Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2025, Volume 31, Number 4, 458–464
DOI: 10.7546/nifs.2025.31.4.458-464

Coincidence of intuitionistic fuzzy observables

Katarína Čunderlíková

Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia
e-mail: cunderlikova.lendelova@gmail.com

Received: 28 September 2025 **Revised:** 18 October 2025
Accepted: 20 October 2025 **Online First:** 27 November 2025

Abstract: The aim of this paper is to define the notion of coincidence for intuitionistic fuzzy observables. We study the almost everywhere coincidence using intuitionistic fuzzy state $\mathbf{m}$ or intuitionistic fuzzy probability $\mathcal{P}$. We further show the connection between $\mathbf{m}$-almost everywhere coincidence and $\mathcal{P}$-almost everywhere coincidence of intuicionistic fuzzy observables.
Keywords: Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Intuitionistic fuzzy probability, Almost everywhere coincidence, Joint intuitionistic fuzzy observable, Product.
2020 Mathematics Subject Classification: 60A86, 60A10.

1 Introduction and elementary notions

The notion of coincidence of observables was studied by M. Jurečková on MV-algebras with product in papers [4, 5]. Later in [6] B. Riečan and M. Jurečková defined a coincidence of observables on fuzzy quantum logic. Since the intuitionistic fuzzy sets are a generalization of the fuzzy sets, so we will try to define a coincidence of intuitionistic fuzzy observables. In this section we explain the basic terms from intuitionistic fuzzy probability theory, which we will use later in the paper.

The notion of intuitionistic fuzzy sets was introduced by K. T. Atanassov in 1983 in paper [1]. By **intuitionistic fuzzy set** A on Ω he understand a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.

Similarly as an **intuitionistic fuzzy event** he understand an intuitionistic fuzzy set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable (see [2, 3, 11]). The family of all intuitionistic fuzzy events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$.

In this paper we will work with Lukasiewicz binary operations $\oplus$ and $\odot$ on $\mathcal{F}$, which are given by these equalities

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),\\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))\end{aligned}$$

for $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$. The partial ordering $\leq$ on $\mathcal{F}$ is defined as follows:

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In the next section we will study the coincidence of intuitionistic fuzzy observables using an intuitionistic fuzzy state $\mathbf{m}$ or using an intuitionistic fuzzy probability $\mathcal{P}$. By **an intuitionistic fuzzy state** we mean a mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ which is satisfying the following three conditions (see [12]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

By **intuitionistic fuzzy probability** we understand a mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, which is satisfying the following conditions (see [8, 9]):

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.

There $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ means that $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$, but $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ means $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$.

Remark that the intuitionistic fuzzy probability $\mathcal{P}$ can be expressed using two intuitionistic fuzzy states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, where $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$ (see [10]). Such $[\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ is an interval, then $\mathcal{P}^\flat(\mathbf{A}) \leq \mathcal{P}^\sharp(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{F}$.

Remind that the intuitionistic fuzzy observable in the intuitionistic fuzzy space has the same role as the random variable in the classical probability space. By the notion of **n-dimensional intuitionistic fuzzy observable** we mean each mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions (see [12]):

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\varnothing) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \varnothing$ and $A, B \in \mathcal{B}(R^n)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R^n)$, $n \in N$, then $x(A_n) \nearrow x(A)$.

For $n = 1$ we say simply about intuitionistic fuzzy observable. There $\mathcal{B}(R)$ is the σ-algebra of the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$, (see [13]).

2 Coincidence of intuitionistic fuzzy observables

In this section we define the notion of almost everywhere coincidence for intuitionistic fuzzy observables. We will use two approaches, one with and one without the use of a joint intuitionistic fuzzy observable. Throughout the text we will use the notation "IF" for intuitionistic fuzzy.

Now we recall what we mean by an joint IF-observable. **An joint IF-observable** of two IF-observables x, y is a two-dimensional IF-observable $h : \mathcal{B}(R^2) \to \mathcal{F}$, which is satisfying the following condition:

$$h(C \times D) = x(C) * y(D)$$

for each $C, D \in \mathcal{B}(R)$, see [11]. There $*$ is the product operation on the family of IF-events $\mathcal{F}$ given by $\mathbf{A} * \mathbf{B} = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big) = \big(\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B\big)$ for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ and $\cdot$ is a multiplication, see [7].

In [7] the product $*$ on $\mathcal{F}$ was defined as a binary operation, which is satisfying the following conditions:

(i) $(1_\Omega, 0_\Omega) * \mathbf{A} = \mathbf{A}$;

(ii) the operation $*$ is commutative and associative;

(iii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\mathbf{C} * (\mathbf{A} \oplus \mathbf{B}) = (\mathbf{C} * \mathbf{A}) \oplus (\mathbf{C} * \mathbf{B})$ and $(\mathbf{C} * \mathbf{A}) \odot (\mathbf{C} * \mathbf{B}) = (0_\Omega, 1_\Omega)$;

(iv) if $\mathbf{A}_n \searrow (0_\Omega, 1_\Omega)$, $\mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$, then $\mathbf{A}_n * \mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$

for each $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A}_n, \mathbf{B}_n \in \mathcal{F}$, $n \in N$.

2.1 Almost everywhere coincidence using IF-state

In this subsection, we will study the almost everywhere coincidence of IF-observables with using IF-state $\mathbf{m}$.

Definition 1. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathbf{m} : \mathcal{F} \to [0, 1]$ be an IF-state. We say that the IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathbf{m}$-almost everywhere if and only if*

$$\mathbf{m}\Big(h\big(\{(u, v) \in R^2 \,;\, u = v\}\big)\Big) = 1,$$

where $h : \mathcal{B}(R) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y. We write that $x = y$ $\mathbf{m}$-almost everywhere.

The next theorem shows a formulation of coincidence of IF-observables without using the joint IF-observable.

Theorem 1. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathbf{m} : \mathcal{F} \to [0, 1]$ be an IF-state. Two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathbf{m}$-almost everywhere if and only if*

$$\mathbf{m}\Big(x\big((-\infty, u)\big) * y\big((u, \infty)\big)\Big) = 0 \;\; \textit{and} \;\; \mathbf{m}\Big(x\big((u, \infty)\big) * y\big((-\infty, u)\big)\Big) = 0$$

for each $u \in R$.

Proof. " $\Rightarrow$ " Let $x = y$ $\mathbf{m}$-almost everywhere. Then by Definition 1

$$\mathbf{m}\Big(h\big(\{(u,v) \in R^2\,; u = v\}\big)\Big) = 1, \tag{1}$$

where $h : \mathcal{B}(R) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y.

But

$$\big((-\infty, u) \times (u, \infty)\big) \cap \{(u,v) \in R^2\,; u = v\} = \varnothing$$

and

$$\big((u, \infty) \times (-\infty, u)\big) \cap \{(u,v) \in R^2\,; u = v\} = \varnothing.$$

Therefore using (1) we have

$$\begin{aligned} 0 &= \mathbf{m}\Big(h\big((-\infty,u) \times (u,\infty)\big)\Big) = \mathbf{m}\Big(x\big((-\infty,u)\big) * y\big((u,\infty)\big)\Big), \\ 0 &= \mathbf{m}\Big(h\big((u,\infty) \times (-\infty,u)\big)\Big) = \mathbf{m}\Big(x\big((u,\infty)\big) * y\big((-\infty,u)\big)\Big) \end{aligned}$$

for each $u \in R$.

" $\Leftarrow$ " Let $\forall u \in R$

$$\mathbf{m}\Big(x\big((-\infty,u)\big) * y\big((u,\infty)\big)\Big) = 0 \text{ and } \mathbf{m}\Big(x\big((u,\infty)\big) * y\big((-\infty,u)\big)\Big) = 0. \tag{2}$$

Denote

$$A_n = \bigcup_{i=-\infty}^{\infty} \left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \right).$$

Then $A_n \supset A_{n+1}$ for each $n \in N$ and

$$\mathbf{m}\Big(h\big(\{(u,v) \in R^2\,; u = v\}\big)\Big) = \lim_{n\to\infty} \mathbf{m}\big(h(A_n)\big).$$

For the complement A_n^c of the set A_n holds

$$A_n^c \subset \left(\bigcup_{i=-\infty}^{\infty} \left(\left(-\infty, \frac{i-1}{2^n} \right) \times \left(\frac{i-1}{2^n}, \infty \right) \right) \right) \cup \left(\bigcup_{i=-\infty}^{\infty} \left(\left(\frac{i-1}{2^n}, \infty \right) \times \left(-\infty, \frac{i-1}{2^n} \right) \right) \right). \tag{3}$$

Therefore using (1), (2) and (3) we obtain

$$\begin{aligned} \mathbf{m}\big(h(A_n^c)\big) \leq & \left(\sum_{i=-\infty}^{\infty} \mathbf{m}\left(x\left(\left(-\infty, \frac{i-1}{2^n} \right) \right) * y\left(\left(\frac{i-1}{2^n}, \infty \right) \right) \right) \right) + \\ & + \left(\sum_{i=-\infty}^{\infty} \mathbf{m}\left(x\left(\left(\frac{i-1}{2^n}, \infty \right) \right) * y\left(\left(-\infty, \frac{i-1}{2^n} \right) \right) \right) \right) = 0 \end{aligned}$$

and hence $\mathbf{m}\big(h(A_n)\big) = 1$.

Finally

$$\mathbf{m}\Big(h\big(\{(u,v) \in R^2\,; u = v\}\big)\Big) = \lim_{n\to\infty} \mathbf{m}\big(h(A_n)\big) = 1$$

and by Definition 1 we have that $x = y$ $\mathbf{m}$-almost everywhere. $\square$

2.2 Almost everywhere coincidence using IF-probability

In this subsection, we will explain an almost everywhere coincidence of IF-observables in connection with the IF-probability $\mathcal{P}$. We will show how $\mathcal{P}$-almost everywhere coincidence and $\mathbf{m}$-almost everywhere coincidence are related, too.

Definition 2. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathcal{P}$ be an IF-probability. We say that the IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathcal{P}$-almost everywhere if and only if*

$$\mathcal{P}\Big(h\big(\{(u,v) \in R^2 \,; u = v\}\big)\Big) = [1,1] = 1,$$

where $h : \mathcal{B}(R) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y. We write that $x = y$ $\mathcal{P}$-almost everywhere.

Theorem 2. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathcal{P}$ be an IF-probability. The IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathcal{P}$-almost everywhere if and only if they coincide $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere, where $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$ are the IF-states.*

Proof. Let $(\mathcal{F}, *)$ be the IF-space with product $*$, $\mathcal{P}$ be an IF-probability and $x, y : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables. The IF-probability $\mathcal{P}$ can be expressed using two IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, where $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$.

Using Definition 2 the IF-observables x, y coincide $\mathcal{P}$-almost everywhere if and only if

$$\begin{aligned}\mathcal{P}\Big(h\big(\{(u,v) \in R^2; u = v\}\big)\Big) &= \Big[\mathcal{P}^\flat\Big(h\big(\{(u,v) \in R^2; u = v\}\big)\Big), \mathcal{P}^\sharp\Big(h\big(\{(u,v) \in R^2 \,; u = v\}\big)\Big)\Big] \\ &= [1,1] = 1,\end{aligned}$$

where $h : \mathcal{B}(R) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y. Therefore

$$\begin{aligned}\mathcal{P}^\flat\Big(h\big(\{(u,v) \in R^2 \,; u = v\}\big)\Big) &= 1, \\ \mathcal{P}^\sharp\Big(h\big(\{(u,v) \in R^2 \,; u = v\}\big)\Big) &= 1.\end{aligned}$$

Hence using Definition 1 we obtain that the IF-observables x, y coincide $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere. $\square$

Theorem 3. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathcal{P}$ be an IF-probability. Two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathcal{P}$-almost everywhere if and only if*

$$\mathcal{P}\Big(x\big((-\infty, u)\big) * y\big((u, \infty)\big)\Big) = [0,0] = 0 \;\text{ and }\; \mathcal{P}\Big(x\big((u, \infty)\big) * y\big((-\infty, u)\big)\Big) = [0,0] = 0$$

for each $u \in R$.

Proof. Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathcal{P}$ be an IF-probability. The IF-probability $\mathcal{P}$ can be expressed using two IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, where $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$.

Using Theorem 2 we have that two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathcal{P}$-almost everywhere if and only if they coincide $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere. For this reason

$$\mathcal{P}^\flat\Big(x\big((-\infty, u)\big) * y\big((u, \infty)\big)\Big) = 0 \;\text{ and }\; \mathcal{P}^\flat\Big(x\big((u, \infty)\big) * y\big((-\infty, u)\big)\Big) = 0,$$

$$\mathcal{P}^\sharp\Big(x\big((-\infty,u)\big)*y\big((u,\infty)\big)\Big)=0 \text{ and } \mathcal{P}^\sharp\Big(x\big((u,\infty)\big)*y\big((-\infty,u)\big)\Big)=0$$

for each $u \in R$, see Theorem 1.

Hence

$$\begin{aligned}\mathcal{P}\Big(x\big((-\infty,u)\big)*y\big((u,\infty)\big)\Big) &= \Big[\mathcal{P}^\flat\Big(x\big((-\infty,u)\big)*y\big((u,\infty)\big)\Big),\mathcal{P}^\sharp\Big(x\big((-\infty,u)\big)*y\big((u,\infty)\big)\Big)\Big]\\ &= [0,0]=0\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}\Big(x\big((u,\infty)\big)*y\big((-\infty,u)\big)\Big) &= \Big[\mathcal{P}^\flat\Big(x\big((u,\infty)\big)*y\big((-\infty,u)\big)\Big),\mathcal{P}^\sharp\Big(x\big((u,\infty)\big)*y\big((-\infty,u)\big)\Big)\Big]\\ &= [0,0]=0\end{aligned}$$

for each $u \in R$. □

3 Conclusion

The present paper deals with the study of almost everywhere coincidence of intuitionistic fuzzy observables in relation to intuitionistic fuzzy state $\mathbf{m}$ and intuitionistic fuzzy probability $\mathcal{P}$. The notion of almost everywhere coincidence is crucial for invariant observables. In further research, we will try to show the existence of an invariant intuitionistic fuzzy observable in a modification of the Individual ergodic theorem for intuitionistic fuzzy observables.

Acknowledgements

This publication was supported by grant VEGA 2/0122/23.

References

[1] Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *VII ITKR Session, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted in: International Journal Bioautomation*, 20(S1), S1–S6.

[2] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Physica Verlag, Heidelberg.

[3] Atanassov, K. T. (2012). *On Intuitionistic Fuzzy Sets*. Springer, Berlin.

[4] Jurečková, M. (2001). The coincidence of observables on MV-algebras with product. In: *Komorníková, M., & Mesiar, R. (Eds.). Proceeding Uncertainty Modeling*, STU, Bratislava, Slovakia, pp. 160–165.

[5] Jurečková, M. (2003). The addition to the ergodic theorem on probability MV-algebras with product. *Soft Computing*, 7, 472–475.

[6] Jurečková, M., & Riečan, B. (2005). Invariant observables and the individual ergodic theorem. *International Journal of Theoretical Physics*, 44, 1587–1597.

[7] Lendelová, K. (2006). Conditional IF-probability. *Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling, Proceedings of the 3rd International Conference on Soft Methods in Probability and Statistics*, 5–7 September 2006, Bristol, UK, 37, 275–283.

[8] Riečan, B. (2003). A descriptive definition of the probability on intuitionistic fuzzy sets. *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology, EUSFLAT '2003*, 10–12 September 2003, Zittau, Germany, 210–213.

[9] Riečan, B. (2004). Representation of probabilities on IFS events. *Soft Methodology and Random Information Systems, Proceedings of the 2nd International Conference on Soft Methods in Probability and Statistics, SMPS'2004*, 2–4 September 2004, Oviedo, Spain, 243–248.

[10] Riečan, B. (2005). On the probability on IF-sets and MV-algebras. *Notes on Intuitionistic Fuzzy Sets*, 11(6), 21–25.

[11] Riečan, B. (2006). On the probability and random variables on IF events. *Applied Artifical Intelligence, Proceedings of 7th FLINS Conference*, 29–31 August 2006, Genova, Italy, 138–145.

[12] Riečan, B. (2012). Analysis of fuzzy logic models. *Intelligent Systems*, INTECH, 219–244.

[13] Riečan, B., & Neubrunn, T. (1997). *Integral, Measure, and Ordering*. Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava.