Notes on Intuitionistic Fuzzy Sets
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Rational divergence measures on intuitionistic fuzzy sets

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Abstract: This paper investigates the theoretical foundations and practical applications of rational divergence measures within the framework of fuzzy set theory. Unlike traditional distance metrics, rational divergence measures are characterized by their functional form, typically expressed as a ratio of membership functions or set cardinalities, which allows for a more nuanced quantification of the "informational gap" between fuzzy sets.

The study explores the axiomatic properties of these measures, focusing on their ability to handle non-linearity and uncertainty in complex data structures. By examining rational forms of divergence, such as those derived from the Jaccard-like ratios or specialized f-divergences, this work demonstrates how these measures overcome the limitations of standard Euclidean distances in high-dimensional fuzzy spaces. Key emphasis is placed on their role in multi-criteria decision-making (MCDM) and pattern recognition, where the rational expression of divergence provides a more stable and intuitive measure of dissimilarity.

Furthermore, the paper provides a comparative analysis of different rational divergence formulations, evaluating their sensitivity to membership fluctuations and their performance in clustering algorithms. The results suggest that rational divergence measures offer superior discriminative power, making them a tool for modeling expert knowledge and imprecise information in modern intelligent systems.

Many authors investigated the possibilities how two fuzzy sets can be compared. The basic study of fuzzy sets theory was introduced by Lotfi Zadeh in 1965. We discuss the divergences defined on more general objects, namely intuitionistic fuzzy sets (IFSs). We have focused on special class of divergences, since some restriction conditions are necessary. This approach to the divergence measure is motivated by class of the rational similarity measures between fuzzy subsets expressed using some set operations (namely intersection, complement, difference, and symmetric difference) and their scalar cardinalities. In this study, we have considered the value of divergence between IFSs as a $\sum$-count of two scalar cardinalities, i.e. as a pair of real numbers.
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1 Introduction

The history of Intuitionistic Fuzzy Sets (IFS) began in 1983 when the Bulgarian mathematician Krassimir Atanassov introduced the concept as a significant extension of Lotfi Zadeh's original fuzzy set theory [3]. While Zadeh's model focused solely on a degree of membership, Atanassov realized that human reasoning often involves a degree of non-membership that is not simply the complement of the first. For each point in the universe X a degree of membership and a degree of non-membership are assigned. A major milestone occurred in 1986 with the publication of the foundational paper that formally defined the core constraint: the sum of the membership and non-membership degrees must be between 0 and 1. The remaining value was identified as the hesitation margin, representing uncertainty or lack of information. IFS is a tool of multi-criteria decision-making, pattern recognition, and the development of even more complex systems like intuitionistic fuzzy sets of second type. Atanassov defined an **intuitionistic fuzzy set** (IF-set) as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where μ_A and ν_A are membership (non-membership) functions $\mu_A, \nu_A : X \to [0,1]$, such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$ and $\mu_A(x), \nu_A(x)$ are membership and non-membership degrees, respectively, of the element $x \in X$ in the set A. The family of all intuitionistic fuzzy sets defined in the universe X will be denoted by the symbol $IFS(X)$. Of course, each fuzzy set can be considered as a special case of an IF-set, where $\nu_A(x) = 1 - \mu_A(x)$ and $\pi_A(x) = 0$.

The concept of triangular norms originated from the need to generalize the triangle inequality in metric spaces. In 1942, Austrian mathematician Karl Menger introduced the idea in his work on probabilistic metric spaces. Menger's goal was to replace fixed distance values with probability distribution functions. His original axioms for t-norms were initially weaker than the modern standards we use today. The formal, modern axiomatization was developed by Berthold Schweizer and Abe Sklar between 1958 and 1961. Schweizer and Sklar defined the t-norms as associative and commutative binary operations in the interval $[0, 1]$. They established that the number 1 must serve as a neutral element of the operation. During the 1960s, research focused primarily on their role in mathematical analysis and probability. A major turning point occurred with the start of fuzzy set theory by Lotfi Zadeh in 1965. Researchers realized that t-norms are the perfect tool to model the logical "and" connective (conjunction). Due to fuzzy logic, the t-norms became crucial for modeling uncertainty and linguistic variables. In the 1980s, parameterized families like the Frank and Hamacher t-norms were introduced.

The triangular norm (t-norm) is a function $T : [0,1]\times[0,1] \to [0,1]$ that satisfies the following conditions:

(T1) $T(a,b) = T(b,a)$, for all $a, b \in [0,1]$ (commutativity),

(T2) $T(T(a,b),c) = T(a,T(b,c))$, for all $a, b, c \in [0,1]$ (associativity),

(T3) $b \leq c \Rightarrow T(a,b) \leq T(a,c)$, for all $a, b, c \in [0,1]$ (monotonicity),

(T4) $T(a,1) = a$, for all $a \in [0,1]$ (boundary condition).

The basic t-norms are the following:

- minimum t-norm: $T_M(a,b) = \min(a,b)$, for all $a, b \in [0,1]$,
- product t-norm: $T_P(a,b) = a \cdot b$, for all $a, b \in [0,1]$,
- Łukasiewicz t-norm: $T_L(a,b) = \max(a+b-1, 0)$, for all $a, b \in [0,1]$,
- drastic t-norm:
$$T_D(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1. \\ 0, & \text{otherwise .} \end{cases}$$

For these basic t-norms, we have $T_D \leq T_L \leq T_P \leq T_M$ and $T_D \leq T \leq T_M$ for any t-norm T.

Changing the neutral element from 1 to 0, we obtain the triangular conorm (t-conorm), a function used to model a disjunction in fuzzy logic and the union of fuzzy sets. T and S are dual since $T(a,b) = 1 - S(1-a, 1-b)$ is fulfilled. For $A, B \in \mathcal{F}(X)$ we define:

- intersection of A and B: $A \cap_T B(x) = T(A(x), B(x))$, for all $x \in X$;
- union of A and B: $A \cup_S B(x) = S(A(x), B(x))$, for all $x \in X$.

In similar way, we define the union, intersection and complement of $IFS(X)$:

(i) union of A and B:
$$A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\},$$
where $\mu_{A\cup B}(x) = S(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = T(\nu_A(x), \nu_B(x))$.

(ii) intersection of A and B:
$$A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cup B}(x)\rangle \mid x \in X\},$$
where $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cup B}(x) = S(\nu_A(x), \nu_B(x))$.

(iii) complement of A:
$$A^c = \{\langle x, \mu_{A^c}(x), \nu_{A^c}(x)\rangle \mid x \in X\},$$
where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$.

2 Divergence measures

In the literature, several measures of comparison between fuzzy sets can be found. The evolution of dissimilarity and divergence measures within fuzzy set theory represents a bridge between classical information theory and approximate reasoning. Traditional divergence measures were originally designed for probability distributions to quantify information loss. With the start of fuzzy set theory by Lotfi Zadeh in 1965, a need arose to measure the difference between imprecise concepts. Dissimilarity measures emerged as a way to quantify the degree to which two fuzzy sets deviate from one another. Unlike standard metrics, a divergence measure in fuzzy logic does not necessarily satisfy the triangle inequality. Early approaches focused on fuzzy entropy, introduced by De Luca and Termini in 1972, to measure the "fuzziness" of a single set. The concept of fuzzy divergence was later formalized to compare two distinct fuzzy membership functions. In the 1990s, Bhandari and Pal adapted the Kullback-Leibler divergence specifically for fuzzy distributions. A key development was the definition of axiomatic requirements for fuzzy divergence, which ensured mathematical consistency. These axioms typically require the measure to be non-negative and reach zero only when the sets are identical. Bouchon-Meunier contributed significantly by exploring measures that account for the semantic proximity of linguistic variables. In 1996, Bouchon-Meunier [4] tried to define a general measure of comparison for fuzzy sets. The introduction of Intuitionistic Fuzzy Sets (IFS) by Atanassov required new types of divergence measure. Viedma and Montes provided a formal axiomatic framework for divergence between fuzzy sets in the early 2000s. The Bregman divergence was also adapted for fuzzy sets, offering a geometric perspective based on convex functions. More measures have been introduced to compare fuzzy sets [1, 20, 21]. A detailed study on that can be found in [5]. The most common comparison measures are dissimilarities [12].

The restriction associated with the definition of the measure of dissimilarity is only given for sets such that $A \subseteq B \subseteq C$, but there are many sets that are not comparable to $\subseteq$. The concept of a divergence measure proposed by Susana Montes [14] provides a formal axiomatic framework to measure the difference between fuzzy sets, distinguishing it from general distance or dissimilarity measures. According to Montes, a function D is a divergence measure if it satisfies three fundamental axioms for any fuzzy sets. The divergence between a set and itself is zero. The divergence between A and B is the same as between B and A. Finally, adding common information to both sets (through union) or extracting common information (through intersection) should never increase the divergence between them.

So we propose the following([11, 14]):

Definition 1. *Let (X, T, S) be a triple with X a universe and T and S any t-norm and t-conorm, respectively. A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a **divergence measure** with respect to (X, T, S) if and only if for all $A, B \in \mathcal{F}(X)$, D satisfies the following conditions:*

(D1) $D(A, A) = 0$;

(D2) $D(A, B) = D(B, A)$;

(D3) $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$, *for all* $C \in \mathcal{F}(X)$, *where the union and intersection are defined by means of* S *and* T, *respectively.*

Locality is the most important property of some divergence measures, which allows us to compute the divergence point-by-point. The definition of a local divergence measure was introduced in [14] as follows.

Definition 2. *A divergence measure* D *is local if for all* $A, B \in \mathcal{F}(X)$ *and for all* $x \in X$ *we have the following:* $D(A, B) - D(A \cup \{x\}, B \cup \{x\}) = h(A(x), B(x))$, *where* h *is a function from* $[0, 1] \times [0, 1]$ *to* $\mathbb{R}$.

Theorem 1 (Representation Theorem)**.** *Let* (X, T, S) *be a triple with* X *a finite universe and* T *and* S *any t-norm and t-conorm, respectively. Let* D *be a divergence associated to* X. D *is local if and only if*

$$D(A, B) \;=\; \sum_{x \in X} h(A(x), B(x)),$$

where h *is a map from* $[0, 1] \times [0, 1]$ *into* $\mathbb{R}$ *such that the following conditions are satisfied:*

(1) for all $a \in [0, 1];\ h(a, a) = 0$*;*

(2) for all $a, b \in [0, 1];\ h(a, b) = h(b, a)$*;*

(3) for all $a, b, c \in [0, 1]$*;* $h(a, b) \geq \max\{h(T(a, c), T(b, c)), h(S(a, c), S(b, c))\}$.

The proof can be found in [11].

We obtain a particular form of local divergence measure D if the function h is constructed by means of a suitable distance in $[0, 1]$.

Example 1. *For any pair of fuzzy sets in* X *we define the function* D *using the Hamming distance as follows:*

$$D(A, B) = \sum_{x \in X} |A(x) - B(x)|.$$

According to [11]*, the map* D *is a local divergence measure, if we work on* (X, T_M, S_M), (X, T_P, S_P) *or* (X, T_L, S_L)*. However, the map* D *is not a divergence measure since* (X, T_D, S_D) *to be considered.*

The basic study related to the topic can be found in [15, 16].

In the context of Intuitionistic Fuzzy Sets (IFS), distance measures are more complex than in standard fuzzy sets because they must account for three components: the degree of membership μ, the degree of non-membership ν and the hesitancy degree π. For the case of IFSs the axiomatic definitions of a distance (metric) are described as follows:

Definition 3. *A distance (metric)* d *in an intuitionistic fuzzy set A in a universe of discourse X is a real function* $d : IFS(X) \times IFS(X) \to \mathbb{R}$*, which satisfies the following conditions for* $A, B, C \in IFS(X)$*:*

(d1) $d(A,B) \geq 0$ *(non-negativity),*

(d2) $d(A,B) = 0 \Leftrightarrow A = B$ *(coincidence),*

(d3) $d(A,B) = d(B,A)$ *(symmetry),*

(d4) $d(A,B) + d(B,C) \geq d(A,C)$ *(triangle inequality).*

In the literature (see [15, 17]) different measures can be found, namely Type 1-Distance measures (based on the Hamming distance, normalized Hamming distance, Euclidean distance, normalized Euclidean distance) and Type 2-Distance measures (based on fuzzy implications). These approaches have been thoroughly developed by Szmidt [16].

Analogously, for the similarity measure. A similarity measure S for Intuitionistic Fuzzy Sets (IFS) quantifies the degree of closeness between two sets, typically ranging from 0 (completely different) to 1 (identical). It is mathematically related to distance, but it specifically emphasizes the overlap of membership and non-membership degrees. To be valid, a similarity measure must satisfy axioms of symmetry, boundedness, and the condition that $S(A,B) = 1$ if and only if $A = B$. Popular methods include cosine similarity, which treats IFSs as vectors in a 3D space (μ, ν, π) and the set-theoretic approach based on the ratio of intersection to union. These measures are crucial in pattern recognition and medical diagnosis, where they help identify which known category most closely matches a new, uncertain observation.

Definition 4. *A similarity measure S in an intuitionistic fuzzy set A in a universe of discourse X is a real function $S : IFS(X) \times IFS(X) \to \mathbb{R}$, which satisfies the following conditions for $A, B, C \in IFS(X)$:*

(S1) $0 \leq S(A,B) \leq 1$,

(S2) $S(A,B) = 1$ *if and only if* $A = B$,

(S3) $S(A,B) = S(B,A)$,

(S4) *if* $A \subseteq B \subseteq C$, *then* $S(A,C) \leq S(A,B)$ *and* $S(A,C) \leq S(B,C)$.

We present another concept of divergence measure originally based on fuzzy sets. Some generalizations of the divergence measure between two intuitionistic fuzzy sets were presented in a similar way (see more in [10]).

Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is the finite universe and (X, T_M, S_M) is the triple. We present some examples of IF-divergence measures based on Hamming (D_{HM}) and Hausdorff distance (D_{HD}), respectively:

$$D_{HM}(A,B) = \frac{1}{2n} \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \right),$$

$$D_{HD}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \max \left\{ |\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)| \right\}.$$

An additional theoretical approach was described in [10]. In the next text, we give some computational studies for two applications (namely pattern recognition and decision making) in order to see how our result can differ depending on the triple (X, T, S) and the weighted vector α being used. The results are presented in the next two sections.

3 Rational divergence measures on IFSs

A rational similarity measure is a specific class of similarity functions for Intuitionistic Fuzzy Sets (IFS) that is constructed as a ratio of information overlap to total information. Unlike simple linear distances, these measures are often expressed as fractional functions, such as the ratio between the intersection and the union of two fuzzy sets. One of the most famous examples is the Jaccard-like similarity, which measures how much “common ground” two IFSs share relative to their combined membership and non-membership degrees. These measures are called “rational” because they maintain a proportional relationship between the components, ensuring that small changes in membership values do not lead to disproportionately large jumps in similarity. They are particularly robust in pattern recognition because they effectively normalize the data, making the comparison independent of the scale of the universe of discourse. In the context of IFSs, a rational measure must simultaneously process the membership μ, non-membership ν and hesitancy π values within its numerator and denominator. One major advantage of these measures is their ability to handle edge cases where one set might be empty or entirely uncertain without producing undefined results. Researchers prefer rational measures in decision-making (like the TOPSIS method) because they provide a more intuitive “percentage of similarity” than standard Euclidean distances. Furthermore, they satisfy the core axioms of symmetry and boundedness, ensuring that the result always stays between 0 and 1. Ultimately, rational similarity measures provide a more nuanced “relative” comparison, which is crucial when dealing with high levels of vague or incomplete information.

Motivated by [6] and [7] we propose a class of rational divergence measures that meets conditions (1)–(3) for the divergence measure. We recall that the binary fuzzy relation R defined on $X \times X$ is a T-equivalence if and only if it satisfies the following properties for each $x, y, z \in X$ and the t-norm T:

(a) reflexivity: $R(x, x) = 1$,
(b) symmetry: $R(x, y) = R(y, x)$,
(c) T-transitivity: $T(R(x, y), R(y, z)) \leq R(x, z)$.

Sometimes, the reflexivity condition can be replaced by a weaker one named local reflexivity: $R(x, x) \geq R(x, y)$ for fixed $x \in X$ and each $y \in X$.

In [6] the following class of rational similarity measures is proposed:

$$S(A, B) = \frac{a_1.\alpha_{A,B} + b_1.\beta_{A,B} + c_1.\gamma_{A,B} + d_1.\delta_{A,B}}{a_2.\alpha_{A,B} + b_2.\beta_{A,B} + c_2.\gamma_{A,B} + d_2.\delta_{A,B}},$$

where:

$$\begin{aligned}
\alpha_{A,B} &= \min\{|A \setminus B|, |B \setminus A|\}, \\
\beta_{A,B} &= \max\{|A \setminus B|, |B \setminus A|\}, \\
\gamma_{A,B} &= |A \cap B|, \\
\delta_{A,B} &= |(A \cup B)^c|,
\end{aligned}$$

and $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \{0, 1\}$.

Reflexive similarity measures can be identified by condition $c_1 = c_2, d_1 = d_2$. Some of them have some special properties. The similarity measure R is self-complementary if and only if $c_1 = d_1$ and $c_2 = d_2$. For each similarity measure, we can create a complementary similarity measure R^c by changing the coefficients $c_1 \leftrightarrow d_1$ and $c_2 \leftrightarrow d_2$, respectively, where $R^c(A, B) = R(A^c, B^c)$. $R^c = R$ if and only if R is self-complementary. The most famous examples are the Jaccard coefficient $R(A, B) = \frac{|A \cap B|}{|A \cup B|}$ or the Simple matching coefficient $R(A, B) = \frac{|(A \Delta B)^c|}{n}$.

Finally, 28 similarity measures were verified and 13 of them are T-transitive for at least one t-norm $T \in \{T_D, T_L, T_P, T_M\}$ (including also nonreflexive and complementary similarity measures) according to [6]. For all 9 reflexive T-transitive similarity measures which are also T-equivalences were selected. The reflexive T-transitive measures discussed in [2] and [6] are suitable candidates for fuzzification.

We have studied all 9 rational similarity measures that are also T-equivalences. All of them have been changed so that the conditions (1) – (3) for the divergence measure are fulfilled. In the following text, we introduce 9 rational measures $d(A, B)$ derived from T-equivalences, which will be fuzzified in the second step. The explicit expressions of these measures based on cardinalities are the following:

(1) $d_1(A, B) = \frac{\max\{|A \setminus B|, |B \setminus A|\}}{|A \cap B|}$, where $A \cap B \neq \varnothing$,

(2) $d_2(A, B) = \frac{\max\{|A \setminus B|, |B \setminus A|\}}{|(A \Delta B)^c|}$, where $(A \Delta B)^c \neq \varnothing$,

(3) $d_3(A, B) = \frac{|A \Delta B|}{|A \cap B|}$, where $A \cap B \neq \varnothing$,

(4) $d_4(A, B) = \frac{|A \Delta B|}{|(A \Delta B)^c|}$, where $(A \Delta B)^c \neq \varnothing$,

(5) $d_5(A, B) = \frac{\max\{|A|, |B|\} - \min\{|A|, |B|\}}{\min\{|A|, |B|\}}$, where $A \neq \varnothing$ and $B \neq \varnothing$,

(6) $d_6(A, B) = \frac{\max\{|A \setminus B|, |B \setminus A|\}}{\min\{|A \setminus B|, |B \setminus A|\}}$, where $A \setminus B \neq \varnothing$ and $B \setminus A \neq \varnothing$,

(7) $d_7(A, B) = \frac{\max\{|A \setminus B|, |B \setminus A|\}}{\min\{|A|, |B|\}}$, where $A \neq \varnothing$ and $B \neq \varnothing$,

(8) $d_8(A, B) = \frac{\max\{|A \setminus B|, |B \setminus A|\}}{\min\{|(A \setminus B)^c|, |(B \setminus A)^c|\}}$, where $A \setminus B \neq \varnothing$ and $B \setminus A \neq \varnothing$,

(9) $d_9(A, B) = 0$.

The rational measures introduced based on the cardinalities quantify the difference between the sets only in the crisp case. There is a question how the measures could be fuzzified. Some of the ideas are described in [2]. For our purposes, we do it the following way. Moreover, only the parametric family of Frank t-norms with parameter p, for which $0 \leq p \leq +\infty$, will be considered:

$$T_p^F(a, b) = \begin{cases} T_M(a, b) = \min\{a, b\}, & \text{if } p = 0, \\ T_P(a, b) = a.b, & \text{if } p = 1, \\ T_L(a, b) = \max\{a + b - 1, 0\}, & \text{if } p = +\infty, \\ \log_p\left(1 + \frac{(p^a - 1).(p^b - 1)}{p - 1}\right), & \text{otherwise.} \end{cases}$$

The measure of a fuzzy set is the form of its $\sum$-count (sigma count) introduced by De Luca and Termini [8] as a simple extension of the concept of cardinality of crisp sets. As mentioned above, intuitionistic fuzzy sets (IFS) can be better modeled by introducing the hesitation part. Here we define the cardinality of an IFS by extending the notion of $\sum$-count stated above.

Definition 5. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be the finite set. For any $A \in IFS(X)$, the sigma-count of A, denoted by the $\sum$-count A be defined by the following formula:*

$$\sum\text{-}count\ A = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n}(1-\nu_A(x_i))\right].$$

Just to simplify the notation used in the next paragraph, for $A \in IFS(X)$ we will use only:

$$|A| = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n}(1-\nu_A(x_i))\right].$$

Let $A, B \in IFS(X)$, let T be a t-norm from the family of Frank t-norms and S be a t-conorm such that T, S are dual. The complement of the set A we denote by A^c. Let $\mu_A(x_i), \mu_B(x_i)$ be the membership values and $\nu_A(x_i), \nu_B(x_i)$ be the non-membership values, respectively, and $|X| = n$. Then we define the following:

(a)

$$|A| = \left[\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} (1-\nu_A(x_i))\right]$$

(b)

$$|A^c| = |X| - |A| = \left[\sum_{i=1}^{n} (1-\mu_A(x_i)), \sum_{i=1}^{n} \nu_A(x_i)\right]$$

(c)

$$|A \cap B| = \left[\sum_{i=1}^{n} T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^{n} (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]$$

(d)

$$|A \cup B| = \left[\sum_{i=1}^{n} S(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^{n} (1 - T(\nu_A(x_i), \nu_B(x_i)))\right]$$

(e)

$$\begin{aligned}|A \setminus B| &= |A| - |A \cap B| \\ &= \left[\sum_{i=1}^{n} (\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^{n} (S(\nu_A(x_i), \nu_B(x_i)) - \nu_A(x_i))\right];\end{aligned}$$

(f)

$$|(A \backslash B)^c| = \left[\sum_{i=1}^{n}(1-\mu_A(x_i)+T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^{n}(1-S(\nu_A(x_i), \nu_B(x_i))+\nu_A(x_i))\right]$$

(g)

$$\begin{aligned}|B \setminus A| &= |B| - |A \cap B| \\ &= \left[\sum_{i=1}^{n}(\mu_B(x_i) - T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^{n}(S(\nu_A(x_i), \nu_B(x_i)) - \nu_B(x_i))\right];\end{aligned}$$

(h)

$$|(B\setminus A)^c| = \left[\sum_{i=1}^{n}(1 - \mu_B(x_i) + T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^{n}(1 - S(\nu_A(x_i), \nu_B(x_i)) + \nu_B(x_i))\right]$$

(i)

$$\begin{aligned}|A\Delta B| &= |A \cup B| - |A \cap B| = |A \setminus B| + |B \setminus A| \\ &= \left[\sum_{i=1}^{n}\ (S(\mu_A(x_i), \mu_B(x_i)) - T(\mu_A(x_i), \mu_B(x_i))), \dots\right] \\ &\quad \left[\dots, \sum_{i=1}^{n}\ (S(\nu_A(x_i), \nu_B(x_i)) - T(\nu_A(x_i), \nu_B(x_i)))\right];\end{aligned}$$

(j)

$$\begin{aligned}|(A\Delta B)^c| &= \left[\sum_{i=1}^{n}\ (1 - S(\mu_A(x_i), \mu_B(x_i)) + T(\mu_A(x_i), \mu_B(x_i))), \dots\right] \\ &\quad \left[\dots, \sum_{i=1}^{n}\ (1 - S(\nu_A(x_i), \nu_B(x_i)) + T(\nu_A(x_i), \nu_B(x_i)))\right].\end{aligned}$$

The rational measure is superior for handling uncertainty because it treats the relationship between variables as a proportional ratio rather than a simple geometric distance. Linear measures (like Hamming) just subtract values. If one person is certain and the other is guessing, the distance is just a number. Rational measures (like Jaccard) use multiplication in the numerator. If one value is high (0.9) and the other is uncertain (0.1), their product is very small (0.09), drastically lowering the similarity. This correctly reflects that the two vectors have very little "common ground." The denominator of a rational measure represents the total information (the union). If both vectors have high uncertainty, the denominator grows, which naturally "dilutes" the similarity. This prevents the system from giving a high similarity score to two vectors just because they are "equally confused." In standard geometric measures, two completely different IFS sets can sometimes result in the same distance. Rational measures are stricter. They ensure that similarity only stays high if the distribution of membership, non-membership, and hesitancy is nearly identical, not just the sum of their differences. In applications like medical diagnosis, uncertainty is a lack of evidence. Rational measures treat high hesitancy as a penalty. This creates a much sharper boundary for decision making, ensuring that a "match" is only confirmed when both vectors share high membership values.

In the following proposition, we give an explicit expression of the divergence measures $D_1 - D_9$, which are adapted from the previous measures $d_1 - d_9$ and also applicable in a fuzzy case.

Proposition 1. *Let (X, T, S) be a triple such that $|X| = n$, $T = T_M$ and $S = S_M$. Then the maps $D_i : IFS(X) \times IFSF(X) \to \mathbb{R}$ for $i \in \{1, \ldots, 9\}$ are divergence measures.*

Moreover, some special cases assigning zero in the denominators of $D_1 - D_8$ for some triple (X, T, S) depending on the parameter p from Frank's family of t-norms T_p^F must be excluded; the complete list of these conditions is shown in Table 1.

(1)

$$\begin{aligned} &D_1(A,B) \\ &= \frac{\left[\max\left(\sum_{i=1}^n (\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^n (\mu_B(x_i) - T(\mu_A(x_i), \mu_B(x_i)))\right), \ldots\right]}{\left[\sum_{i=1}^n T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]} \\ &= \frac{\left[\ldots, \max\left(\sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - \nu_A(x_i)), \sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - \nu_B(x_i))\right)\right]}{\left[\sum_{i=1}^n T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]} \\ &= \frac{\left[\max\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]}{\left[\sum_{i=1}^n T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]} \end{aligned}$$

(2)

$$\begin{aligned} &D_2(A,B) \\ &= \frac{\left[\max\left(\sum_{i=1}^n (\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))), \sum_{i=1}^n (\mu_B(x_i) - T(\mu_A(x_i), \mu_B(x_i)))\right), \ldots\right]}{\left[\sum_{i=1}^n (1 - S(\mu_A(x_i), \mu_B(x_i)) + T(\mu_A(x_i), \mu_B(x_i))), \ldots\right]} \\ &= \frac{\left[\ldots, \max\left(\sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - \nu_A(x_i)), \sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - \nu_B(x_i))\right)\right]}{\left[\ldots, \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)) + T(\nu_A(x_i), \nu_B(x_i)))\right]} \\ &= \frac{\left[\max\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]}{\left[\sum_{i=1}^n (1 - S(\mu_A(x_i), \mu_B(x_i)) + T(\mu_A(x_i), \mu_B(x_i))), \ldots\right]} \\ &= \frac{\left[\max\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]}{\left[\ldots, \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)) + T(\nu_A(x_i), \nu_B(x_i)))\right]} \end{aligned}$$

(3)

$$\begin{aligned} D_3(A,B) &= \frac{\left[\sum_{i=1}^n (S(\mu_A(x_i), \mu_B(x_i)) - T(\mu_A(x_i), \mu_B(x_i))), \ldots\right]}{\left[\sum_{i=1}^n T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]} \\ &= \frac{\left[\ldots, \sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - T(\nu_A(x_i), \nu_B(x_i)))\right]}{\left[\sum_{i=1}^n T(\mu_A(x_i), \mu_B(x_i)), \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)))\right]} \end{aligned}$$

(4)

$$\begin{aligned} D_4(A,B) &= \frac{\left[\sum_{i=1}^n (S(\mu_A(x_i), \mu_B(x_i)) - T(\mu_A(x_i), \mu_B(x_i))), \ldots\right]}{\left[\sum_{i=1}^n (1 - S(\mu_A(x_i), \mu_B(x_i)) + T(\mu_A(x_i), \mu_B(x_i))), \ldots\right]} \\ &= \frac{\left[\ldots, \sum_{i=1}^n (S(\nu_A(x_i), \nu_B(x_i)) - T(\nu_A(x_i), \nu_B(x_i)))\right]}{\left[\ldots, \sum_{i=1}^n (1 - S(\nu_A(x_i), \nu_B(x_i)) + T(\nu_A(x_i), \nu_B(x_i)))\right]} \end{aligned}$$

(5)

$$\begin{aligned} &D_5(A,B) \\ &= \frac{\left[\max\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right) - \min\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \ldots\right]}{\left[\min\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \max\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]} \\ &= \frac{\left[\ldots, \min\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right) - \max\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]}{\left[\min\left(\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\right), \max\left(\sum_{i=1}^n \nu_A(x_i), \sum_{i=1}^n \nu_B(x_i)\right)\right]} \end{aligned}$$

(6)

$$D_6(A,B)$$

$$= \frac{\left[\max\left(\sum_{i=1}^n \ (\mu_A(x_i) - T(\mu_A(x_i),\mu_B(x_i))), \sum_{i=1}^n \ (\mu_B(x_i) - T(\mu_A(x_i),\mu_B(x_i)))\right),\dots\right]}{\left[\min\left(\sum_{i=1}^n \ (\mu_A(x_i) - T(\mu_A(x_i),\mu_B(x_i))), \sum_{i=1}^n \ (\mu_B(x_i) - T(\mu_A(x_i),\mu_B(x_i)))\right),\dots\right]}$$

$$= \frac{\left[\dots,\max\left(\sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_A(x_i)), \sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_B(x_i))\right)\right]}{\left[\dots,\min\left(\sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_A(x_i)), \sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_B(x_i))\right)\right]}$$

$$= \frac{\left[\max\left(\sum_{i=1}^n \ \mu_A(x_i), \sum_{i=1}^n \ \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \ \nu_A(x_i), \sum_{i=1}^n \ \nu_B(x_i)\right)\right]}{\left[\min\left(\sum_{i=1}^n \ \mu_A(x_i), \sum_{i=1}^n \ \mu_B(x_i)\right), \max\left(\sum_{i=1}^n \ \nu_A(x_i), \sum_{i=1}^n \ \nu_B(x_i)\right)\right]}$$

(7)

$$D_7(A,B) = D_6(A,B)$$

(8)

$$D_8(A,B)$$

$$= \frac{\left[\max\left(\sum_{i=1}^n \ (\mu_A(x_i) - T(\mu_A(x_i),\mu_B(x_i))), \sum_{i=1}^n \ (\mu_B(x_i) - T(\mu_A(x_i),\mu_B(x_i)))\right),\dots\right]}{\left[\min\left(\sum_{i=1}^n \ (1-\mu_A(x_i) + T(\mu_A(x_i),\mu_B(x_i))), \sum_{i=1}^n \ (1-\mu_B(x_i) + T(\mu_A(x_i),\mu_B(x_i)))\right),\dots\right]}$$

$$= \frac{\left[\dots,\max\left(\sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_A(x_i)), \sum_{i=1}^n \ (S(\nu_A(x_i),\nu_B(x_i)) - \nu_B(x_i))\right)\right]}{\left[\dots,\min\left(\sum_{i=1}^n \ (1-S(\nu_A(x_i),\nu_B(x_i)) + \nu_A(x_i)), \sum_{i=1}^n \ (1-S(\nu_A(x_i),\nu_B(x_i)) + \nu_B(x_i))\right)\right]}$$

$$= \frac{\left[\max\left(\sum_{i=1}^n \ \mu_A(x_i), \sum_{i=1}^n \ \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \ \nu_A(x_i), \sum_{i=1}^n \ \nu_B(x_i)\right)\right]}{\left[\max\left(\sum_{i=1}^n \ \mu_A(x_i), \sum_{i=1}^n \ \mu_B(x_i)\right), \min\left(\sum_{i=1}^n \ \nu_A(x_i), \sum_{i=1}^n \ \nu_B(x_i)\right)\right]} = 1$$

(9)

$$D_9(A,B) = 0.$$

Table 1. Restriction conditions for rational divergence measures for $p \geq 0$ to be excluded.

	$\boldsymbol{p \geq 0}$
$\boldsymbol{D_1(A,B)}$	$(\forall i)(\min\{\mu_A(x_i),\mu_B(x_i)\} = 0, \max\{\nu_A(x_i),\nu_B(x_i)\} = 1)$
$\boldsymbol{D_2(A,B)}$	$(\forall i)(\min\{\mu_A(x_i),\mu_B(x_i)\} \in \{0,1\}, \max\{\nu_A(x_i),\nu_B(x_i)\} \in \{0,1\})$
$\boldsymbol{D_3(A,B)}$	$(\forall i)(\min\{\mu_A(x_i),\mu_B(x_i)\} = 0, \max\{\nu_A(x_i),\nu_B(x_i)\} = 1)$
$\boldsymbol{D_4(A,B)}$	$(\forall i)(\min\{\mu_A(x_i),\mu_B(x_i)\} \in \{0,1\}, \max\{\nu_A(x_i),\nu_B(x_i)\} \in \{0,1\})$
$\boldsymbol{D_5(A,B)}$	$(\forall i)(\mu_A(x_i) = 0, \nu_A(x_i) = 1)$ *or* $(\forall i)(\mu_B(x_i) = 0, \nu_B(x_i) = 1)$
$\boldsymbol{D_6(A,B)}$	$(\forall i)(\mu_A(x_i) = 0, \nu_A(x_i) = 1)$ *or* $(\forall i)(\mu_B(x_i) = 0, \nu_B(x_i) = 1)$

Proof 1. *We present this proof for the membership values $\mu_A(x_i), \mu_B(x_i), \mu_C(x_i)$, analogously this ideas can be extended also for the non-membership values $\nu_A(x_i), \nu_B(x_i), \nu_C(x_i)$.*
In the first step, we check conditions for which the measures D_i having zero in the denominator, and therefore these IFSs A, B must be excluded. For example, take D_1.

$$\sum_{i=1}^n T_p^F(\mu_A(x_i),\mu_B(x_i)) = 0 \Leftrightarrow T_p^F(\mu_A(x_i),\mu_B(x_i)) = 0 \, \textit{for all } i \in \{1,\dots,n\}\,.$$

- *for* $p = 0$*:*
$$T_p^F = T_M$$
and
$$T_p^F(\mu_A(x_i), \mu_B(x_i)) = 0 \Leftrightarrow \min\{\mu_A(x_i), \mu_B(x_i)\} = 0,$$
- *for* $p = 1$*:*
$$T_p^F = T_P$$
and
$$T_p^F(\mu_A(x_i), \mu_B(x_i)) = 0 \Leftrightarrow \mu_A(x_i) \cdot \mu_B(x_i) = 0 \Leftrightarrow \min\{\mu_A(x_i), \mu_B(x_i)\} = 0,$$
- *for* $0 < p < \infty,\ p \neq 1$*:*
$$T_p^F(\mu_A(x_i), \mu_B(x_i)) = 0 \Leftrightarrow \log_p\left(1 + \frac{(p^{\mu_A(x_i)} - 1).(p^{\mu_B(x_i)} - 1)}{p - 1}\right) = 0$$
$$\Leftrightarrow (p^{\mu_A(x_i)} - 1).(p^{\mu_B(x_i)} - 1) = 0 \Leftrightarrow p^{\mu_A(x_i)} = 1 \text{ or } p^{\mu_B(x_i)} = 1$$
$$\Leftrightarrow \mu_A(x_i) = 0 \text{ or } \mu_B(x_i) = 0 \min\{\mu_A(x_i), \mu_B(x_i)\} = 0,$$
- *for* $p = \infty$*:*
$$T_p^F = T_L$$
and
$$T_p^F(\mu_A(x_i), \mu_B(x_i)) = 0 \Leftrightarrow \max\{\mu_A(x_i) + \mu_B(x_i) - 1, 0\} = 0 \Leftrightarrow \mu_A(x_i) + \mu_B(x_i) \leq 1.$$

The other cases can be done similarly, the results are scheduled in the Table 1.

In the second step, we must verify that the maps $D_1 - D_6$ *are really divergence measures for a triple* (X, T, S)*, where* $T = T_M$ *and* $S = S_M$*. It is evident that* $D_i(A, B) = 0$ *if and only if* $A = B$ *and* $D_i(A, B) = D_i(B, A)$ *for all* $i \in \{1, \ldots, 6\}$*. It remains to show the third condition from Definition 1. We will do it for the maps* D_1 *and* D_5*, the other can be proven in a similar way. Obviously, since the map* D_7 *can be identified with* D_6*,* $D_8 = 1$ *and* $D_9 = 0$*, therefore, we do not consider it in the following ideas.*

To show that the map D_1 *is a divergence we can divide it into six cases.*

(i) If $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$*, then*
$$T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))$$
$$= \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))$$
and
$$T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) = T(\mu_A(x_i), \mu_B(x_i)).$$
Therefore
$$\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}$$
$$= \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.$$

(ii) If $\mu_A(x_i) \leq \mu_C(x_i) < \mu_B(x_i)$*, then*

$$\begin{aligned}
&T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) \\
&= \mu_A(x_i) - T(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i) - \mu_A(x_i) = 0 \\
&= \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))
\end{aligned}$$

and

$$\begin{aligned}
&T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) \\
&= T(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i) = T(\mu_A(x_i), \mu_B(x_i)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))} \\
&= \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.
\end{aligned}$$

(iii) If $\mu_C(x_i) < \mu_A(x_i) \leq \mu_B(x_i)$*, then*

$$\begin{aligned}
&T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) \\
&= \mu_C(x_i) - T(\mu_C(x_i), \mu_C(x_i)) = \mu_C(x_i) - \mu_C(x_i) = 0 \\
&= \mu_A(x_i) - \mu_A(x_i) = \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))
\end{aligned}$$

and

$$\begin{aligned}
&T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) = T(\mu_C(x_i), \mu_C(x_i)) \\
&= \mu_C(x_i) \leq \mu_A(x_i) = T(\mu_A(x_i), \mu_B(x_i)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))} = 0 \\
&= \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.
\end{aligned}$$

(iv) If $\mu_B(x_i) < \mu_A(x_i) \leq \mu_C(x_i)$*, then*

$$\begin{aligned}
&T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) \\
&= \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))
\end{aligned}$$

and

$$T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) = T(\mu_A(x_i), \mu_B(x_i)).$$

Therefore

$$\begin{aligned}
&\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))} \\
&= \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.
\end{aligned}$$

(v) If $\mu_B(x_i) \leq \mu_C(x_i) < \mu_A(x_i)$, then

$$T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))$$
$$= \mu_C(x_i) - T(\mu_C(x_i), \mu_B(x_i)) = \mu_C(x_i) - \mu_B(x_i) \leq \mu_A(x_i) - \mu_B(x_i)$$
$$= \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))$$

and

$$T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))$$
$$= T(\mu_C(x_i), \mu_B(x_i)) = \mu_B(x_i) = T(\mu_A(x_i), \mu_B(x_i)).$$

Therefore

$$\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}$$
$$\leq \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.$$

(vi) If $\mu_C(x_i) < \mu_B(x_i) < \mu_A(x_i)$, then

$$T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))$$
$$= \mu_C(x_i) - T(\mu_C(x_i), \mu_C(x_i)) = \mu_C(x_i) - \mu_C(x_i) = 0$$
$$= \mu_A(x_i) - \mu_A(x_i) = \mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))$$

and

$$T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))) = T(\mu_C(x_i), \mu_C(x_i))$$
$$= \mu_C(x_i) \leq \mu_B(x_i) = T(\mu_A(x_i), \mu_B(x_i)).$$

Therefore

$$\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))} = 0$$
$$= \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}.$$

We have shown the inequality

$$\frac{T(\mu_A(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}$$
$$\leq \frac{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))}$$

in all six cases.
Similarly, the following inequality can be proved:

$$\frac{T(\mu_B(x_i), \mu_C(x_i)) - T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}{T(T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i)))}$$
$$\leq \frac{\mu_B(x_i) - T(\mu_A(x_i), \mu_B(x_i))}{T(\mu_A(x_i), \mu_B(x_i))},$$

hence the following relationship is fulfilled:

$$\frac{\max\left\{T(\mu_A(x_i),\mu_C(x_i)) - T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\dots\right\}}{T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))}$$

$$\leq \frac{\max\left\{\dots,T(\mu_B(x_i),\mu_C(x_i)) - T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))\right\}}{T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))}$$

$$\leq \frac{\max\left\{\mu_A(x_i) - T(\mu_A(x_i),\mu_B(x_i)),\mu_B(x_i) - T(\mu_A(x_i),\mu_B(x_i))\right\}}{T(\mu_A(x_i),\mu_B(x_i))}.$$

Applying the previous result to all n elements of the universal set X we obtain:

$$\frac{\max\left\{\sum_{i=1}^{n}\left(T(\mu_A(x_i),\mu_C(x_i)) - T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))\right),\dots\right\}}{\sum_{i=1}^{n} T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))}$$

$$\leq \frac{\max\left\{\dots,\sum_{i=1}^{n}\left(T(\mu_B(x_i),\mu_C(x_i)) - T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))\right)\right\}}{\sum_{i=1}^{n} T(T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i)))}$$

$$\leq \frac{\max\left\{\sum_{i=1}^{n}\left(\mu_A(x_i) - T(\mu_A(x_i),\mu_B(x_i))\right),\sum_{i=1}^{n}\left(\mu_B(x_i) - T(\mu_A(x_i),\mu_B(x_i))\right)\right\}}{\sum_{i=1}^{n} T(\mu_A(x_i),\mu_B(x_i))}.$$

We have proven that $D_1(A\cap C, B\cap C) \leq D_1(A,B)$. Similarly can be proved the inequality $D_1(A\cup C, B\cup C) \leq D_1(A,B)$. We conclude that the map D_1 is a divergence measure.

Now we are going to prove that the map D_5 is a divergence measure. Without loss of generality we can assume $\mu_B(x_i) \leq \mu_A(x_i)$. Three following cases must be considered: $\mu_B(x_i) \leq \mu_A(x_i) < \mu_C(x_i)$ or $\mu_B(x_i) \leq \mu_C(x_i) \leq \mu_A(x_i)$ or $\mu_C(x_i) < \mu_B(x_i) \leq \mu_A(x_i)$. In all cases we have $|T(\mu_A(x_i),\mu_C(x_i)) - T(\mu_B(x_i),\mu_C(x_i))| \leq |\mu_A(x_i) - \mu_B(x_i)|$ by Example 1. In detail, we have:

- *if $\mu_B(x_i) \leq \mu_A(x_i) < \mu_C(x_i)$, then*

$$|T(\mu_A(x_i),\mu_C(x_i)) - T(\mu_B(x_i),\mu_C(x_i))| = |\mu_A(x_i) - \mu_B(x_i)|$$

  and

$$\min\left\{T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i))\right\} = \min\left\{\mu_A(x_i),\mu_B(x_i)\right\}.$$

  Therefore

$$\frac{|T(\mu_A(x_i),\mu_C(x_i)) - T(\mu_B(x_i),\mu_C(x_i))|}{\min\left\{T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i))\right\}} = \frac{|\mu_A(x_i) - \mu_B(x_i)|}{\min\left\{\mu_A(x_i),\mu_B(x_i)\right\}}.$$

- *if $\mu_B(x_i) \leq \mu_C(x_i) \leq \mu_A(x_i)$, then*

$$|T(\mu_A(x_i),\mu_C(x_i)) - T(\mu_B(x_i),\mu_C(x_i))| = |\mu_C(x_i) - \mu_B(x_i)| \leq |\mu_A(x_i) - \mu_B(x_i)|$$

  and

$$\begin{aligned}\min\left\{T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i))\right\} &= \min\left\{\mu_C(x_i),\mu_B(x_i)\right\}\\ &= \mu_B(x_i) = \min\left\{\mu_A(x_i),\mu_B(x_i)\right\}.\end{aligned}$$

  Therefore,

$$\frac{|T(\mu_A(x_i),\mu_C(x_i)) - T(\mu_B(x_i),\mu_C(x_i))|}{\min\left\{T(\mu_A(x_i),\mu_C(x_i)),T(\mu_B(x_i),\mu_C(x_i))\right\}} \leq \frac{|\mu_A(x_i) - \mu_B(x_i)|}{\min\left\{\mu_A(x_i),\mu_B(x_i)\right\}}.$$

- *if $\mu_C(x_i) < \mu_B(x_i) \leq \mu_A(x_i)$, then*

$$|T(\mu_A(x_i), \mu_C(x_i)) - T(\mu_B(x_i), \mu_C(x_i))| = |\mu_C(x_i) - \mu_C(x_i)| = 0 \leq |\mu_A(x_i) - \mu_B(x_i)|$$

and

$$\begin{aligned}\min\{T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))\} &= \min\{\mu_C(x_i), \mu_C(x_i)\}\\ &= \mu_C(x_i) \leq b_i = \min\{\mu_A(x_i), \mu_B(x_i)\}.\end{aligned}$$

Therefore

$$\frac{|T(\mu_A(x_i), \mu_C(x_i)) - T(\mu_B(x_i), \mu_C(x_i))|}{\min\{T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))\}} = 0 \leq \frac{|\mu_A(x_i) - \mu_B(x_i)|}{\min\{\mu_A(x_i), \mu_B(x_i)\}}.$$

We have shown the inequality

$$\frac{|T(\mu_A(x_i), \mu_C(x_i)) - T(\mu_B(x_i), \mu_C(x_i))|}{\min\{T(\mu_A(x_i), \mu_C(x_i)), T(\mu_B(x_i), \mu_C(x_i))\}} \leq \frac{|\mu_A(x_i) - \mu_B(x_i)|}{\min\{\mu_A(x_i), \mu_B(x_i)\}}$$

in all three cases.

Applying the previous result to all n elements of the universal set X we obtain:

$$\frac{|\sum_{i=1}^n T(\mu_A(x_i), \mu_C(x_i)) - \sum_{i=1}^n T(\mu_B(x_i), \mu_C(x_i))|}{\min\{\sum_{i=1}^n T(\mu_A(x_i), \mu_C(x_i)), \sum_{i=1}^n T(\mu_B(x_i), \mu_C(x_i))\}} \leq \frac{|\sum_{i=1}^n \mu_A(x_i) - \sum_{i=1}^n \mu_B(x_i)|}{\min\{\sum_{i=1}^n \mu_A(x_i), \sum_{i=1}^n \mu_B(x_i)\}}.$$

We have proven that $D_5(A\cap C, B\cap C) \leq D_5(A, B)$. Similarly, the inequality $D_5(A\cup C, B\cup C) \leq D_5(A, B)$ can be proved. We conclude that the map D_5 is a divergence measure.

For the case $A = B = \varnothing$, we define axiomatically $D(A, B) = 0$.

We introduce some tools for comparing the divergences in the following way. Let us suppose that

$$D_1 = \frac{[a_1, b_1]}{[c_1, d_1]},\ D_2 = \frac{[a_2, b_2]}{[c_2, d_2]}$$

then we define:

$$D_1 \leq D_2 \iff \frac{a_1}{c_1} \leq \frac{a_2}{c_2} \text{ and } \frac{b_1}{d_1} \leq \frac{a_2}{c_2}.$$

Let us study some important properties of these divergences focused on the monotonicity and local property.

Proposition 2. *For the divergences $D_1 - D_9$ the following relations are fulfilled for each t-norm $T \in \langle T_L, T_M\rangle$:*

(i) *$D_2 \leq D_4 \leq D_3$ and $D_2 \leq D_1 \leq D_3$, but the divergences D_1 and D_4 are not comparable,*
(ii) *D_5 is not comparable to any other D_i.*

Proof 2. *(i) $T(\mu_A(x_i), \mu_B(x_i)) \geq \mu_A(x_i) + \mu_B(x_i) - 1$ for a Łukasiewicz t-norm T_L, while $T_L(\mu_A(x_i), \mu_B(x_i)) = 0$ if $\mu_A(x_i) + \mu_B(x_i) - 1 < 0$ and $T_L(\mu_A(x_i), \mu_B(x_i)) = \mu_A(x_i) + \mu_B(x_i) - 1$ in the other case. Since $T_L(\mu_A(x_i), \mu_B(x_i)) \geq \mu_A(x_i) + \mu_B(x_i) - 1$ we have $T(\mu_A(x_i), \mu_B(x_i)) \geq \mu_A(x_i) + \mu_B(x_i) - 1$ for each $T \in \langle T_L, T_M \rangle$. It is equivalent to*

$$1 - \mu_A(x_i) - \mu_B(x_i) + 2T(\mu_A(x_i), \mu_B(x_i)) \geq T(\mu_A(x_i), \mu_B(x_i)).$$

If we apply it to all elements of the universal set X, we get

$$\sum_{i=1}^{n} (1 - \mu_A(x_i) - \mu_B(x_i) + 2T(\mu_A(x_i), \mu_B(x_i))) \geq \sum_{i=1}^{n} T(\mu_A(x_i), \mu_B(x_i)).$$

Since the denominator of the divergence D_2 increases and the nominator we keep without change, we get $D_2 \leq D_1$. Similarly, we can show $D_4 \leq D_3$. Since $T(\mu_A(x_i), \mu_B(x_i)) \leq \mu_A(x_i)$ and $T(\mu_A(x_i), \mu_B(x_i)) \leq \mu_B(x_i)$, adding "$+\mu_B(x_i) - 2T(\mu_A(x_i), \mu_B(x_i))$" to the first inequality and "$+\mu_A(x_i) - 2T(\mu_A(x_i), \mu_B(x_i))$" to the second inequality, it follows that

$$\max\{\mu_A(x_i) - T(\mu_A(x_i), \mu_B(x_i)), \mu_B(x_i) - T(\mu_A(x_i), \mu_B(x_i))\}$$
$$\leq \mu_A(x_i) + \mu_B(x_i) - 2T(\mu_A(x_i), \mu_B(x_i)).$$

It shows that $D_1 \leq D_3$ and $D_2 \leq D_4$.

Now we show that the divergences D_1 and D_4 are not comparable. We give one counterexample. Consider the intuitionistic fuzzy sets given as follows:

$$\begin{aligned} A_1 &= \{\langle x, 0.8, 0.1\rangle, \langle y, 1, 0\rangle, \langle z, 0.9, 0.1\rangle\} \\ A_2 &= \{\langle x, 0.2, 0.4\rangle, \langle y, 0.7, 0.2\rangle, \langle z, 1, 0\rangle\} \\ B_1 &= \{\langle x, 0.5, 0.4\rangle, \langle y, 0.6, 0.3\rangle, \langle z, 0.4, 0.3\rangle\} \\ B_2 &= \{\langle x, 0.9, 0\rangle, \langle y, 0, 1\rangle, \langle z, 0.6, 0.1\rangle\}. \end{aligned}$$

We compute the divergences D_1 and D_4 for the minimum t-norm T_M and the maximum t-conorm S_M:

$$D_1(A_1, B_1) = \frac{[2.7, 0.2]}{[1.5, 2]}, \; D_4(A_1, B_1) = \frac{[1.2, 0.8]}{[1.8, 2.2]}$$
$$D_1(A_2, B_2) = \frac{[1.9, 0.6]}{[0.8, 1.5]}, \; D_4(A_2, B_2) = \frac{[1.8, 1.3]}{[1.2, 1.7]}$$

The divergences D_1 and D_4 are not comparable can be concluded.

(ii) Similarly, it can be shown that the divergences D_4 and D_5 are not comparable, also. Consider again the intuitionistic fuzzy sets given in part (i) and for example, computing the divergences D_4 and D_5 for the minimum t-norm T_M and the maximum t-conorm S_M we obtain:

$$D_4(A_1, B_1) = \frac{[1.2, 0.8]}{[1.8, 2.2]}, \; D_5(A_1, B_1) = \frac{[1.2, -0.8]}{[1.5, 1]}$$
$$D_4(A_2, B_2) = \frac{[1.8, 1.3]}{[1.2, 1.7]}, \; D_5(A_2, B_2) = \frac{[0.6, -0.5]}{[1.5, 1.1]}$$

The divergences D_3, D_4 have a local property, since both can be expressed as a sum $\sum_{i=1}^{n} h(\mu_A(x_i), \mu_B(x_i))$, where the function $h : [0,1] \times [0,1] \to \mathbb{R}$ fulfills conditions (1)–(3) from the definition of local divergence. The first and second are trivial, the third follows from the monotonicity of the t-norm T. The other divergences need not be local in general.

4 Conclusion

We give an alternative approach on how to measure the difference of two IFSs.

In our considerations, we have restricted ourselves to the case when the universe X is finite. In case of an infinite universe X we cannot use the concept of cardinality as above. It would be appropriate to find another approach to define a divergence measure that could also be used in the case of an infinite universe X.

A complete characterization of all divergence measures is still an open problem. In future work, we will specify a class of some interesting properties that could be fulfilled by divergence D, namely its local property.

We have considered the value of divergence between IFSs as a $\sum$-count of two scalar cardinalities, i.e., as a pair of real numbers. The question of how this concept should be extended so that the values of divergence could be represented as fuzzy numbers may be a quite interesting problem that requires a new approach.

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