Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy operations and operators: Some observations

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Abstract: Since its inception, the concept of intuitionistic fuzzy sets is getting popularity day by day. In addition to several operations, so many operators are there to excel the development of this thought. Few of such operators have been focused here. The key objective of this paper is to study those operators over intuitionistic fuzzy sets and to explore some major properties on it.
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1 Introduction

The notion of fuzzy set was introduced and developed by Zadeh [12] in 1965. In fuzzy set concept, a membership function is defined to assign each element of the reference system, a real value in the interval $[0, 1]$. The membership value of an element is 1 indicates that the element belong to that class whereas the membership value of an element is 0 indicates that the element does not belong to the class. In fuzzy set theory, the concept of non-membership function and the hesitation margin are overlooked. In 1983, Atanassov [1] rectified this concept and presented a new conception namely intuitionistic fuzzy sets as an extension of fuzzy sets accommodating both membership and non-membership functions along with hesitation margin. It is to be noted that in intuitionistic fuzzy set theory, the sum of the membership function and non-membership function is a value between 0 and 1. Moreover the value of hesitation margin also lies between 0 and 1. The notion of modal operators were first introduced by Atanassov [2] in 1986. Modal operators $(\Box, \Diamond)$ defined over the set of all intuitionistic fuzzy sets that convert every intuitionistic fuzzy set into a fuzzy set. Many mathematicians and researchers [5–11] are working hard to develop and enrich this concept.

The operators $D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}$ and $H^*_{\alpha,\beta}$ are also defined by Atanassov [4]. In this paper, we try to investigate various properties of these operators using modal operators.

2 Preliminaries

Throughout this paper, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS, respectively.

Definition 2.1 [12] Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A : x \to [0,1]$ is the membership function of the fuzzy set A. Fuzzy set is a collection of objects with graded membership i.e. having degrees of membership.

Definition 2.2 [2] Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A, \nu_A : x \to [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$,$0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of $x \in A$. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ that is $\pi_A : x \to [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$. In other words, $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 2.3 [2] Let A, B be two IFSs in X. The basic operations are defined as follows:

1. $A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$.
2. $A = B \iff \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$.
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.
4. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
5. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
6. $A \oplus B = \{\langle x, (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)), \nu_A(x)\nu_B(x)\rangle : x \in X\}$.
7. $A \otimes B = \{\langle x, \mu_A(x)\mu_B(x), (\nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)),\rangle : x \in X\}$.
8. $A - B = \{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle : x \in X\}$.
9. $A \triangle B = \{\langle x, \max[\min(\mu_A(x), \nu_B(x)), \min(\mu_B(x), \nu_A(x))],$
   $\min[\max(\nu_A(x), \mu_B(x)), \max(\nu_B(x), \mu_A(x))]\rangle : x \in X\}$.
10. $A \times B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle : x \in X\}$.

Definition 2.4 [3] Let A, B and C be IFSs in X. The algebraic laws are as follows:

1. $(A^c)^c = A$. [Complementary law]
2. $(i)A \cup A = A, (ii)A \cap A = A$. [Idempotent law]
3. $(i)A \cup B = B \cup A, (ii)A \cap B = B \cap A$. [Commutative law]
4. $(i)(A \cup B) \cup C = A \cup (B \cup C),\ (ii)(A \cap B) \cap C = A \cap (B \cap C)$. [Associative laws]
5. $(i)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
   $(ii)A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [Distributive laws]
6. $(i)(A \cup B)^c = A^c \cap B^c, (ii)(A \cap B)^c = A^c \cup B^c$. [De Morgan's laws]
7. $(i)A \cap (A \cup B) = A, (ii)A \cup (A \cap B) = A$. [Absorption laws]
8. $(i)A \oplus B = B \oplus A, (ii)A \otimes B = B \otimes A$.
9. $(i)A \oplus (B \oplus C) = (A \oplus B) \oplus C, \quad (ii)A \otimes (B \otimes C) = (A \otimes B) \otimes C$.
10. $(i)(A \oplus B)^c = A^c \otimes B^c, (ii)(A \otimes B)^c = A^c \oplus B^c$.
11. $(i)A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C), \quad (ii)A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$.
12. $(i)A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C), \quad (ii)A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$.

Distributive laws hold for both right and left distributions.

Definition 2.5 Let A and B be two IFSs in a nonempty set X. Then

1. (see [8]) $A \ominus B = \{\langle x, \frac{1}{2}[\mu_A(x) + \mu_B(x)], \frac{1}{2}[\nu_A(x) + \nu_B(x)]\rangle : x \in X\}$;
2. (see [4]) $A\$B = \{\langle x, (\mu_A(x).\mu_B(x))^{\frac{1}{2}}, (\nu_A(x).\nu_B(x))^{\frac{1}{2}}\rangle : x \in X\}$.

Definition 2.6 [4] Let X be a nonempty set. If A is an IFS drawn from X, then,

1. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;
2. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$.

For a proper IFS, $\Box A \subset A \subset \Diamond A$ and $\Box A \neq A \neq \Diamond A$.

Definition 2.7 [4] Let $\alpha, \beta \in [0, 1]$ and $A \in IFSX$. Then the following operators can be defined as

1. $D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x)\rangle : x \in X\}$.
2. $F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$..
3. $G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.
4. $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.
5. $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

3 Main results

Theorem 3.1. *Let X be a nonempty set. If A and B are any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[\Box\Diamond(D_\alpha(A \cup B))]^C = \Diamond\Box[D_\alpha(A \cup B)]^C$;
2. $[\Diamond\Box(D_\alpha(A \cup B))]^C = \Box\Diamond[D_\alpha(A \cup B)]^C$;
3. $[\Box\Diamond(D_\alpha(A \cap B))]^C = \Diamond\Box[D_\alpha(A \cap B)]^C$;
4. $[\Diamond\Box(D_\alpha(A \cap B))]^C = \Box\Diamond[D_\alpha(A \cap B)]^C$;
5. $[\Box\Diamond(F_{\alpha,\beta}(A \cup B))]^C = \Diamond\Box[F_{\alpha,\beta}(A \cup B)]^C$;
6. $[\Diamond\Box(F_{\alpha,\beta}(A \cup B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \cup B)]^C$;
7. $[\Box\Diamond(F_{\alpha,\beta}(A \cap B))]^C = \Diamond\Box[F_{\alpha,\beta}(A \cap B)]^C$;
8. $[\Diamond\Box(F_{\alpha,\beta}(A \cap B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \cap B)]^C$;
9. $[\Box\Diamond(G_{\alpha,\beta}(A \cup B))]^C = \Diamond\Box[G_{\alpha,\beta}(A \cup B)]^C$;
10. $[\Diamond\Box(G_{\alpha,\beta}(A \cup B))]^C = \Box\Diamond[G_{\alpha,\beta}(A \cup B)]^C$;
11. $[\Box\Diamond(G_{\alpha,\beta}(A \cap B))]^C = \Diamond\Box[G_{\alpha,\beta}(A \cap B)]^C$;
12. $[\Diamond\Box(G_{\alpha,\beta}(A \cap B))]^C = \Box\Diamond[G_{\alpha,\beta}(A \cap B)]^C$;
13. $[\Box\Diamond(H_{\alpha,\beta}(A \cup B))]^C = \Diamond\Box[H_{\alpha,\beta}(A \cup B)]^C$;
14. $[\Diamond\Box(H_{\alpha,\beta}(A \cup B))]^C = \Box\Diamond[H_{\alpha,\beta}(A \cup B)]^C$;
15. $[\Box\Diamond(H_{\alpha,\beta}(A \cap B))]^C = \Diamond\Box[H_{\alpha,\beta}(A \cap B)]^C$;
16. $[\Diamond\Box(H_{\alpha,\beta}(A \cap B))]^C = \Box\Diamond[H_{\alpha,\beta}(A \cap B)]^C$;
17. $[\Box\Diamond(H^*_{\alpha,\beta}(A \cup B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A \cup B)]^C$;
18. $[\Diamond\Box(H^*_{\alpha,\beta}(A \cup B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A \cup B)]^C$;
19. $[\Box\Diamond(H^*_{\alpha,\beta}(A \cap B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A \cap B)]^C$;
20. $[\Diamond\Box(H^*_{\alpha,\beta}(A \cap B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A \cap B)]^C$.

Proof. We will prove the cases (1) and (6) in detail.
Proof of (1). Now

$$
\begin{aligned}
(A \cup B) &= \langle \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle \\
D_\alpha(A \cup B) &= \langle \mu_{A \cup B}(x) + \alpha\pi_{A \cup B}(x), \nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x) \rangle \\
\Diamond D_\alpha(A \cup B) &= \langle 1 - (\nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x)), \nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x) \rangle \\
[\Box\Diamond D_\alpha(A \cup B)] &= \langle 1 - (\nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x)), (\nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x)) \rangle \\
[\Box\Diamond D_\alpha(A \cup B)]^C &= \langle (\nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x)), 1 - (\nu_{A \cup B}(x) + (1-\alpha)\pi_{A \cup B}(x)) \rangle
\end{aligned}
$$

Again,

$$[D_\alpha(A \cup B)]^C = \langle \nu_{A\cup B}(x) + (1-\alpha)\pi_{A\cup B}(x), \mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x)\rangle$$
$$\Box[D_\alpha(A \cup B)]^C = \langle \nu_{A\cup B}(x) + (1-\alpha)\pi_{A\cup B}(x), 1 - (\nu_{A\cup B}(x) + (1-\alpha)\pi_{A\cup B}(x))\rangle$$
$$\Diamond\Box[D_\alpha(A \cup B)]^C = \langle \nu_{A\cup B}(x) + (1-\alpha)\pi_{A\cup B}(x), 1 - (\nu_{A\cup B}(x) + (1-\alpha)\pi_{A\cup B}(x))\rangle.$$

Hence,

$$[\Box\Diamond(D_\alpha(A \cup B))]^C = \Diamond\Box[D_\alpha(A \cup B)]^C.$$

Similarly, Statements (2) to (5) can be proved.

Proof of (6). Let us have $\alpha + \beta \leq 1$.
Now,

$$F_{\alpha,\beta}(A \cup B) = \langle \mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x), \nu_{A\cup B}(x) + \beta\pi_{A\cup B}(x)\rangle.$$
$$\Box(F_{\alpha,\beta}(A \cup B)) = \langle \mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x), 1 - (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x))\rangle.$$
$$\Diamond\Box(F_{\alpha,\beta}(A \cup B)) = \langle \mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x), 1 - (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x))\rangle.$$
$$[\Diamond\Box(F_{\alpha,\beta}(A \cup B))]^C = \langle 1 - (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x)), (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x))\rangle.$$

Again,

$$[(F_{\alpha,\beta}(A \cup B))]^C = \langle \nu_{A\cup B}(x) + \beta\pi_{A\cup B}(x), \mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x)\rangle.$$
$$\Diamond[(F_{\alpha,\beta}(A \cup B))]^C = \langle 1 - (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x)), (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x))\rangle.$$
$$\Box\Diamond[(F_{\alpha,\beta}(A \cup B))]^C = \langle 1 - (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x)), (\mu_{A\cup B}(x) + \alpha\pi_{A\cup B}(x))\rangle.$$

Hence

$$[\Diamond\Box(F_{\alpha,\beta}(A \cup B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \cup B)]^C.$$

Similarly to (6), Statements (7) to (20) can be proved. □

Theorem 3.2 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[\Box\Diamond(D_\alpha(A \oplus B))]^C = \Diamond\Box[D_\alpha(A \oplus B)]^C$;
2. $[\Diamond\Box(D_\alpha(A \oplus B))]^C = \Box\Diamond[D_\alpha(A \oplus B)]^C$;
3. $[\Box\Diamond(D_\alpha(A \otimes B))]^C = \Diamond\Box[D_\alpha(A \otimes B)]^C$;
4. $[\Diamond\Box(D_\alpha(A \otimes B))]^C = \Box\Diamond[D_\alpha(A \otimes B)]^C$;
5. $[\Box\Diamond(F_{\alpha,\beta}(A \oplus B))]^C = \Diamond\Box[F_{\alpha,\beta}(A \oplus B)]^C$;
6. $[\Diamond\Box(F_{\alpha,\beta}(A \oplus B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \oplus B)]^C$;
7. $[\Box\Diamond(F_{\alpha,\beta}(A \otimes B))]^C = \Diamond\Box[F_{\alpha,\beta}(A \otimes B)]^C$;
8. $[\Diamond\Box(F_{\alpha,\beta}(A \otimes B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \otimes B)]^C$;

9. $[\Box\Diamond(G_{\alpha,\beta}(A \oplus B))]^C = \Diamond\Box[G_{\alpha,\beta}(A \oplus B)]^C$;

10. $[\Diamond\Box(G_{\alpha,\beta}(A \oplus B))]^C = \Box\Diamond[G_{\alpha,\beta}(A \oplus B)]^C$;

11. $[\Box\Diamond(G_{\alpha,\beta}(A \otimes B))]^C = \Diamond\Box[G_{\alpha,\beta}(A \otimes B)]^C$;

12. $[\Diamond\Box(G_{\alpha,\beta}(A \otimes B))]^C = \Box\Diamond[G_{\alpha,\beta}(A \otimes B)]^C$;

13. $[\Box\Diamond(H_{\alpha,\beta}(A \oplus B))]^C = \Diamond\Box[H_{\alpha,\beta}(A \oplus B)]^C$;

14. $[\Diamond\Box(H_{\alpha,\beta}(A \oplus B))]^C = \Box\Diamond[H_{\alpha,\beta}(A \oplus B)]^C$;

15. $[\Box\Diamond(H_{\alpha,\beta}(A \otimes B))]^C = \Diamond\Box[H_{\alpha,\beta}(A \otimes B)]^C$;

16. $[\Diamond\Box(H_{\alpha,\beta}(A \otimes B))]^C = \Box\Diamond[H_{\alpha,\beta}(A \otimes B)]^C$;

17. $[\Box\Diamond(H^*_{\alpha,\beta}(A \oplus B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A \oplus B)]^C$;

18. $[\Diamond\Box(H^*_{\alpha,\beta}(A \oplus B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A \oplus B)]^C$;

19. $[\Box\Diamond(H^*_{\alpha,\beta}(A \otimes B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A \otimes B)]^C$;

20. $[\Diamond\Box(H^*_{\alpha,\beta}(A \otimes B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A \otimes B)]^C$.

Proof. Similar to Theorem 3.1. $\Box$

Theorem 3.3 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[\Box\Diamond(D_\alpha(A - B))]^C = \Diamond\Box[D_\alpha(A - B)]^C$;

2. $[\Diamond\Box(D_\alpha(A - B))]^C = \Box\Diamond[D_\alpha(A - B)]^C$;

3. $[\Box\Diamond(D_\alpha(A\Delta B))]^C = \Diamond\Box[D_\alpha(A\Delta B)]^C$;

4. $[\Diamond\Box(D_\alpha(A\Delta B))]^C = \Box\Diamond[D_\alpha(A\Delta B)]^C$;

5. $[\Box\Diamond(F_{\alpha,\beta}(A - B))]^C = \Diamond\Box[F_{\alpha,\beta}(A - B)]^C$;

6. $[\Diamond\Box(F_{\alpha,\beta}(A - B))]^C = \Box\Diamond[F_{\alpha,\beta}(A - B)]^C$;

7. $[\Box\Diamond(F_{\alpha,\beta}(A\Delta B))]^C = \Diamond\Box[F_{\alpha,\beta}(A\Delta B)]^C$;

8. $[\Diamond\Box(F_{\alpha,\beta}(A\Delta B))]^C = \Box\Diamond[F_{\alpha,\beta}(A\Delta B)]^C$;

9. $[\Box\Diamond(G_{\alpha,\beta}(A - B))]^C = \Diamond\Box[G_{\alpha,\beta}(A - B)]^C$;

10. $[\Diamond\Box(G_{\alpha,\beta}(A - B))]^C = \Box\Diamond[G_{\alpha,\beta}(A - B)]^C$;

11. $[\Box\Diamond(G_{\alpha,\beta}(A\Delta B))]^C = \Diamond\Box[G_{\alpha,\beta}(A\Delta B)]^C$;

12. $[\Diamond\Box(G_{\alpha,\beta}(A\Delta B))]^C = \Box\Diamond[G_{\alpha,\beta}(A\Delta B)]^C$;

13. $[\Box\Diamond(H_{\alpha,\beta}(A-B))]^C = \Diamond\Box[H_{\alpha,\beta}(A-B)]^C$;

14. $[\Diamond\Box(H_{\alpha,\beta}(A-B))]^C = \Box\Diamond[H_{\alpha,\beta}(A-B)]^C$;

15. $[\Box\Diamond(H_{\alpha,\beta}(A\Delta B))]^C = \Diamond\Box[H_{\alpha,\beta}(A\Delta B)]^C$;

16. $[\Diamond\Box(H_{\alpha,\beta}(A\Delta B))]^C = \Box\Diamond[H_{\alpha,\beta}(A\Delta B)]^C$;

17. $[\Box\Diamond(H^*_{\alpha,\beta}(A-B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A-B)]^C$;

18. $[\Diamond\Box(H^*_{\alpha,\beta}(A-B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A-B)]^C$;

19. $[\Box\Diamond(H^*_{\alpha,\beta}(A\Delta B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A\Delta B)]^C$;

20. $[\Diamond\Box(H^*_{\alpha,\beta}(A\Delta B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A\Delta B)]^C$.

Proof. Similar to Theorem 3.1. $\Box$

Theorem 3.4 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[\Box\Diamond(D_\alpha(A \ominus B))]^C = \Diamond\Box[D_\alpha(A \ominus B)]^C$;

2. $[\Diamond\Box(D_\alpha(A \ominus B))]^C = \Box\Diamond[D_\alpha(A \ominus B)]^C$;

3. $[\Box\Diamond(D_\alpha(A\$B))]^C = \Diamond\Box[D_\alpha(A\$B)]^C$;

4. $[\Diamond\Box(D_\alpha(A\$B))]^C = \Box\Diamond[D_\alpha(A\$B)]^C$;

5. $[\Box\Diamond(F_{\alpha,\beta}(A \ominus B))]^C = \Diamond\Box[F_{\alpha,\beta}(A \ominus B)]^C$;

6. $[\Diamond\Box(F_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond[F_{\alpha,\beta}(A \ominus B)]^C$;

7. $[\Box\Diamond(F_{\alpha,\beta}(A\$B))]^C = \Diamond\Box[F_{\alpha,\beta}(A\$B)]^C$;

8. $[\Diamond\Box(F_{\alpha,\beta}(A\$B))]^C = \Box\Diamond[F_{\alpha,\beta}(A\$B)]^C$;

9. $[\Box\Diamond(G_{\alpha,\beta}(A \ominus B))]^C = \Diamond\Box[G_{\alpha,\beta}(A \ominus B)]^C$;

10. $[\Diamond\Box(G_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond[G_{\alpha,\beta}(A \ominus B)]^C$;

11. $[\Box\Diamond(G_{\alpha,\beta}(A\$B))]^C = \Diamond\Box[G_{\alpha,\beta}(A\$B)]^C$;

12. $[\Diamond\Box(G_{\alpha,\beta}(A\$B))]^C = \Box\Diamond[G_{\alpha,\beta}(A\$B)]^C$;

13. $[\Box\Diamond(H_{\alpha,\beta}(A \ominus B))]^C = \Diamond\Box[H_{\alpha,\beta}(A \ominus B)]^C$;

14. $[\Diamond\Box(H_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond[H_{\alpha,\beta}(A \ominus B)]^C$;

15. $[\Box\Diamond(H_{\alpha,\beta}(A\$B))]^C = \Diamond\Box[H_{\alpha,\beta}(A\$B)]^C$;

16. $[\Diamond\Box(H_{\alpha,\beta}(A\$B))]^C = \Box\Diamond[H_{\alpha,\beta}(A\$B)]^C$;

17. $[\Box\Diamond(H^*_{\alpha,\beta}(A \ominus B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A \ominus B)]^C$;

18. $[\Diamond\Box(H^*_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A \ominus B)]^C$;

19. $[\Box\Diamond(H^*_{\alpha,\beta}(A\$B))]^C = \Diamond\Box[H^*_{\alpha,\beta}(A\$B)]^C$;

20. $[\Diamond\Box(H^*_{\alpha,\beta}(A\$B))]^C = \Box\Diamond[H^*_{\alpha,\beta}(A\$B)]^C$.

Proof. We will prove Statement (1).
Now

$$\begin{aligned}
D_\alpha(A \ominus B) &= \langle \mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x), \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)\rangle \\
\Diamond(D_\alpha(A \ominus B)) &= \langle 1 - (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)), \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)\rangle \\
[\Box\Diamond(D_\alpha(A \ominus B))] &= \langle 1 - (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)), (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x))\rangle \\
[\Box\Diamond(D_\alpha(A \ominus B))]^C &= \langle (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)), 1 - (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x))\rangle.
\end{aligned}$$

Again,

$$\begin{aligned}
[D_\alpha(A \ominus B)]^C &= \langle \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x), \mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x)\rangle \\
\Box[D_\alpha(A \ominus B)]^C &= \langle \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x), 1 - (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x))\rangle \\
\Diamond\Box[D_\alpha(A \ominus B)]^C &= \langle \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x), 1 - (\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x))\rangle.
\end{aligned}$$

Hence,

$$[\Box\Diamond(D_\alpha(A \ominus B))]^C = \Diamond\Box[D_\alpha(A \ominus B)]^C.$$

Similarly the other parts of the Theorem, that is Statements (2) to (20), can be proved.

Theorem 3.5 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[(\Box(D_\alpha(A))) \cup (\Diamond(D_\alpha(B)))]^C = (\Diamond[D_\alpha(A)]^C) \cap (\Box[D_\alpha(B)]^C)$;

2. $[(\Diamond(D_\alpha(A))) \cup (\Box(D_\alpha(B)))]^C = (\Box[D_\alpha(A)]^C) \cap (\Diamond[D_\alpha(B)]^C)$;

3. $[(\Box(D_\alpha(A))) \cap (\Diamond(D_\alpha(B)))]^C = (\Diamond[D_\alpha(A)]^C) \cup (\Box[D_\alpha(B)]^C)$;

4. $[(\Diamond(D_\alpha(A))) \cap (\Box(D_\alpha(B)))]^C = (\Box[D_\alpha(A)]^C) \cup (\Diamond[D_\alpha(B)]^C)$;

5. $[(\Box(F_{\alpha,\beta}(A))) \cup (\Diamond(F_{\alpha,\beta}(B)))]^C = (\Diamond[F_{\alpha,\beta}(A)]^C) \cap (\Box[F_{\alpha,\beta}(B)]^C)$;

6. $[(\Diamond(F_{\alpha,\beta}(A))) \cup (\Box(F_{\alpha,\beta}(B)))]^C = (\Box[F_{\alpha,\beta}(A)]^C) \cap (\Diamond[F_{\alpha,\beta}(B)]^C)$;

7. $[(\Box(F_{\alpha,\beta}(A))) \cap (\Diamond(F_{\alpha,\beta}(B)))]^C = (\Diamond[F_{\alpha,\beta}(A)]^C) \cup (\Box[F_{\alpha,\beta}(B)]^C)$;

8. $[(\Diamond(F_{\alpha,\beta}(A))) \cap (\Box(F_{\alpha,\beta}(B)))]^C = (\Box[F_{\alpha,\beta}(A)]^C) \cup (\Diamond[F_{\alpha,\beta}(B)]^C)$;

9. $[(\Box(G_{\alpha,\beta}(A))) \cup (\Diamond(G_{\alpha,\beta}(B)))]^C = (\Diamond[G_{\alpha,\beta}(A)]^C) \cap (\Box[G_{\alpha,\beta}(B)]^C)$;

10. $[(\Diamond(G_{\alpha,\beta}(A))) \cup (\Box(G_{\alpha,\beta}(B)))]^C = (\Box[G_{\alpha,\beta}(A)]^C) \cap (\Diamond[G_{\alpha,\beta}(B)]^C)$;

11. $[(\Box(G_{\alpha,\beta}(A))) \cap (\Diamond(G_{\alpha,\beta}(B)))]^C = (\Diamond[G_{\alpha,\beta}(A)]^C) \cup (\Box[G_{\alpha,\beta}(B)]^C)$;

12. $[(\Diamond(G_{\alpha,\beta}(A))) \cap (\Box(G_{\alpha,\beta}(B)))]^C = (\Box[G_{\alpha,\beta}(A)]^C) \cup (\Diamond[G_{\alpha,\beta}(B)]^C)$;

13. $[(\Box(H_{\alpha,\beta}(A))) \cup (\Diamond(H_{\alpha,\beta}(B)))]^C = (\Diamond[H_{\alpha,\beta}(A)]^C) \cap (\Box[H_{\alpha,\beta}(B)]^C)$;

14. $[(\Diamond(H_{\alpha,\beta}(A))) \cup (\Box(H_{\alpha,\beta}(B)))]^C = (\Box[H_{\alpha,\beta}(A)]^C) \cap (\Diamond[H_{\alpha,\beta}(B)]^C)$;

15. $[(\Box(H_{\alpha,\beta}(A))) \cap (\Diamond(H_{\alpha,\beta}(B)))]^C = (\Diamond[H_{\alpha,\beta}(A)]^C) \cup (\Box[H_{\alpha,\beta}(B)]^C)$;

16. $[(\Diamond(H_{\alpha,\beta}(A))) \cap (\Box(H_{\alpha,\beta}(B)))]^C = (\Box[H_{\alpha,\beta}(A)]^C) \cup (\Diamond[H_{\alpha,\beta}(B)]^C)$;

17. $[(\Box(H^*_{\alpha,\beta}(A))) \cup (\Diamond(H^*_{\alpha,\beta}(B)))]^C = (\Diamond[H^*_{\alpha,\beta}(A)]^C) \cap (\Box[H^*_{\alpha,\beta}(B)]^C)$;

18. $[(\Diamond(H^*_{\alpha,\beta}(A))) \cup (\Box(H^*_{\alpha,\beta}(B)))]^C = (\Box[H^*_{\alpha,\beta}(A)]^C) \cap (\Diamond[H^*_{\alpha,\beta}(B)]^C)$;

19. $[(\Box(H^*_{\alpha,\beta}(A))) \cap (\Diamond(H^*_{\alpha,\beta}(B)))]^C = (\Diamond[H^*_{\alpha,\beta}(A)]^C) \cup (\Box[H^*_{\alpha,\beta}(B)]^C)$;

20. $[(\Diamond(H^*_{\alpha,\beta}(A))) \cap (\Box(H^*_{\alpha,\beta}(B)))]^C = (\Box[H^*_{\alpha,\beta}(A)]^C) \cup (\Diamond[H^*_{\alpha,\beta}(B)]^C)$.

Proof. (1). Now,

$$\begin{aligned}[\Box D_\alpha(A)] \cup [\Diamond D_\alpha(B)] &= \langle \mu_A(x) + \alpha\pi_A(x), 1\text{–}(\mu_A(x) + \alpha\pi_A(x))\rangle \\ &\quad \cup \langle 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)), \nu_B(x) + (1-\alpha)\pi_B(x)\rangle \\ &= \langle \max[\mu_A(x) + \alpha\pi_A(x), 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x))], \\ &\quad \min[1\text{–}(\mu_A(x) + \alpha\pi_A(x)), \nu_B(x) + (1-\alpha)\pi_B(x)\rangle.\end{aligned}$$

So,

$$\begin{aligned}[(\Box(D_\alpha(A)) \cup (\Diamond(D_\alpha(B))]^C = \langle &\min[1\text{–}(\mu_A(x) + \alpha\pi_A(x)), \nu_B(x) + (1-\alpha)\pi_B(x)], \\ &\max[\mu_A(x) + \alpha\pi_A(x), 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x))\rangle.\end{aligned}$$

Again,

$$\begin{aligned}\Diamond[D_\alpha(A)]^C \cap \Box[D_\alpha(B)]^C &= \langle 1\text{–}(\mu_A(x) + \alpha\pi_A(x)), \mu_A(x) + \alpha\pi_A(x)\rangle \\ &\quad \cap \langle \nu_B(x) + (1-\alpha)\pi_B(x), 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x))\rangle \\ &= \langle \min[1\text{–}(\mu_A(x) + \alpha\pi_A(x)), \nu_B(x) + (1-\alpha)\pi_B(x)], \\ &\quad \max[\mu_A(x) + \alpha\pi_A(x), 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x))\rangle.\end{aligned}$$

Therefore,

$$[(\Box(D_\alpha(A))) \cup (\Diamond(D_\alpha(B)))]^C = (\Diamond[D_\alpha(A)]^C) \cap (\Box[D_\alpha(B)]^C).$$

Hence the proof of (1).
Similarly we can prove the other parts of this theorem that is (2) to (20). $\Box$

Theorem 3.6. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then*

1. $[(\Box(D_\alpha(A))) \oplus (\Diamond(D_\alpha(B)))]^C = (\Diamond[D_\alpha(A)]^C) \otimes (\Box[D_\alpha(B)]^C)$;
2. $[(\Diamond(D_\alpha(A))) \oplus (\Box(D_\alpha(B)))]^C = (\Box[D_\alpha(A)]^C) \otimes (\Diamond[D_\alpha(B)]^C)$;
3. $[(\Box(D_\alpha(A))) \otimes (\Diamond(D_\alpha(B)))]^C = (\Diamond[D_\alpha(A)]^C) \oplus (\Box[D_\alpha(B)]^C)$;
4. $[(\Diamond(D_\alpha(A))) \otimes (\Box(D_\alpha(B)))]^C = (\Box[D_\alpha(A)]^C) \oplus (\Diamond[D_\alpha(B)]^C)$;
5. $[(\Box(F_{\alpha,\beta}(A))) \oplus (\Diamond(F_{\alpha,\beta}(B)))]^C = (\Diamond[F_{\alpha,\beta}(A)]^C) \otimes (\Box[F_{\alpha,\beta}(B)]^C)$;
6. $[(\Diamond(F_{\alpha,\beta}(A))) \oplus (\Box(F_{\alpha,\beta}(B)))]^C = (\Box[F_{\alpha,\beta}(A)]^C) \otimes (\Diamond[F_{\alpha,\beta}(B)]^C)$;
7. $[(\Box(F_{\alpha,\beta}(A))) \otimes (\Diamond(F_{\alpha,\beta}(B)))]^C = (\Diamond[F_{\alpha,\beta}(A)]^C) \oplus (\Box[F_{\alpha,\beta}(B)]^C)$;
8. $[(\Diamond(F_{\alpha,\beta}(A))) \otimes (\Box(F_{\alpha,\beta}(B)))]^C = (\Box[F_{\alpha,\beta}(A)]^C) \oplus (\Diamond[F_{\alpha,\beta}(B)]^C)$;
9. $[(\Box(G_{\alpha,\beta}(A))) \oplus (\Diamond(G_{\alpha,\beta}(B)))]^C = (\Diamond[G_{\alpha,\beta}(A)]^C) \otimes (\Box[G_{\alpha,\beta}(B)]^C)$;
10. $[(\Diamond(G_{\alpha,\beta}(A))) \oplus (\Box(G_{\alpha,\beta}(B)))]^C = (\Box[G_{\alpha,\beta}(A)]^C) \otimes (\Diamond[G_{\alpha,\beta}(B)]^C)$;
11. $[(\Box(G_{\alpha,\beta}(A))) \otimes (\Diamond(G_{\alpha,\beta}(B)))]^C = (\Diamond[G_{\alpha,\beta}(A)]^C) \oplus (\Box[G_{\alpha,\beta}(B)]^C)$;
12. $[(\Diamond(G_{\alpha,\beta}(A))) \otimes (\Box(G_{\alpha,\beta}(B)))]^C = (\Box[G_{\alpha,\beta}(A)]^C) \oplus (\Diamond[G_{\alpha,\beta}(B)]^C)$;
13. $[(\Box(H_{\alpha,\beta}(A))) \oplus (\Diamond(H_{\alpha,\beta}(B)))]^C = (\Diamond[H_{\alpha,\beta}(A)]^C) \otimes (\Box[H_{\alpha,\beta}(B)]^C)$;
14. $[(\Diamond(H_{\alpha,\beta}(A))) \oplus (\Box(H_{\alpha,\beta}(B)))]^C = (\Box[H_{\alpha,\beta}(A)]^C) \otimes (\Diamond[H_{\alpha,\beta}(B)]^C)$;
15. $[(\Box(H_{\alpha,\beta}(A))) \otimes (\Diamond(H_{\alpha,\beta}(B)))]^C = (\Diamond[H_{\alpha,\beta}(A)]^C) \oplus (\Box[H_{\alpha,\beta}(B)]^C)$;
16. $[(\Diamond(H_{\alpha,\beta}(A))) \otimes (\Box(H_{\alpha,\beta}(B)))]^C = (\Box[H_{\alpha,\beta}(A)]^C) \oplus (\Diamond[H_{\alpha,\beta}(B)]^C)$;
17. $[(\Box(H^*_{\alpha,\beta}(A))) \oplus (\Diamond(H^*_{\alpha,\beta}(B)))]^C = (\Diamond[H^*_{\alpha,\beta}(A)]^C) \otimes (\Box[H^*_{\alpha,\beta}(B)]^C)$;
18. $[(\Diamond(H^*_{\alpha,\beta}(A))) \oplus (\Box(H^*_{\alpha,\beta}(B)))]^C = (\Box[H^*_{\alpha,\beta}(A)]^C) \otimes (\Diamond[H^*_{\alpha,\beta}(B)]^C)$;
19. $[(\Box(H^*_{\alpha,\beta}(A))) \otimes (\Diamond(H^*_{\alpha,\beta}(B)))]^C = (\Diamond[H^*_{\alpha,\beta}(A)]^C) \oplus (\Box[H^*_{\alpha,\beta}(B)]^C)$;
20. $[(\Diamond(H^*_{\alpha,\beta}(A))) \otimes (\Box(H^*_{\alpha,\beta}(B)))]^C = (\Box[H^*_{\alpha,\beta}(A)]^C) \oplus (\Diamond[H^*_{\alpha,\beta}(B)]^C)$.

Proof. Similar to the Theorem 3.5. $\Box$

Remark 3.7 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then*

1. $(\Box(H^*_{\alpha,\beta}(A))) \oplus (\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A \oplus B)$;
2. $(\Diamond(H^*_{\alpha,\beta}(A))) \oplus (\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A \oplus B)$;

3. $(\Box(H^*_{\alpha,\beta}(A))) \otimes (\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A \otimes B)$;
4. $(\Diamond(H^*_{\alpha,\beta}(A))) \otimes (\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A \otimes B)$;
5. $(\Box(H^*_{\alpha,\beta}(A)))\triangle(\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A\triangle B)$;
6. $(\Diamond(H^*_{\alpha,\beta}(A)))\triangle(\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A\triangle B)$;
7. $(\Box(H^*_{\alpha,\beta}(A))) - (\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A - B)$;
8. $(\Diamond(H^*_{\alpha,\beta}(A))) - (\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A - B)$;
9. $(\Box(H^*_{\alpha,\beta}(A))) \times (\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A \times B)$;
10. $(\Diamond(H^*_{\alpha,\beta}(A))) \times (\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A \times B)$;
11. $(\Box(H^*_{\alpha,\beta}(A)))\$(\Box(H^*_{\alpha,\beta}(B))) \neq \Box H^*_{\alpha,\beta}(A\$B)$;
12. $(\Diamond(H^*_{\alpha,\beta}(A)))\$(\Diamond(H^*_{\alpha,\beta}(B))) \neq \Diamond H^*_{\alpha,\beta}(A\$B)$.

Counterexample. Let us consider an example. Suppose that $A = \langle 0.7, 0.2, 0.1\rangle$ and $B = \langle 0.6, 0.3, 0.1\rangle$. Here, we consider $\alpha = 0.2$ and $\beta = 0.4$. Then

1. $(\Box H^*_{\alpha,\beta}(A)) \oplus (\Box H^*_{\alpha,\beta}(B)) = \langle 0.2432, 0.7568\rangle$ and $\Box H^*_{\alpha,\beta}(A \oplus B) = \langle 0.176, 0.824\rangle$. So,
$$(\Box H^*_{\alpha,\beta}(A)) \oplus (\Box H^*_{\alpha,\beta}(B)) \neq \Box H^*_{\alpha,\beta}(A \oplus B).$$
Similarly we can show that:
2. $(\Diamond H^*_{\alpha,\beta}(A)) \oplus (\Diamond H^*_{\alpha,\beta}(B)) = \langle 0.7531, 0.2468\rangle$ and $\Diamond H^*_{\alpha,\beta}(A \oplus B) = \langle 0.6344, 0.3656\rangle$;
3. $(\Box H^*_{\alpha,\beta}(A)) \otimes (\Box H^*_{\alpha,\beta}(B)) = \langle 0.0168, 0.9832\rangle$ and $\Box H^*_{\alpha,\beta}(A \otimes B) = \langle 0.084, 0.916\rangle$;
4. $(\Diamond H^*_{\alpha,\beta}(A)) \otimes (\Diamond H^*_{\alpha,\beta}(B)) = \langle 0.2508, 0.7592\rangle$ and $\Diamond H^*_{\alpha,\beta}(A \otimes B) = \langle 0.3696, 0.6304\rangle$;
5. $(\Box H^*_{\alpha,\beta}(A))\Delta(\Box H^*_{\alpha,\beta}(B)) = \langle 0.14, 0.86\rangle$ and $\Box H^*_{\alpha,\beta}(A\Delta B) = \langle 0.06, 0.94\rangle$;
6. $(\Diamond H^*_{\alpha,\beta}(A))\Delta(\Diamond H^*_{\alpha,\beta}(B)) = \langle 0.532, 0.468\rangle$ and $\Diamond H^*_{\alpha,\beta}(A\Delta B) = \langle 0.264, 0.736\rangle$;
7. $(\Box H^*_{\alpha,\beta}(A)) - (\Box H^*_{\alpha,\beta}(B)) = \langle 0.14, 0.86\rangle$ and $\Box H^*_{\alpha,\beta}(A - B) = \langle 0.06, 0.94\rangle$;
8. $(\Diamond H^*_{\alpha,\beta}(A)) - (\Diamond H^*_{\alpha,\beta}(B)) = \langle 0..532, 0.468\rangle$ and $\Diamond H^*_{\alpha,\beta}(A - B) = \langle 0.264, 0.736\rangle$;
9. $(\Box H^*_{\alpha,\beta}(A)) \times (\Box H^*_{\alpha,\beta}(B)) = \langle 0.0168, 0.7568\rangle$ and $\Box H^*_{\alpha,\beta}(A \times B) = \langle 0.084, 0.916\rangle$;
10. $(\Diamond H^*_{\alpha,\beta}(A)) \times (\Diamond H^*_{\alpha,\beta}(B)) = \langle 0.2508, 0.2468\rangle$ and $\Diamond H^*_{\alpha,\beta}(A \times B) = \langle 0.5976, 0.4024\rangle$;
11. $(\Box H^*_{\alpha,\beta}(A))\$(\Box H^*_{\alpha,\beta}(B)) = \langle 0.1296, 0.8699\rangle$ and $\Box H^*_{\alpha,\beta}(A\$B) = \langle 0.1296, 0.8704\rangle$;
12. $(\Diamond H^*_{\alpha,\beta}(A))\$(\Diamond H^*_{\alpha,\beta}(B)) = \langle 0.5008, 0.4968\rangle$ and $\Diamond H^*_{\alpha,\beta}(A\$B) = \langle 0.5049, 0.4951\rangle$.

The results of the above Remark 3.7 are also true for the operators $D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}$ and $H_{\alpha,\beta}$ that means if $H^*_{\alpha,\beta}$ is replaced by any of the operators $D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}$ or $H_{\alpha,\beta}$, then the remark stated above is also true.

4 Conclusion

It is well-known that modal operators and other operators are playing a noteworthy role in intuitionistic fuzzy set theory. Here, in this paper, some new equalities are established in intuitionistic fuzzy sets with the help of modal operators. These results will obviously extend a new aspect for developing the concept.

References

[1] Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *VII ITKR's Session, Sofia*.

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[3] Atanassov, K. T. (1994). New operations defined over Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 61(2), 137–142.

[4] Atanassov, K. T. (1997). Some operators on intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy sets*, 3(4), 28–33.

[5] Bhattacharya, J. (2021). Some results on certain properties of intuitionistic fuzzy sets. *Journal of Fuzzy Extension and Applications*, 2(4), 377–387.

[6] Bhattacharya, J. (2022). Several significant equalities on intuitionistic fuzzy operators. *Notes on Intuitionistic Fuzzy sets*, 28(2), 132–148.

[7] Bhattacharya, J. (2023). Some new results on intuitionistic fuzzy operators. *Notes on Intuitionistic Fuzzy sets*, 29(3), 247–260.

[8] Buhaescu, T. (1989). Some observations on intuitionistic fuzzy relations. *Itinerant Seminar on Functional Equations, Approximation and Convexity*, Cluj-Napoca, Romania, 111–118.

[9] Citil, M., & Tugrul, F. (2018). Some new equalities on the intuitionistic fuzzy modal operators. *Sakarya University Journal of Science*, 22(6), 1524–1531.

[10] Kozae, A. M., Elshenawy, A., & Omran, M. (2015). Intuitionistic fuzzy set and its application in selecting specialization: A case study for engineering students. *International Journal of Mathematical Analysis and Applications*, 2(6), 74–78.

[11] Tarsuslu, S., Citil, M., Dimirbas, E., & Aydin, M. (2017). Some modal operators with intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 23(4), 20–28.

[12] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.