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Introduction to intuitionistic fuzzy partial differential equations

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Abstract: This paper consider solutions to elementary intuitionistic fuzzy partial differential equations.
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1 Introduction

We begin this section with defining the notation we will use in the paper and then we specify the type of elementary intuitionistic fuzzy partial diferential equation we wish to solve. In the next section we present the concept of intuitionistic fuzzy solution, and we give a sufficient condition for the solution to exist. The third section contains some examples.

We place a bar over a capital letter to denote a

Let us now first define the elementary partial differential equation we are interested in. Let $I_1 = [0, M_1]$ and $I_2 = [0, M_2]$, for some M_1 $(M_2) > 0$, be two intervals. $F(x, y, k)$ will be a continuous function for $(x, y) \in I_1 \times I_2$ and $k = (k_1, \dots, k_n)$ a vector of constants with k_i in interval J_i, $1 \le i \le n$. the operator $\varphi(D_x, D_y)$ will be a polynomial, with constant coefficients, in D_x and D_y where $D_x\,(D_y)$ stands for the "partial" with respect to $x\,(y)$. Also $U(x, y)$ is a continuous function, having continuous partials[1] with respect to both x and y, with $(x, y) \in I_1 \times I_2$. The partial differential equation is

$$\varphi(D_x, D_y)\, U(x, y) = F(x, y, k) \tag{1}$$

subject to certain boundary conditions. These boundary conditions can come in a variety of forms such as

$$\begin{aligned} &U(0, y) = c_1, U(x, 0) = c_2, U(M_1, y) = c_3, \dots \\ &U(0, y) = g_1(y; c_4), U(x, 0) = f_1(x; c_5), \dots \\ &U_x(x, 0) = f_2(x; c_6), U_y(0, y) = g_2(y; c_7; c_8), \dots \end{aligned}$$

At this point we will not give any explicit structure to the boundary conditions except to saay they depend on constants $c_1, \dots, c_m$ with the c_i in intervals L_i, $1 \le i \le m$. Let $c = (c_1, \dots, c_m)$.

We assume the problem, with certain boundary conditions, has aa solution

$$U(x, y) = G(x, y, k, c) \tag{2}$$

for continuous G[2] with $(x, y) \in I_1 \times I_2$, $k \in J = \Pi J_i$ and $c \in L = \Pi L_i$.

[1] We really need to assume that $\varphi(D_x, D_y)\, U(x, y)$ is continuous for $(x, y) \in I_1 \times I_2$.
[2] $\varphi(D_x, D_y)\, G(x, y, k, c)$ is continuous for $(x, y) \in I_1 \times I_2$, $k \in J$, $c \in L$.

Now we can say more about what we mean by "elementary" partial differential equations. By "elementary" we mean that the solution G in Eq. (1) is not defined in terms of a series. That is, there are no Fourier series used to define G. Since in this paper we will interested for G with intuitionistic fuzzy parameters we do not wish to consider Fourier series in intuitionistic fuzzy sets concept. We need the solution G to be fairly simple. So, we also assume that Bessel functions and Legendre functions are not used in G.

The constants k_i and c_i are not known exactly so there will be uncertainty in their values. We will model this uncertainty using intuitionistic fuzzy numbers. So, we will substitute intuitionistic fuzzy numbers $\bar{K}_i$ for k_i, $\bar{K}_i$ in J_i, $1 \le i \le n$, and substitute intuitionistic fuzzy numbers $\bar{C}_i$ for c_i, $\bar{C}_i$ in L_i, $1 \le i \le m$.

The intuitionistic fuzzy partial differential equation is

$$\varphi\left(D_x, D_y\right) \bar{U}(x, y)=\bar{F}\left(x, y, \bar{K}\right) \tag{3}$$

where $\bar{K}=\left(\bar{K}_1, \ldots, \bar{K}_n\right)$ for $\bar{K}_i$ an intuitionistic fuzzy number in J_i, $\le i \le n$. The function U becomes $\bar{U}$ where $\bar{U}$ maps $I_1 \times I_2$ into intuitionistic fuzzy numbers. That is, $\bar{U}(x, y)=\bar{Z}$ where $\bar{Z}$ is an intuitionistic fuzzy number.

By the following operator, we can obtain fuzzy sets from intuitionistic fuzzy sets

$$\begin{aligned}
\bar{K}_i^{\beta} &= \left\{\left\langle x, \mu_{\bar{K}_i}(x)+\beta \pi_{\bar{K}_i}(x), \nu_{\bar{K}_i}(x)+(1-\beta) \pi_{\bar{K}_i}(x)\right\rangle : x \in \mathbb{R}\right\} \\
\bar{C}_i^{\beta} &= \left\{\left\langle x, \mu_{\bar{C}_i}(x)+\beta \pi_{\bar{C}_i}(x), \nu_{\bar{C}_i}(x)+(1-\beta) \pi_{\bar{C}_i}(x)\right\rangle : x \in \mathbb{R}\right\}
\end{aligned}$$

where $\beta \in [0,1]$. Then the fuzzified form of the Eq. (1) is

$$\varphi\left(D_x, D_y\right) \bar{U}^{\beta}(x, y)=\bar{F}^{\beta}\left(x, y, \bar{K}^{\beta}\right) \tag{4}$$

subject to certain boundary conditions. The boundary conditions can be of the form

$$\begin{aligned}
&\bar{U}^{\beta}(0, y)=\bar{C}_1^{\beta}, \bar{U}^{\beta}(x, 0)=\bar{C}_2^{\beta}, \bar{U}^{\beta}\left(M_1, y\right)=\bar{C}_3^{\beta}, \ldots \\
&\bar{U}^{\beta}(0, y)=\bar{g}_1^{\beta}\left(y ; \bar{C}_4^{\beta}\right), \bar{U}^{\beta}(x, 0)=\bar{f}_1^{\beta}\left(x ; \bar{C}_5^{\beta}\right), \ldots \\
&\bar{U}_x^{\beta}(x, 0)=\bar{f}_2^{\beta}\left(x ; \bar{C}_6^{\beta}\right), \bar{U}_y^{\beta}(0, y)=\bar{g}_2^{\beta}\left(y ; \bar{C}_7^{\beta} ; \bar{C}_8^{\beta}\right), \ldots
\end{aligned}$$

The $\bar{g}_i^{\beta}$ and $\bar{f}_i^{\beta}$ are the extensions principle extensions of $\bar{g}_i$ and $\bar{f}_i$, respectively. Let $\bar{C}^{\beta}=\left(\bar{C}_1^{\beta}, \ldots, \bar{C}_m^{\beta}\right)$ with $\bar{C}_i^{\beta}$ triangular fuzzy number in L_i, $1 \le i \le m$. we wish to solve the problem given in Eq. (4).

Finally, we fuzzify G in Eq. (2). Let $\bar{Y}^{\beta}(x, y)=\bar{G}^{\beta}\left(x, y, \bar{K}^{\beta}, \bar{C}^{\beta}\right)$ where $\bar{Y}^{\beta}$ is computed using the extension principle.

Let the α-cuts of $\bar{K}^{\beta}$ and $\bar{C}^{\beta}$.

$$\bar{K}^{\beta}[\alpha]=\Pi \bar{K}_i^{\beta}[\alpha] \qquad \bar{C}^{\beta}[\alpha]=\Pi \bar{C}_i^{\beta}[\alpha]$$

2 Solution concept

From [3] the solution is presented as follows. For all α we have

$$\bar{Y}^{\beta}(x, y)[\alpha]=\left[y_1^{\beta}(x, y, \alpha), y_2^{\beta}(x, y, \alpha)\right]$$

and

$$\bar{F}^{\beta}\left(x,y,\bar{K}\right)[\alpha]=\left[\bar{F}_1^{\beta}\left(x,y,\alpha\right),\bar{F}_2^{\beta}\left(x,y,\alpha\right)\right]$$

We know that [4]

$$y_1^{\beta}\left(x,y,\alpha\right)=\min\left\{G^{\beta}\left(x,y,k,c\right):k\in\bar{K}^{\beta}[\alpha],\ c\in\bar{C}^{\beta}[\alpha]\right\} \tag{5}$$

$$y_2^{\beta}\left(x,y,\alpha\right)=\max\left\{G^{\beta}\left(x,y,k,c\right):k\in\bar{K}^{\beta}[\alpha],\ c\in\bar{C}^{\beta}[\alpha]\right\} \tag{6}$$

and

$$\bar{F}_1^{\beta}\left(x,y,\alpha\right)=\min\left\{F^{\beta}\left(x,y,k\right):k\in\bar{K}^{\beta}[\alpha]\right\} \tag{7}$$

$$\bar{F}_2^{\beta}\left(x,y,\alpha\right)=\max\left\{F^{\beta}\left(x,y,k\right):k\in\bar{K}^{\beta}[\alpha]\right\} \tag{8}$$

for all x, y and α.

Assume that the $y_i^{\beta}\left(x,y,\alpha\right)$ have continuous partials so that $\varphi\left(D_x,D_y\right)y_i^{\beta}\left(x,y,\alpha\right)$ is continuous for all $(x,y)\in I_1\times I_2$ and all α, $i=1,2$. Define

$$\Gamma^{\beta}\left(x,y,\alpha\right)=\left[\varphi\left(D_x,D_y\right)y_1^{\beta}\left(x,y,\alpha\right),\varphi\left(D_x,D_y\right)y_2^{\beta}\left(x,y,\alpha\right)\right] \tag{9}$$

for all $(x,y)\in I_1\times I_2$ and all α. If, for each fixed $(x,y)\in I_1\times I_2$, $\Gamma^{\beta}\left(x,y,\alpha\right)$ defines the α-cut of fuzzy number, then we will say that $\bar{Y}^{\beta}\left(x,y\right)$ is differentiable and we write

$$\varphi\left(D_x,D_y\right)\bar{Y}^{\beta}\left(x,y\right)=\Gamma^{\beta}\left(x,y,\alpha\right) \tag{10}$$

then $\bar{Y}^{\beta}\left(x,y\right)$ is a solution in the sense of [3] (without the boundary conditions) if $\bar{Y}^{\beta}\left(x,y\right)$ is differentiable and

$$\varphi\left(D_x,D_y\right)\bar{Y}^{\beta}\left(x,y\right)=\bar{F}^{\beta}\left(x,y,\bar{K}^{\beta}\right) \tag{11}$$

or the following equations must hold

$$\varphi\left(D_x,D_y\right)y_1^{\beta}\left(x,y,\alpha\right)=F_1^{\beta}\left(x,y,\alpha\right) \tag{12}$$

$$\varphi\left(D_x,D_y\right)y_2^{\beta}\left(x,y,\alpha\right)=F_2^{\beta}\left(x,y,\alpha\right) \tag{13}$$

for all $(x,y)\in I_1\times I_2$ and all α.

And we say that $\bar{Y}^{\beta}\left(x,y\right)$ is a solution in the sense of [3] satisying the boundary conditions.

Proposition 1 *Assume $\bar{Y}^{\beta}\left(x,y\right)$ is differentiable. If for all $i\in\{1,\ldots,n\}$ $G\left(x,y,k\right)$ and $F\left(x,y,k\right)$ are both increasing (or both decreasing) in k_i, for $(x,y)\in I_1\times I_2$ and $k\in J$, then $\bar{Y}\left(x,y\right)$ is a solution in the sense of Buckely and Feuring [3]*

Proof :

For simplicity assume $n=2$ and $G\left(x,y,k\right)$ is increasing in k_1, $F\left(x,y,k\right)$ is increasing in k_1, $G\left(x,y,k\right)$ is decreasing in k_2 and $F\left(x,y,k\right)$ is also decreasing in k_2. Then from (5) - (8) we have

$$y_1^{\beta}\left(x,y,\alpha\right)=G\left(x,y,k_{11}^{\beta}\left(\alpha\right),k_{22}^{\beta}\left(\alpha\right)\right) \tag{14}$$

$$y_2^{\beta}\left(x,y,\alpha\right)=G\left(x,y,k_{12}^{\beta}\left(\alpha\right),k_{21}^{\beta}\left(\alpha\right)\right) \tag{15}$$

$$F_1^{\beta}(x,y,\alpha) = F\left(x,y,k_{11}^{\beta}(\alpha), k_{22}^{\beta}(\alpha)\right) \tag{16}$$

$$F_2^{\beta}(x,y,\alpha) = F\left(x,y,k_{12}^{\beta}(\alpha), k_{21}^{\beta}(\alpha)\right) \tag{17}$$

for all α where $\bar{K}_1^{\beta}[\alpha] = \left[k_{11}^{\beta}(\alpha), k_{12}^{\beta}(\alpha)\right]$ and $\bar{K}_2^{\beta}[\alpha] = \left[k_{21}^{\beta}(\alpha), k_{22}^{\beta}(\alpha)\right]$

Now G solves the partial differential equation (1), which means

$$\varphi(D_x, D_y)\, G\left(x,y,k_1^{\beta}, k_2^{\beta}\right) = F\left(x,y,k_1^{\beta}, k_2^{\beta}\right)$$

for all $(x,y) \in I_1 \times I_2$, $k_1^{\beta} \in J_1$ and $k_2^{\beta} \in J_2$. But $k_{1j}^{\beta}(\alpha) \in J_1$ and $k_{2j}^{\beta}(\alpha) \in J_2$ for all α, $j = 1, 2$, so

$$\varphi(D_x, D_y)\, y_1^{\beta}(x,y,\alpha) = F_1^{\beta}(x,y,\alpha)$$

$$\varphi(D_x, D_y)\, y_2^{\beta}(x,y,\alpha) = F_2^{\beta}(x,y,\alpha)$$

for all $(x,y) \in I_1 \times I_2$ and all α, β. Hence, (12) and (13) hold and $\bar{Y}^{\beta}(x,y)$ is a solution in the sense of [3].

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