1st Int. Workshop on IFSs, Mersin, 14 Nov. 2014
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 20, 2014, No. 5, 9–16

Matrix representation of the second type of intuitionistic fuzzy modal operators

Gökhan Çuvalcıoğlu[1], Sinem Yılmaz[1] and Krassimir T. Atanassov[2]

[1] Department Of Mathematics, University Of Mersin
Mersin, Turkey
e-mails: gcuvalcioglu@gmail.com,
sinemnyilmaz@gmail.com

[2] Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria,
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Burgas-8010, Bulgaria
e-mail: krat@bas.bg

Abstract: Intuitionistic Fuzzy Modal Operator was defined in 1999, then these operators were generalized [7]. After these studies, some authors defined modal operators which are called one type and two type modal operators on Intuitionistic Fuzzy Sets. In this study, we will examine Intuitionistic Fuzzy Operators with matrices and we will examine algebraic structures of them.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators.
AMS Classification: Primary: 05C38, 15A15; Secondary: 05A15, 15A18.

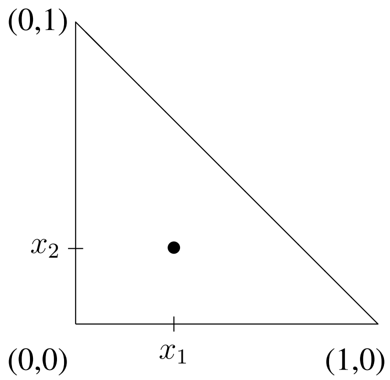
1 Introduction

The concept of fuzzy sets was introduced by Zadeh [14] as an extension of crisp sets by expanding the truth value set to the real unit interval $[0, 1]$. Let X be a set. The function $\mu : X \to [0, 1]$ is called a fuzzy set over $X(FS(X))$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$.Intuitionistic fuzzy sets have been introduced in [1], as an extension of fuzzy sets. If X is a universal then a intuitionistic fuzzy set A, the membership and non-membership degree for each $x \in X$ respectively, $\mu_A(x)(\mu_A : X \to [0,1])$ and $\nu_A(x)(\nu_A{:}X \to [0,1])$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$. While the sum of membership degree and non-membership degree is 1 on FS, this sum is less than 1 on IFS.

Definition 1. *Let $L = [0,1]$ then*

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$" and with different geometrical interpretations. The most important from them is the following:



For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as following;
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x)$, $(\nu_A : X \to [0,1])$ is called the "degree of non- membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 3. *[1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1]Let* $A \in IFS$ *and let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *then the above set is callede the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$$

2 Matrix representation of the intuitionistic fuzzy modal operators

The modal operators have been known to be important tools for IFSs. The notion of Intuitionistic Fuzzy Operators (IFO) was discussed in [1,7]. After than new Intuitionistic Fuzzy Operators were defined by several autors [4, 6, 9, 10, 12, 13] and some properties of these operators were studied. In this study, we will examine IFOs with matrices and we will examine algebraic structures of them.

Let us define that $(\max)(\min)\{a, b\}$ has property P if and only if $(\max\{a, b\}$ has property P and $\wedge(\min\{a, b\}$ has property P.

Let for brevity $(a_{i,j})$ denote a matrice with elements, denoted also by a and let $M_{3\times3}(\mathbb{R})$ be the set of (3×3)-matrices with elements – real numbers.

Let X be a fixed set. Then Ω and Γ are defining as following;

$$\Omega = \{\Theta \mid \Theta : IFS(X) \to IFS(X) \text{ is an IFMO}\}$$

$$\Gamma = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times3}(\mathbb{R}) \ \& \ 0 \le (\max)(\min)\{a_{1,1} + a_{1,2}, a_{2,1} + a_{2,2}\} \le 1 \ \& \ 0 \le a_{3,1} + a_{3,2} \le 1\}.$$

Definition 5. *Let* X *be a set and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.

The mapping $\varphi_A : \Gamma \to \Omega$,

$$\varphi_A((a_{i,j})) = \{\langle x, a_{1,1}\mu_A(x) + a_{2,1}\nu_A(x) + a_{3,1}, a_{1,2}\mu_A(x) + a_{2,2}\nu_A(x) + a_{3,2}\rangle : x \in X \ \& \ 0 \le (\max)(\min)\{a_{1,1} + a_{12}, a_{2,1} + a_{2,2}\} + a_{3,1} + a_{3,2} \le 1 \ \& \ 0 \le a_{3,1} + a_{3,2} \le 1\}.$$

After this we show the second type of IFMOs with matrices as follows.

Let $a_{1,1}, a_{2,1}, a_{3,1}, a_{1,2}, a_{2,2}, a_{3,2} \in [0, 1]$ satisfy inequalities

$$0 \le (\max)(\min)\{a_{1,1} + a_{12}, a_{2,1} + a_{2,2}\} + a_{3,1} + a_{3,2} \le 1$$

and

$$0 \le a_{3,1} + a_{3,2} \le 1.$$

Then

$$\Theta(A) = \left\{\left\langle x, a_{1,1}\,\mu_A\,(x) + a_{2,1}\,\nu_A(x) + a_{3,1},\ a_{1,2}\,\mu_A\,(x) + a_{2,2}\,\nu_A\,(x) + a_{3,2}\right\rangle : x \in X\right\}$$

$$= \left[\begin{array}{ccc} \mu_A\,(x) & \nu_A\,(x) & 1 \end{array}\right] \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{array}\right]$$

It is clear that for the present case, sets

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times2}(\mathbb{R})\ \&\ (a_{i,j}) = \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{array}\right]\}$$

and

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times2}(\mathbb{R})\ \&\ (a_{i,j}) = \left[\begin{array}{ccc} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 1 \end{array}\right]\}$$

are equal.

For brevity, in this paper, if $\varphi_A((a_{i,j})) = \Theta$, then we will use the notation

$$\Theta = \left[\begin{array}{ccc} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 1 \end{array}\right].$$

Example 1. *[7, 9] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0, 1]$.*

1. $\boxplus_\alpha A = \left[\begin{array}{ccc} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 1-\alpha & 1 \end{array}\right]$

2. $\boxtimes_\alpha A = \left[\begin{array}{ccc} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 1-\alpha & 0 & 1 \end{array}\right]$

3. $E_{\alpha,\beta}(A) = \left[\begin{array}{ccc} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta(1-\alpha) & \alpha(1-\beta) & 1 \end{array}\right]$

Example 2. *[7]Let X be universal and $A \in IFS(X)$,$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$,*

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1,$$
$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

Then,

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \left[\begin{array}{ccc} \alpha & -\zeta & 0 \\ -\varepsilon & \beta & 0 \\ \gamma & \delta & 1 \end{array}\right]$$

Thanks to this property we can show whether the new intuitionistic fuzzy operator satisfies conditions or not.

Theorem 1. $(\Gamma, .)$ is a monoid with multiplication operation over matrices.
Proof. (i) Let $(a_{i,j}), (b_{i,j}) \in \Gamma$. Then

$$\begin{aligned}
& b_{1,2}(a_{2,1}+a_{2,2})+b_{1,1}(a_{1,1}+a_{1,2})+b_{3,1}(a_{1,1}+a_{1,2})+b_{3,2}(a_{2,1}+a_{2,2})+a_{3,1}+a_{3,2} \\
= \; & (b_{1,2}+b_{3,2})(a_{2,1}+a_{2,2})+(b_{1,1}+b_{3,1})(a_{1,1}+a_{1,2})+a_{3,1}+a_{3,2} \\
\leq \; & (b_{1,2}+b_{3,2})(1-a_{3,1}-a_{3,2})+(b_{1,1}+b_{3,1})(1-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2} \\
\leq \; & (b_{1,2}+b_{3,2})(1-a_{3,1}-a_{3,2})+(1-b_{1,1}-b_{3,1})(1-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2}=1
\end{aligned}$$

and

$$\begin{aligned}
& b_{1,2}(a_{2,1}+a_{2,2})+b_{1,1}(a_{1,1}+a_{1,2})+b_{3,1}(a_{1,1}+a_{1,2})+b_{3,2}(a_{2,1}+a_{2,2})+a_{3,1}+a_{3,2} \\
= \; & (b_{1,2}+b_{3,2})(a_{2,1}+a_{2,2})+(b_{1,1}+b_{3,1})(a_{1,1}+a_{1,2})+a_{3,1}+a_{3,2} \\
\geq \; & (b_{1,2}+b_{3,2})(-a_{3,1}-a_{3,2})+(-b_{1,2}-b_{3,2})(-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2} \\
\geq \; & a_{3,1}+a_{3,2} \geq 0
\end{aligned}$$

(ii)

$$\begin{aligned}
& b_{2,1}(a_{1,1}+a_{2,2})+b_{2,2}(a_{2,1}+a_{2,2})+b_{3,1}(a_{1,1}+a_{1,2})+b_{3,2}(a_{2,1}+a_{2,2})+a_{3,1}+a_{3,2} \\
= \; & (b_{2,1}+b_{3,1})(a_{1,2}+a_{1,1})+(b_{2,2}+b_{3,2})(a_{2,1}+a_{2,2})+a_{3,1}+a_{3,2} \\
\leq \; & (b_{2,1}+b_{3,1})(1-a_{3,1}-a_{3,2})+(b_{2,2}+b_{3,2})(1-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2} \\
\leq \; & (1-b_{2,2}-b_{3,2})(1-a_{3,1}-a_{3,2})+(b_{2,2}+b_{3,2})(1-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2}=1
\end{aligned}$$

and

$$\begin{aligned}
& (b_{2,1}+b_{3,1})(a_{1,1}+a_{1,2})+(b_{2,2}+b_{3,2})(a_{2,1}+a_{2,2})+a_{3,1}+a_{3,2} \\
\geq \; & (-b_{2,2}-b_{3,2})(-a_{3,1}-a_{3,2})+(b_{1,2}+b_{3,2})(-a_{3,1}-a_{3,2})+a_{3,1}+a_{3,2} \\
\geq \; & a_{3,1}+a_{3,2} \geq 0
\end{aligned}$$

So, "." is a binary operation on Γ.

Associativity is clear and the identity element is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

This completes the proof. □

Theorem 2: Let X be a set and $A \in IFS(X)$. If $(a_{ij}), (b_{ij}) \in \Gamma$,

$$\varphi_A((a_{ij})(b_{ij})) = \varphi_A((b_{ij})) \circ \varphi_A((a_{ij})).$$

Proof: Let $(a_{i,j}), (b_{i,j}) \in \Gamma$,

$$\varphi_A((a_{ij})(b_{ij}))$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{11}b_{1,2} + a_{1,2}b_{2,2} & 0 \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} & 0 \\ a_{3,1}b_{1,1} + a_{3,2}b_{2,1} + b_{31} & a_{3,1}b_{1,2} + a_{3,2}b_{2,2} + b_{3,2} & 1 \end{bmatrix}$$

$$= \Theta_1(A) \circ \Theta_2(A) = \varphi_A((b_{i,j})) \circ \varphi_A((a_{i,j})).$$

This completes the proof. □

Corollary 1. $(\Omega, \circ)$ *is monoid.*

Example 3. *If* $\alpha, \beta \in [0, 1]$ *satisfy inequality* $\alpha + \beta \leq 1$ *and if* $\boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta} \in \Omega$*, then*

$$\begin{aligned} \boxtimes_{\alpha,\beta} \circ \boxplus_{\alpha,\beta} &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \beta & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ \alpha\beta & \beta & 1 \end{bmatrix}, \end{aligned}$$

because

$$\max(\alpha^2, \alpha^2) + \alpha\beta + \beta = \alpha^2 + \alpha\beta + \beta \leq 1,$$

$$\min(\alpha^2, \alpha^2) + \alpha\beta + \beta = \alpha^2 + \alpha\beta + \beta \geq 0$$

and

$$\alpha\beta + \beta \leq 1.$$

Example 4. *We can see that*

$$\boxplus_{\alpha,\beta,\gamma} A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & 1 \end{bmatrix}$$

and with first column operation on coefficient matrix we get

$$(\boxplus_{\alpha,\beta,\gamma} A)^c = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 1 \end{bmatrix}$$

Corollary 2. *As seen from the definitions of matrix for one type and uni type intuitionistic fuzzy operators, if we change the location of* a_{31} *and* a_{32} *we can see an other intuitionistic fuzzy operator.*

Example 5. *For* $\boxplus_{\alpha,\beta}$ *and* $Z^{\omega}_{\alpha,\beta}$ *with change the location of* a_{31} *and* a_{32}*, we get* $\boxtimes_{\alpha,\beta}$ *and* $Z^{\omega}_{\beta\alpha}$*, respectively.*

$$\boxplus_{\alpha,\beta} A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \beta & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \beta & 0 & 1 \end{bmatrix} = \boxtimes_{\alpha,\beta}$$

and

$$Z^{\omega}_{\alpha,\beta} = \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta\omega(1-\alpha) & \alpha\omega(1-\beta) & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \alpha\omega(1-\beta) & \beta\omega(1-\alpha) & 1 \end{bmatrix} = Z^{\omega}_{\beta,\alpha}$$

3 Open problems

1. Can there be given the ordering relation on the sets of intuitionistic fuzzy operators, with relation over sets of matrices?

2. Can we get class of operations with submonoids?

3. Can we change the above (standard) matrices with index matrices (see [8]) and will we obtain some advantage?

References

[1] Atanassov, K., Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] Atanassov, K. T., *Intuitionistic Fuzzy Sets*, Springer, Heidelberg, 1999.

[3] Atanassov, K. T., Remark on Two Operations Over Intuitionistic Fuzzy Sets, *Int. J. of Uncertanity, Fuzziness and Knowledge Syst.*, Vol. 9, 2001, No. 1, 71–75.

[4] Atanassov, K. T., The most general form of one type of intuitionistic fuzzy modal operators, *Notes on Intuitionistic Fuzzy Sets*, Vol. 12, 2006, No. 2, 36–38.

[5] Atanassov, K. T., Some properties of the operators from one type of intuitionistic fuzzy modal operators, *Advanced Studies on Contemporary Mathematics*, Vol. 15, 2007, No. 1, 13–20.

[6] Atanassov, K. T., The most general form of one type of intuitionistic fuzzy modal operators, Part 2, *Advanced Studies on Contemporary Mathematics*, Vol. 14, 2008, No. 1, 27–32.

[7] Atanassov, K. T., *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[8] Atanassov, K., *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham, 2014.

[9] Çuvalcıoğlu, G., Some Properties of $E_{\alpha,\beta}$ operator, *Advanced Studies on Contemporary Mathematics*, Vol. 14, 2007, No. 2, 305–310.

[10] Çuvalcıoğlu, G., On the Diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets: Last Expanding with $Z^{\omega,\theta}_{\alpha,\beta}$, *Iranian J. of Fuzzy Systems*, Vol. 10, 2013, No. 1, 89–106.

[11] Çuvalcıoğlu, G., The Extension of Modal Operators' Diagram with Last Operators, *Notes on Intuitionistic Fuzzy Sets*, Vol. 19, 2013, No. 3, 56–61.

[12] Dencheva, K., Extension of intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$, *Proc. of the Second Int. IEEE Symp. Intelligent Systems*, Varna, June 22-24 2004, Vol. 3, 21–22.

[13] Doycheva, B., Inequalities with intuitionistic fuzzy topological and Gökhan Çuvalcıoğlu's operators, *Advanced Studies on Contemporary Mathematics*, Vol. 14, 2008, No. 1, 20–22.

[14] Zadeh, L. A., Fuzzy Sets, *Information and Control*, Vol. 8, 1965, 338–353.