max(c, e)\,\mathrm{sg}(b - \min(d, f)) \\ &= -\max(c, e)\,\mathrm{sg}(b - \min(d, f)) \leq 0. \end{aligned}$$

Thus, in all cases,

$$D \leq 0.$$

Consequently, $\overline{\mathrm{sg}}(D) = 1$ and $\mathrm{sg}(D) = 0$. Hence

$$\Big\langle \overline{\mathrm{sg}}(D),\ \overline{\mathrm{sg}}(b - \min(d, f))\ \mathrm{sg}(D)\Big\rangle = \langle 1, 0\rangle.$$

Therefore, formula (2) is a tautology for the intuitionistic fuzzy implication $\to_{62}$.

On the other hand, as shown in the proof of Theorem 3, formula (2) is not a tautology for $\to_1$. The remaining intuitionistic fuzzy implications can be checked in the same manner. $\square$

All the next theorems are proved in the same manner.

2.2 Second case

Second, we will discuss the case, when operations intuitionistic fuzzy conjunction and disjunction are generated by the respective intuitionistic fuzzy implication ($\to$) and the intuitionistic fuzzy negation ($\neg$) generated by it through formulas (see [5]):

$$\neg\langle a, b\rangle = \langle a, b\rangle \to \langle 0, 1\rangle,$$
$$\langle a, b\rangle \& \langle c, d\rangle = \neg(\langle a, b\rangle \to \neg\langle c, d\rangle),$$
$$\langle a, b\rangle \vee \langle c, d\rangle = \neg\langle a, b\rangle \to \langle c, d\rangle.$$

Theorem 5. *For* $i = 1–9, 11–29, 46–53, 55–57, 61, 64, 66, 71, 74–77, 79, 81, 91, 94, 97, 100–102, 104, 105, 107, 109–113, 119–122, 124–128, 133–137, 147–149, 151, 153, 158–161, 166, 167, 169–172, 174–177, 179, 180, 182, 184, 185, 187–189, 191$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 6. *For* $i = 2, 3, 8, 11, 14 - 16, 19, 20, 23, 47, 48, 52, 55 - 57, 74, 77, 97, 153, 171, 172, 174 - 177, 179, 180$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

Theorem 7. *For* $i = 1-6, 8-9, 11-38, 40-53, 55-57, 61-66, 68, 69, 71, 72, 74-77, 79-85, 88-91, 93, 94, 97-107, 109-117, 119-122, 124-137, 139, 141, 146-149, 151, 153, 158, 159, 160, 161, 166 - 191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 8. *For* $i = 2, 3, 8, 11, 14-16, 19, 20, 23, 24, 31, 32, 34, 37, 40-45, 47-49, 52, 55-57, 62, 63, 65, 68, 69, 74, 77, 83, 84, 88, 90, 93, 97, 153, 171-186, 190, 191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

2.3 Third case

Third, we will discuss the case, when operations intuitionistic fuzzy conjunction and disjunction are generated by the respective intuitionistic fuzzy implication ($\to$) and the intuitionistic fuzzy negation ($\neg$) generated by it through formulas (see, [6]):

$$\neg\langle a, b\rangle = \langle a, b\rangle \to \langle 0, 1\rangle,$$
$$\langle a, b\rangle \& \langle c, d\rangle = \neg(\neg\neg\langle a, b\rangle \to \neg\langle c, d\rangle),$$
$$\langle a, b\rangle \vee \langle c, d\rangle = \neg\langle a, b\rangle \to \neg\neg\langle c, d\rangle.$$

Theorem 9. *For* $i = 1 - 9, 11 - 29, 46 - 53, 55 - 57, 61, 64, 66, 71, 74 - 77, 79, 81, 91, 94, 97, 100 - 102, 104, 105, 107, 109 - 113, 119 - 122, 124 - 128, 133 - 137, 141, 147 - 149, 151, 153, 158 - 161, 166, 167, 169 - 172, 174 - 177, 179, 180, 182, 184, 185, 187 - 189, 191$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 10. *For* $i = 2, 3, 8, 11, 14-16, 19, 20, 23, 47, 48, 52, 55-57, 74, 77, 97, 153, 171-177, 179, 180$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

Theorem 11. *For* $i = 1-6, 8, 9, 11-38, 40-53, 55-57, 61-66, 68, 69, 71, 72, 74-77, 79-85, 88-91, 93, 94, 97-117, 119-122, 124-137, 141, 146-149, 151, 153, 158-161, 166-191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 12. *For* $i = 2, 3, 8, 11, 14-16, 19, 20, 23, 24, 31, 32, 34, 37, 40-45, 47-49, 52, 55-57, 62, 63, 65, 68, 69, 74, 77, 83, 84, 88, 90, 93, 97, 153, 171-186, 190, 191$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

2.4 Fourth case

Fourth, we will discuss the case, when operations intuitionistic fuzzy conjunction and disjunction are generated by the respective intuitionistic fuzzy implication ($\to$) and standard intuitionistic fuzzy negation ($\neg$), as in Case 2.1. In this case, $\neg\neg\langle a, b\rangle = \langle a, b\rangle$ and therefore, formulas for conjunction and disjunction coincide with these from Case 2.2.

Theorem 13. *For* $i = 1, 4-7, 9, 12, 13, 17, 18, 21, 25, 28, 29, 46, 49-51, 53, 61, 64, 66, 71, 75-77, 79, 81, 91, 94, 97, 100-102, 109-113, 118-122, 124-128, 135-137, 139, 151, 153, 158-161, 166, 167, 169, 182, 184, 187, 188$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 14. *For* $i = 77, 97$ *formula (1) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

Theorem 15. *For* $i = 1, 3-6, 8, 9, 11, 13, 14, 17, 18, 20-23, 25, 27-38, 42, 44, 45, 61-66, 68, 69, 71, 72, 74-77, 79-85, 88-90, 100-105, 109-117, 124-133, 139, 141, 146, 147, 151, 153, 158-161, 166-170, 181-183, 185-188$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as an IFT for every three IFPs* A, B *and* C.

Theorem 16. *For* $i = 3, 8, 11, 14, 20, 23, 32, 34, 37, 42, 63, 65, 68, 69, 74, 83, 84, 88, 90, 181-183, 185, 186$ *formula (2) is valid for the intuitionistic fuzzy implication* $\to_i$ *as a tautology for every three IFPs* A, B *and* C.

3 Conclusion

In [2], following the criteria described there, some of the existing intuitionistic fuzzy implications were determined as "best". These implications $\to_i$ have indices $i = 1-5, 9, 11, 13, 14, 17, 18, 20, 22-24, 27-29, 35, 61, 71, 74, 76, 77, 79, 81, 100-102, 105, 109, 110, 112, 125, 127, 166, 167, 170, 176, 177, 180, 186$. Each one of these implications satisfies at least one of each of the four groups of implications.

Therefore, the number of the "best" intuitionistic fuzzy implications will not decrease after the present research.

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