Notes on Intuitionistic Fuzzy Sets
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Some new extension principles for intuitionistic fuzzy sets

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Abstract: Intuitionistic fuzzy sets (IFS) are more applicable and reliable to deal with the real life situations under uncertain environment. The extension principle is a vital concept in fuzzy field specially in fuzzy arithmetic. In this article, a pair of extension principles namely minimal extension principle and average extension principle are proposed for IFS with some associated properties. Moreover, some properties of general extension principle for IFSs are explored. Finally, arithmetic operations for IFSs based on the average extension principle are developed with examples.

Keywords: Intuitionistic fuzzy set, Extension principle, Minimal extension principle, Average extension principle, Arithmetic operations.

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1 Introduction

Uncertainty is an important entity in almost all the fields under medical science, engineering, business, economics etc. Fuzzy set theory [23] which was the first concept to deal with uncertainty and takes into account membership degree and the non-membership degree is the complement of the membership degree. However, in real life situation, this linguistic negation does not satisfy the logical negation all the time and there arise some hesitation. To overcome these difficulties, in 1983, K.T. Atanassov [1] developed the concept of intuitionistic fuzzy set

(IFS), where the non-membership degree is not equal to the complement of the membership degree due to the fact that some kind of hesitation or lack of knowledge is present while defining the membership function. So the IFS is the important generalization of fuzzy set theory, where the sum of the membership degree and the non-membership degree is less than or equal to one. Extension principle [24] is one of the most basic concept of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets. The useful application of cut sets are in fuzzy algebra [15, 16], fuzzy reasoning [8,14], fuzzy measure [7, 9, 17, 18] and so on. The cut sets on fuzzy sets are described in [20] by using the neighborhood relations between fuzzy point and fuzzy set. Based on the cut sets, the decomposition theorems and representation theorems can be established [24]. Li [12] and Zou [26] presented the concepts of upper cut sets and lower cut sets of IFSs respectively. The theoretical discussion and applications of IFSs are developed rapidly [5, 6, 10, 11, 13, 19, 20, 21, 22, 25]. In the two books of Atanassov [3, 4] modal operators and a series of their extensions were described.

In this article, we propose two definitions of extension principle for IFS namely minimal extension principle and average extension principle with some properties. Also some properties of the general extension principle for IFS are explored. Finally, the arithmetic operations of IFS based on the average extension principle are also illustrated.

2 Preliminaries

In this section, basic definitions are discussed which will be supported in the rest of the paper.

Definition 2.1. [23] A fuzzy set (FS) A in X is given by $A=\left\{\left(x,\mu_A\left(x\right)\right):x\in X\right\}$, with $\mu_A:X\to\left[0,1\right]$ and $0\le\mu\left(x\right)\le 1;\forall x\in X$.

Definition 2.2. [1, 2] An IFS A in X is defined by $A=\left\{\left(x,\mu_A\left(x\right),\nu_A\left(x\right)\right):x\in X\right\}$, with $\mu_A:X\to\left[0,1\right]$, $\nu_A:X\to\left[0,1\right]$ and $0\le\mu_A\left(x\right)+\nu_A\left(x\right)\le 1;\forall x\in X$ and the hesitant value, $\pi_A\left(x\right)=1-\left(\mu_A\left(x\right)+\nu_A\left(x\right)\right)$.

Definition 2.3. [1, 2, 3] Let $A,B\in IFS\left(X\right)$, then

i. $A\subseteq B$ iff $\forall x\in X,\mu_A\left(x\right)\le\mu_B\left(x\right)$ and $\nu_A\left(x\right)\ge\nu_B\left(x\right)$;

ii. $A=B$ iff $\forall x\in X,\mu_A\left(x\right)=\mu_B\left(x\right)$ and $\nu_A\left(x\right)=\nu_B\left(x\right)$;

iii. $A\cup B=\left\{\left(x,\max\left(\mu_A\left(x\right),\mu_B\left(x\right)\right),\min\left(\nu_A\left(x\right),\nu_B\left(x\right)\right)\right):x\in X\right\}$;

iv. $A\cap B=\left\{\left(x,\min\left(\mu_A\left(x\right),\mu_B\left(x\right)\right),\max\left(\nu_A\left(x\right),\nu_B\left(x\right)\right)\right):x\in X\right\}$;

v. $A^c=\left\{\left(x,\nu_A\left(x\right),\mu_A\left(x\right)\right):x\in X\right\}$.

Definition 2.4. [1] Let, and $\alpha,\beta\in\left[0,1\right]$,$\alpha+\beta\le 1$. Then, $A^{(\alpha,\beta)}=\left\{x:\mu_A\left(x\right)\ge\alpha\ ,\nu_A\left(x\right)\le\beta\right\}$ and $A^{(\alpha,\beta)+}=\left\{x:\mu_A\left(x\right)>\alpha\ ,\nu_A\left(x\right)<\beta\right\}$ are the $\left(\alpha,\beta\right)$-*cut* and the strong $\left(\alpha,\beta\right)$-*cut of A*, respectively.

Definition 2.5. [17] Let, $X \neq \varnothing$ and $Y \neq \varnothing$ and a pair of correspondences are as follows: $\overline{f}: IFS(X) \rightarrow IFS(Y)$ and $\overline{f}^{-1}: IFS(Y) \rightarrow IFS(X)$ defined by

$$\overline{f}(A)(y) = \left(\mu_{\overline{f}(A)}(y), \nu_{\overline{f}(A)}(y)\right),$$

where $A \in IFS(X)$ and

$$\mu_{\overline{f}(A)}(y) = \begin{cases} \vee\left\{\mu_A(x) : x \in \overline{f}^{-1}(y)\right\} & ; \overline{f}^{-1}(y) \neq \varnothing \\ 0 & ; \text{otherwise} \end{cases}$$

$$\nu_{\overline{f}(A)}(y) = \begin{cases} \wedge\left\{\nu_A(x) : x \in \overline{f}^{-1}(y)\right\} & ; \overline{f}^{-1}(y) \neq \varnothing \\ 0 & ; \text{otherwise} \end{cases}.$$

and

$$\overline{f}^{-1}(B)(x) = \left(\mu_{\overline{f}^{-1}(B)}(x), \nu_{\overline{f}^{-1}(B)}(x)\right),$$

where $B \in IFS(Y)$ and $A = \left\{\left(x, \mu_A(x), \nu_A(x)\right) : x \in X\right\} \in IFS(X)$, $\mu_{\overline{f}^{-1}(B)}(x) = \mu_B\left(\overline{f}(x)\right)$ and $\nu_{\overline{f}^{-1}(B)}(x) = \nu_B\left(\overline{f}(x)\right)$.

3 Extension principles for intuitionistic fuzzy sets

In this section, new two extension principles for intuitionistic fuzzy set are established and some associated properties are discussed.

Definition 3.1 (Minimal Extension Principle). Let, $X \neq \varnothing$ and $Y \neq \varnothing$ and a pair of correspondences are as follows: $\underline{f}: IFS(X) \rightarrow IFS(Y)$ and $\underline{f}^{-1}: IFS(Y) \rightarrow IFS(X)$, defined by

$$\underline{f}(A)(y) = \left(\mu_{\underline{f}(A)}(y), \nu_{\underline{f}(A)}(y)\right),$$

where $A \in IFS(X)$ and

$$\mu_{\underline{f}(A)}(y) = \begin{cases} \wedge\left\{\mu_A(x) : x \in \underline{f}^{-1}(y)\right\} & ; \ \underline{f}^{-1}(y) \neq \varnothing \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\nu_{\underline{f}(A)}(y) = \begin{cases} \vee\left\{\nu_A(x) : x \in \underline{f}^{-1}(y)\right\} & ; \ \underline{f}^{-1}(y) \neq \varnothing \\ 0 & ; \ \text{otherwise} \end{cases}.$$

and

$$\underline{f}^{-1}(B)(x) = \left(\mu_{\underline{f}^{-1}(B)}(x), \nu_{\underline{f}^{-1}(B)}(x)\right),$$

where $B \in IFS(Y)$ and $\mu_{\underline{f}^{-1}(B)}(x) = \mu_B\left(\underline{f}(x)\right)$ and $\nu_{\underline{f}^{-1}(B)}(x) = \nu_B\left(\underline{f}(x)\right)$.

Definition 3.2 (Average Extension Principle). Let, $X \neq \varnothing$ and $Y \neq \varnothing$ and a pair of correspondences are as follows:

$$\overline{\overline{f}} : IFS(X) \to IFS(Y) \text{ and } \overline{\overline{f}}^{-1} : IFS(Y) \to IFS(X),$$

defined by $\overline{\overline{f}}(A)(y) = \left(\mu_{\overline{\overline{f}}(A)}(y), \nu_{\overline{\overline{f}}(A)}(y)\right)$, where $A \in IFS(X)$ and

$$\mu_{\overline{\overline{f}}(A)}(y) = \begin{cases} \frac{1}{n}\sum\left\{\mu_A(x) : x \in \overline{\overline{f}}^{-1}(y)\right\}; & \overline{\overline{f}}^{-1}(y) \neq \varnothing \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\nu_{\overline{\overline{f}}(A)}(y) = \begin{cases} \frac{1}{n}\sum\left\{\nu_A(x) : x \in \overline{\overline{f}}^{-1}(y)\right\}; & \overline{\overline{f}}^{-1}(y) \neq \varnothing \\ 0 & ; \text{ otherwise} \end{cases}.$$

and $\overline{\overline{f}}^{-1}(B)(x) = \left(\mu_{\overline{\overline{f}}^{-1}(B)}(x), \nu_{\overline{\overline{f}}^{-1}(B)}(x)\right)$, where $B \in IFS(Y)$ and $\mu_{\overline{\overline{f}}^{-1}(B)}(x) = \mu_B\left(\overline{\overline{f}}(x)\right)$ and $\nu_{\overline{\overline{f}}^{-1}(B)}(x) = \nu_B\left(\overline{\overline{f}}(x)\right)$.

Theorem 3.3. *If $\overline{f} : X \to Y$ and $A, B, A_i \in IFS(X)$, then the following assertions hold:*

1. $A \subseteq B$ *implies* $\overline{f}(A) \subseteq \overline{f}(B)$,
2. $\overline{f}(\cup_{i\in I} A_i) = \left(\cup_{i\in I} \overline{f}(A_i)\right)$,
3. $\overline{f}(\cap_{i\in I} A_i) \subseteq \left(\cap_{i\in I} \overline{f}(A_i)\right)$,
4. $\overline{f}\left(A^{(\alpha,\beta)}\right) \subseteq \left(\overline{f}(A)\right)^{(\alpha,\beta)}$*; equality holds if* $\alpha + \beta = 1$,
5. $\overline{f}\left(A^{(\alpha,\beta)+}\right) = \left(\overline{f}(A)\right)^{(\alpha,\beta)+}$.

Proof.
1. It is trivial.
2. For each $y \in Y$, if $\overline{f}^{-1}(y)$ is not empty, then $\overline{f}(\cup_{i\in I} A_i)(y) = \left(\mu_{\overline{f}(\cup_{i\in I} A_i)}(y), \nu_{\overline{f}(\cap_{i\in I} A_i)}(y)\right)$, where

$$\begin{aligned} \mu_{\overline{f}(\cup_{i\in I} A_i)}(y) &= \vee_{u\in \overline{f}^{-1}(y)} \left\{\mu_{\bigcup_{i\in I} A_i}(x)\right\} \\ &= \vee_{x\in \overline{f}^{-1}(y)} \left\{\vee_{i\in I} \left\{\mu_{A_i}(x)\right\}\right\} \\ &= \vee_{i\in I} \left\{\vee_{x\in \overline{f}^{-1}(y)} \left\{\mu_{A_i}(x)\right\}\right\} \\ &= \vee_{i\in I} \left\{\mu_{\overline{f}(A_i)}(y)\right\} \end{aligned}$$

Therefore, $\mu_{\overline{f}_{(\cup_{i\in I} A_i)}}(y) = \mu_{\bigcup_{i\in I} \overline{f}(A_i)}(y)$.

Similarly,

$$\nu_{\overline{f}(\cap_{i\in I} A_i)}(y) = \underset{x\in \overline{f}^{-1}(y)}{\wedge}\left\{\nu_{\underset{i\in I}{\cap} A_i}(x)\right\}$$
$$= \underset{x\in \overline{f}^{-1}(y)}{\wedge}\left\{\underset{i\in I}{\wedge}\nu_{A_i}(x)\right\}$$
$$= \underset{i\in I}{\wedge}\left\{\underset{x\in \overline{f}^{-1}(y)}{\wedge}\left\{\nu_{A_i}(x)\right\}\right\}$$
$$= \underset{i\in I}{\wedge}\nu_{\overline{f}(A_i)}(y).$$

Therefore, $\nu_{\overline{f}(\cap_{i\in I} A_i)}(y) = \nu_{\underset{i\in I}{\cap}\overline{f}(A_i)}(y)$.

Hence, $\overline{f}(\cup_{i\in I} A_i) = (\cup_{i\in I}\overline{f}(A_i))$.

3. Straightforward.

4. Let, $y\in \overline{f}\left(A^{(\alpha,\beta)}\right)$, if $\overline{f}^{-1}(y)\neq\varnothing$, then there exists $u\in A^{(\alpha,\beta)}$ such that $\overline{f}(x)=y$ and $\mu_A(x)\geq\alpha$ and $\nu_A(x)\leq\beta$.

   From there it follows that $\vee\left\{\mu_A(x): x\in \overline{f}^{-1}(y)\right\}\geq\alpha$ and $\wedge\left\{\nu_A(x): x\in \overline{f}^{-1}(y)\right\}\leq\beta$, i.e., $\mu_{\overline{f}(A)}(y)\geq\alpha$ and $\nu_{\overline{f}(A)}(y)\leq\beta$.

   From there it follows that $y\in\left(\overline{f}(A)\right)^{(\alpha,\beta)}$.

   Therefore, $\overline{f}\left(A^{(\alpha,\beta)}\right)\subseteq\left(\overline{f}(A)\right)^{(\alpha,\beta)}$.

5. For all $y\in Y$,

$$y\in\left(\overline{f}(A)\right)^{(\alpha,\beta)+} \Leftrightarrow \mu_{\overline{f}(A)}(y)>\alpha \text{ and } \nu_{\overline{f}(A)}(y)<\beta$$
$$\Leftrightarrow \underset{x\in \overline{f}^{-1}(y)}{\vee}\mu_A(x)>\alpha \text{ and } \underset{x\in \overline{f}^{-1}(y)}{\wedge}\nu_A(x)<\beta$$
$$\Leftrightarrow (\exists x_0\in X)\left(y=\overline{f}(x_0)\right) \text{ and } \mu_A(x_0)>\alpha, \nu_A(x_0)<\beta$$
$$\Leftrightarrow (\exists x_0\in X)\left(y=\overline{f}(x_0)\right) \text{ and } x_0\in A^{(\alpha,\beta)+}$$
$$\Leftrightarrow x\in \overline{f}\left(A^{(\alpha,\beta)+}\right).$$

   Thus, $\overline{f}\left(A^{(\alpha,\beta)+}\right)=\left(\overline{f}(A)\right)^{(\alpha,\beta)+}$.

This completes the proof. □

Theorem 3.4. *If $\overline{f}: X\to Y$ and $A, B, B_i\in IFS(Y)$, then the following assertions hold:*

1. $A\subseteq B$ *implies* $\overline{f}^{-1}(A)\subseteq\overline{f}^{-1}(B)$,
2. $\overline{f}^{-1}\left(B^c\right)=\left(\overline{f}^{-1}(B)\right)^c$,
3. $\overline{f}^{-1}(\cup_{i\in I} B_i)=\left(\cup_{i\in I}\overline{f}^{-1}(B_i)\right)$,
4. $\overline{f}^{-1}(\cap_{i\in I} B_i)=\left(\cap_{i\in I}\overline{f}^{-1}(B_i)\right)$,

5. $\overline{f}^{-1}\left(B^{(\alpha,\beta)}\right)=\left(\overline{f}^{-1}(B)\right)^{(\alpha,\beta)}$,

6. $\overline{f}^{-1}\left(B^{(\alpha,\beta)+}\right)=\left(\overline{f}^{-1}(B)\right)^{(\alpha,\beta)+}$.

Proof. The proofs of assertions 1 and 2 are trivial.

3. For all $x\in X$, we have $\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)(x)=\left(\mu_{\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)}(x),\nu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x)\right)$, where

$$\begin{aligned}\mu_{\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)}(x)&=\mu_{\bigcup_{i\in I}B_i}\left(\overline{f}(x)\right)\\&=\vee_{i\in I}\left\{\mu_{B_i}\left(\overline{f}(x)\right)\right\}\\&=\vee_{i\in I}\left\{\mu_{\overline{f}^{-1}(B_i)}(x)\right\}\end{aligned}$$

Therefore, $\mu_{\overline{f}^{-1}_{\left(\cup_{i\in I}B_i\right)}}(x)=\mu_{\bigcup_{i\in I}\overline{f}^{-1}(B_i)}(x)$.

Similarly,

$$\begin{aligned}\nu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x)&=\nu_{\bigcap_{i\in I}B_i}\left(\overline{f}(x)\right)\\&=\wedge_{i\in I}\left\{\nu_{B_i}\left(\overline{f}(x)\right)\right\}\\&=\wedge_{i\in I}\left\{\nu_{\overline{f}^{-1}(B_i)}(x)\right\}\end{aligned}$$

Therefore, $\nu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x)=\nu_{\bigcap_{i\in I}\overline{f}^{-1}(B_i)}(x)$.

Hence, $\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)=\left(\cup_{i\in I}\overline{f}^{-1}(B_i)\right)$.

4. For all $\forall x\in X$, we have $\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)(x)=\left(\mu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x),\nu_{\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)}(x)\right)$, where

$$\begin{aligned}\mu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x)&=\mu_{\bigcup_{i\in I}B_i}\left(\overline{f}(x)\right)\\&=\wedge_{i\in I}\left\{\mu_{B_i}\left(f(x)\right)\right\}\\&=\wedge_{i\in I}\left\{\mu_{\overline{f}^{-1}(B_i)}(x)\right\}\end{aligned}$$

Therefore, $\mu_{\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)}(x)=\mu_{\bigcap_{i\in I}\overline{f}^{-1}(B_i)}(x)$.

Respectively,

$$\begin{aligned}\nu_{\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)}(x)&=\nu_{\bigcap_{i\in I}B_i}\left(\overline{f}(x)\right)\\&=\vee_{i\in I}\left\{\nu_{B_i}\left(f(x)\right)\right\}\\&=\vee_{i\in I}\left\{\nu_{\overline{f}^{-1}(B_i)}(x)\right\}\end{aligned}$$

Therefore, $\nu_{\overline{f}^{-1}\left(\cup_{i\in I}B_i\right)}(x)=\nu_{\bigcup_{i\in I}\overline{f}^{-1}(B_i)}(x)$.

Hence, $\overline{f}^{-1}\left(\cap_{i\in I}B_i\right)=\left(\cap_{i\in I}\overline{f}^{-1}(B_i)\right)$.

5. For all $x \in X$, we have,

$$x \in \left(\overline{f}^{-1}(B) \right)^{(\alpha,\beta)} \Leftrightarrow \left\{ x \in X : \mu_{\overline{f}^{-1}(B)}(x) \geq \alpha, \nu_{\overline{f}^{-1}(B)}(x) \leq \beta \right\}$$
$$\Leftrightarrow \left\{ x \in X : \mu_B\left(\overline{f}(x) \right) \geq \alpha, \nu_B\left(\overline{f}(x) \right) \leq \beta \right\}$$
$$\Leftrightarrow \left\{ x \in X : \overline{f}(x) \in B^{(\alpha,\beta)} \right\}.$$

Therefore, $x \in \overline{f}^{-1}\left(B^{(\alpha,\beta)} \right)$.

Thus, $\overline{f}^{-1}\left(B^{(\alpha,\beta)} \right) = \left(\overline{f}^{-1}(B) \right)^{(\alpha,\beta)}$.

6. Trivial.

This completes the proof. □

Theorem 3.5. *If* $\underline{f} : X \to Y$ *and* A, B , $A_i \in IFS(U)$, *then the following assertions hold:*

1. $A \subseteq B$ *implies* $\underline{f}(A) \subseteq \underline{f}(B)$,
2. $\underline{f}(\cap_{i\in I} A_i) = \left(\cap_{i\in I} \underline{f}(A_i) \right)$,
3. $\underline{f}(\cup_{i\in I} A_i) \subseteq \left(\cup_{i\in I} \underline{f}(A_i) \right)$,
4. $\underline{f}\left(A^{(\alpha,\beta)} \right) \subseteq \left(\underline{f}(A) \right)^{(\alpha,\beta)}$,
5. $\underline{f}\left(A^{(\alpha,\beta)+} \right) = \left(\underline{f}(A) \right)^{(\alpha,\beta)+}$.

Proof.

1. Trivial.
2. For all $y \in Y$, if $\underline{f}^{-1} \neq \varnothing$, then $\underline{f}(\cap_{i\in I} A_i)(y) = \left(\mu_{\underline{f}(\cap_{i\in I} A_i)}(y), \nu_{\underline{f}(\cup_{i\in I} A_i)}(y) \right)$, where

$$\mu_{\underline{f}(\cap_{i\in I} A_i)}(y) = \underset{x\in \underline{f}^{-1}(y)}{\wedge} \left\{ \mu_{\cap_{i\in I} A_i}(x) \right\}$$
$$= \underset{x\in \underline{f}^{-1}(y)}{\wedge} \left\{ \underset{i\in I}{\wedge} \left\{ \mu_{A_i}(x) \right\} \right\} \; ;$$
$$= \underset{i\in I}{\wedge} \left\{ \underset{x\in \underline{f}^{-1}(y)}{\wedge} \left\{ \mu_{A_i}(x) \right\} \right\}$$
$$= \underset{i\in I}{\wedge} \left\{ \mu_{\underline{f}(A_i)}(y) \right\} \; .$$

Therefore, $\mu_{\underline{f}(\cap_{i\in I} A_i)}(y) = \mu_{\underline{f}(A_i)}(y)$.

Similarly,

$$\nu_{\underline{f}(\cup_{i\in I} A_i)}(y) = \underset{x\in \underline{f}^{-1}(y)}{\vee} \left\{ \nu_{\cup_{i\in I} A_i}(x) \right\}$$
$$= \underset{x\in \underline{f}^{-1}(y)}{\vee} \left\{ \underset{i\in I}{\vee} \left\{ \nu_{A_i}(x) \right\} \right\} \; ;$$
$$= \underset{i\in I}{\vee} \left\{ \underset{x\in \underline{f}^{-1}(y)}{\vee} \left\{ \nu_{A_i}(x) \right\} \right\}$$
$$= \underset{i\in I}{\vee} \left\{ \left\{ \nu_{\underline{f}(A_i)}(y) \right\} \right\} \; .$$

Therefore, $\nu_{\underline{f}(\cup_{i\in I} A_i)}(y) = \nu_{\cup_{i\in I} \underline{f}(A_i)}(y)$.

Hence, $\underline{f}(\cap_{i\in I} A_i) = (\cap_{i\in I} \underline{f}(A_i))$.

3. Trivial.

4. If $y \in \underline{f}\left(A^{(\alpha,\beta)}\right)$, then there exists $x \in A^{(\alpha,\beta)}$ such that $\underline{f}(x) = y$ and $\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$.

From there it follows that $\wedge\left\{\mu_A(x) : x \in \underline{f}^{-1}(x) \geq \alpha\right\}$ and $\vee\left\{\nu_A(x) : x \in \underline{f}^{-1}(x) \leq \beta\right\}$, i.e., $\mu_{f(A)}(y) \geq \alpha$ and $\nu_{f(A)}(y) \leq \beta$.

From there it follows that $y \in \left(\underline{f}(A)\right)^{(\alpha,\beta)}$.

Therefore, $\underline{f}\left(A^{(\alpha,\beta)}\right) \subseteq \left(\underline{f}(A)\right)^{(\alpha,\beta)}$.

5. If $y \in \left(\underline{f}(A)\right)^{(\alpha,\beta)+}$, then there exists $x \in A^{(\alpha,\beta)+}$ such that $\underline{f}(x) = y$ and $\mu_{\underline{f}(A)}(y) > \alpha$ and $\nu_{\underline{f}(A)}(y) < \beta$.

$$\Leftrightarrow \quad \bigvee_{x\in \underline{f}^{-1}(y)} \mu_A(x) > \alpha \text{ and } \bigwedge_{x\in \underline{f}^{-1}(y)} \nu_A(x) < \beta.$$

$$\Leftrightarrow \quad (\exists x_0 \in X)\left(y = \underline{f}(x_0)\right) \text{ and } \mu_A(x_0) > \alpha \text{ ,} \nu_A(x_0) < \beta$$

$$\Leftrightarrow \quad (\exists x_0 \in X)\left(v = \underline{f}(x_0)\right) \text{ and } x_0 \in A^{(\alpha,\beta)+}$$

$$\Leftrightarrow \quad y \in \underline{f}\left(A^{(\alpha,\beta)+}\right).$$

Thus, $\underline{f}\left(A^{(\alpha,\beta)+}\right) = \left(\underline{f}(A)\right)^{(\alpha,\beta)+}$.

This completes the proof. ☐

Theorem 3.6. *If* $\overline{\overline{f}} : X \to Y$ *and* $A, B, A_i \in IFS(X)$, *then the following assertions hold:*

1. $A \subseteq B$ *implies* $\overline{\overline{f}}(A) \subseteq \overline{\overline{f}}(B)$,
2. $\overline{\overline{f}}\left(\sum_{i\in I} A_i\right) = \left(\sum_{i\in I} \overline{\overline{f}}(A_i)\right)$,
3. $\overline{\overline{f}}\left(A^{(\alpha,\beta)}\right) \subseteq \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)}$,
4. $\overline{\overline{f}}\left(A^{(\alpha,\beta)+}\right) = \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)+}$.

Proof.

1. Trivial.

2. For all $y \in Y$, if $\overline{\overline{f}}^{-1}(y) \neq \varnothing$, then $\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)(y) = \left(\mu_{\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)}(y), \nu_{\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)}(y)\right)$.

Now,

$$\mu_{\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)}(y) = \frac{1}{n}\sum_{x\in \overline{\overline{f}}^{-1}(y)} \left\{\mu_{\sum_{i\in I} A_i}(x)\right\}$$

$$= \frac{1}{n}\sum_{x\in \overline{\overline{f}}^{-1}(y)} \left\{\frac{1}{n}\sum_{i\in I}\mu_{A_i}(x)\right\} \quad ;$$

$$= \frac{1}{n}\sum_{i\in I}\left\{\frac{1}{n}\sum_{x\in \overline{\overline{f}}^{-1}(y)} \left\{\mu_{A_i}(x)\right\}\right\}$$

$$= \frac{1}{n}\sum_{i\in I}\left\{\mu_{\overline{\overline{f}}(A_i)}(y)\right\}$$

Therefore, $\mu_{\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)}(y) = \mu_{\sum_{i\in I}\overline{\overline{f}}(A_i)}(y)$.

Similarly, $\nu_{\overline{\overline{f}}\left(\sum_{i\in I} A_i\right)}(y) = \nu_{\sum_{i\in I}\overline{\overline{f}}(A_i)}(y)$.

Hence, $\overline{\overline{f}}\left(\sum_{i\in I} A_i\right) = \left(\sum_{i\in I}\overline{\overline{f}}(A_i)\right)$.

3. If for all $y \in \overline{\overline{f}}\left(A^{(\alpha,\beta)}\right)$, then $x \in A^{(\alpha,\beta)}$ such that $\overline{\overline{f}}(x) = y$ and from $\mu_A(y) \ge \alpha$ and $\nu_A(y) \le \beta$ it follows that

$$\frac{1}{n}\Sigma\left\{\mu_A(x) : x \in \overline{\overline{f}}^{-1}(x) \ge \alpha\right\} \text{ and } \frac{1}{n}\Sigma\left\{\nu_A(x) : x \in \overline{\overline{f}}^{-1}(x) \le \beta\right\},$$

i.e., from $\mu_{\overline{\overline{f}}(A)}(y) \ge \alpha$ and $\nu_{\overline{\overline{f}}(A)}(y) \le \beta$ it follows that $y \in \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)}$.

Therefore,

$$\overline{\overline{f}}\left(A^{(\alpha,\beta)}\right) \subseteq \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)}.$$

4. Let for all $y \in \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)+}$, then there exists $x \in A^{(\alpha,\beta)+}$ such that $\overline{\overline{f}}(x) = y$ and $\mu_{\overline{\overline{f}}(A)}(y) > \alpha$ and $\nu_{\overline{\overline{f}}(A)}(y) < \beta$

$$\Leftrightarrow \bigvee_{x\in \overline{\overline{f}}^{-1}(y)} \mu_A(x) > \alpha \text{ and } \bigwedge_{x\in \overline{\overline{f}}^{-1}(y)} \nu_A(x) < \beta$$

$$\Leftrightarrow (\exists x_0 \in X)\left(y = \overline{\overline{f}}(x_0)\right) \text{ and } \mu_A(x_0) > \alpha \;, \nu_A(x_0) < \beta$$

$$\Leftrightarrow (\exists x_0 \in X)\left(y = \overline{\overline{f}}(x_0)\right) \text{ and } x_0 \in A^{(\alpha,\beta)+}$$

$$\Leftrightarrow y \in \overline{\overline{f}}\left(A^{(\alpha,\beta)+}\right).$$

Thus, $\overline{\overline{f}}\left(A^{(\alpha,\beta)+}\right) = \left(\overline{\overline{f}}(A)\right)^{(\alpha,\beta)+}$.

This completes the proof. □

4 Arithmetic operations on intuitionistic fuzzy sets

In this section, the arithmetic operations for intuitionistic fuzzy set with respect to average extension principle are discussed with illustrations.

Definition 4.1. Let $A,B \in IFS(X)$. Then $A*B$ (where $* \in \{+\,,-\,,\ .\ ,\diagup\}$) is defined by

$$A*B = \{z, \mu_{A*B}(z), \nu_{A*B}(z)\},$$

where

$$\mu_{A*B}(z) = \frac{1}{n}\sum_{z=x*y}\left[\mu_A(x) \wedge \mu_B(y)\right] \text{ and } \nu_{A*B}(z) = \frac{1}{n}\sum_{z=x*y}\left[\nu_A(x) \vee \nu_B(y)\right]$$

and n represents the number of occurrences of an element during the operation.

Definition 4.2 (Addition). Let $A,B \in IFS(X)$, then $A+B = \{z, \mu_{A+B}(z), \nu_{A+B}(z)\}$, where

$$\mu_{A+B}(z) = \frac{1}{n}\sum_{z=x+y}\left[\mu_A(x) \wedge \mu_B(y)\right] \text{ and } \nu_{A+B}(z) = \frac{1}{n}\sum_{z=x+y}\left[\nu_A(x) \vee \nu_B(y)\right].$$

Example 4.1. Let $A,B \in IFS(X)$, where $A = \{(1,\ 0.5,\ 0.2),\ (2,\ 0.4,\ 0.2)\}$ and $B = \{(2,\ 0.5,\ 0.1),\ (3,\ 0.2,\ 0.4),\ (4,\ 0.6,\ 0.2)\}$.

Therefore,

$$A+B = \left\{\begin{matrix}(1+2,\ \min(0.5,\ 0.5),\ \max(0.2,\ 0.1)),\ (1+3,\ \min(0.5,\ 0.2),\ \max(0.2,\ 0.4)),\\ (1+4,\ \min(0.5,\ 0.6),\ \max(0.2,\ 0.2)),\ (2+2,\ \min(0.4,\ 0.5),\ \max(0.2,\ 0.1)),\\ (2+3,\ \min(0.4,\ 0.2),\ \max(0.2,\ 0.4)),\ (2+4,\ \min(0.4,\ 0.6),\ \max(0.2,\ 0.2))\end{matrix}\right\}$$

$$= \{(3,\ 0.5,\ 0.2),\ (4,\ 0.2,\ 0.4),\ (5,\ 0.5,\ 0.2),\ (4,\ 0.4,\ 0.2),\ (5,\ 0.2,\ 0.4),\ (6,\ 0.4,\ 0.2)\}$$

$$= \left\{(3,\ 0.5,\ 0.2),\ \left(4, \frac{1}{2}(0.2+0.4),\ \frac{1}{2}(0.4+0.2)\right),\right.$$
$$\left.\left(5, \frac{1}{2}(0.5+0.2),\ \frac{1}{2}(0.2+0.4)\right),\ (6,\ 0.4,\ 0.2)\right\}$$

$$= \{(3,\ 0.5,\ 0.2),\ (4,\ 0.3,\ 0.3),\ (5,\ 0.35,\ 0.3),\ (6,\ 0.4,\ 0.2)\}.$$

Definition 4.3 (Subtraction). Let $A,B \in IFS(X)$, then $A-B = \{z, \mu_{A-B}(z), \nu_{A-B}(z)\}$, where

$$\mu_{A-B}(z) = \frac{1}{n}\sum_{z=x-y}\left[\mu_A(x) \wedge \mu_B(y)\right] \text{ and } \nu_{A-B}(z) = \frac{1}{n}\sum_{z=x-y}\left[\nu_A(x) \vee \nu_B(y)\right].$$

Example 4.2. Let $A,B \in IFS(X)$, where $A = \{(1,0.5,0.2),(2,0.4,0.2)\}$ and $B = \{(2,\ 0.5,\ 0.1),\ (3,\ 0.2,\ 0.4),\ (4,\ 0.6,\ 0.2)\}$.

Therefore,

$$A-B=\left\{\begin{array}{l}(1-2,\min(0.5,0.5),\max(0.2,\ 0.1)),(1-3,\min(0.5,0.2),\max(0.2,\ 0.4)),\\(1-4,\min(0.5,0.6),\max(0.2,\ 0.2)),(2-2,\min(0.4,0.5),\max(0.2,\ 0.1)),\\(2-3,\min(0.4,0.2),\max(0.2,\ 0.4)),(2-4,\min(0.4,0.6),\max(0.2,\ 0.2))\end{array}\right\}$$

$$=\{(-1,0.5,0.2),(-2,0.2,0.4),(-3,0.5,0.2),(0,0.4,0.2),(-1,0.2,0.4),(-2,0.4,0.2)\}$$

$$=\left\{\left(-1,\frac{1}{2}(0.5+0.2),\frac{1}{2}(0.2+0.4)\right),\left(-2,\frac{1}{2}(0.2+0.4),\frac{1}{2}(0.4+0.2)\right),(-3,0.5,0.2),(0,0.4,0.2)\right\}$$

$$=\{(-1,0.35,0.3),(-2,0.3,0.3),(-3,0.5,0.2),(0,0.4,0.2)\}.$$

Definition 4.4. (Multiplication) Let $A,B\in IFS(X)$, then $A\times B=\{z,\mu_{A\times B}(z),\nu_{A\times B}(z)\}$, where

$$\mu_{A\times B}(z)=\frac{1}{n}\sum_{z=x\times y}\left[\mu_A(x)\wedge\mu_B(y)\right]\text{ and }\nu_{A\times B}(z)=\frac{1}{n}\sum_{z=x\times y}\left[\nu_A(x)\vee\nu_B(y)\right].$$

Example 4.3. Let $A,B\in IFS(X)$, where $A=\{(1,0.5,0.2),(2,0.4,0.2)\}$ and $B=\{(2,\ 0.5,\ 0.1),(3,\ 0.2,\ 0.4),(4,\ 0.6,\ 0.2)\}$.

Therefore,

$$A\times B=\left\{\begin{array}{l}(1\times 2,\min(0.5,0.5),\max(0.2,\ 0.1)),(1\times 3,\min(0.5,0.2),\max(0.2,\ 0.4)),\\(1\times 4,\min(0.5,0.6),\max(0.2,\ 0.2)),(2\times 2,\min(0.4,0.5),\max(0.2,\ 0.1)),\\(2\times 3,\min(0.4,0.2),\max(0.2,\ 0.4)),(2\times 4,\min(0.4,0.6),\max(0.2,\ 0.2))\end{array}\right\}$$

$$=\{(2,0.5,0.2),(3,0.2,0.4),(4,0.5,0.2),(4,0.4,0.2),(6,0.2,0.4),(8,0.4,0.2)\}$$

$$=\left\{(2,0.5,0.2),(3,0.2,0.4),\left(4,\frac{1}{2}(0.5+0.4),\frac{1}{2}(0.2+0.2)\right),(6,0.2,\ 0.4),(8,0.4,\ 0.2)\right\}$$

$$=\{(2,0.5,0.2),(3,0.2,0.4),(4,0.45,0.1),(6,0.2,0.4),(8,0.4,0.2)\}.$$

Definition 4.5 (Division). Let $A,B\in IFS(X)$, then $A/B=\{x,\mu_{A/B}(x),\nu_{A/B}(x)\}$, where

$$\mu_{A/B}(z)=\frac{1}{n}\sum_{z=x/y}\left[\mu_A(x)\wedge\mu_B(y)\right]\text{ and }\nu_{A/B}(z)=\frac{1}{n}\sum_{z=x/y}\left[\nu_A(x)\vee\nu_B(y)\right].$$

Example 4.4. Let $A,B\in IFS(X)$, where $A=\{(1,0.5,0.2),(2,0.4,0.2)\}$ and $B=\{(2,\ 0.5,\ 0.1),(3,\ 0.2,\ 0.4),(4,\ 0.6,\ 0.2)\}$.

Therefore,

$$\frac{A}{B}=\left\{\begin{array}{l}\left(\frac{1}{2},\min(0.5,0.5),\max(0.2,0.1)\right),\left(\frac{1}{3},\min(0.5,0.2),\max(0.2,0.4)\right),\\ \left(\frac{1}{4},\min(0.5,0.6),\max(0.2,0.2)\right),\left(\frac{2}{2},\min(0.4,0.5),\max(0.2,0.1)\right),\\ \left(\frac{2}{3},\min(0.4,0.2),\max(0.2,0.4)\right),\left(\frac{2}{4},\min(0.4,0.6),\max(0.2,0.2)\right)\end{array}\right\}$$

$$=\left\{\left(\frac{1}{2},0.5,0.2\right),\left(\frac{1}{3},0.2,0.4\right),\left(\frac{1}{4},0.5,0.2\right),(1,0.4,0.2),\left(\frac{2}{3},0.2,0.4\right),\left(\frac{1}{2},0.4,0.2\right)\right\}$$

$$=\left\{\left(\frac{1}{2},\frac{1}{2}(0.5+0.4),\frac{1}{2}(0.2+0.2)\right),\left(\frac{1}{3},0.2,0.4\right),\left(\frac{1}{4},0.5,0.2\right),(1,0.4,0.2),\left(\frac{2}{3},0.2,0.4\right)\right\}$$

$$=\left\{\left(\frac{1}{2},0.45,0.2\right),\left(\frac{1}{3},0.2,0.4\right),\left(\frac{1}{4},0.5,0.2\right),(1,0.4,0.2),\left(\frac{2}{3},0.2,0.4\right)\right\}.$$

5 Conclusions

In this article, we have proposed two new extension principles such as the minimal extension principle and the average extension principle for intuitionistic fuzzy sets. Some related properties of the proposed two extension principles have been established. Finally, the arithmetic operations for intuitionistic fuzzy sets based on average extension principle are described with illustrations

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