Notes on Intuitionistic Fuzzy Sets
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Invariant intuitionistic fuzzy observables

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Abstract: The aim of this contribution is showed that a sequence of Cesaro means of intuitionistic fuzzy observables has an invariant limit $\mathbf{m}$-almost everywhere, where $\mathbf{m}$ is an intuitionistic fuzzy state. We proved that this limit is an invariant intuitionistic fuzzy observable for a special type of intuitionistic fuzzy observables called P-intuitionistic fuzzy observables. We formulated the modification of the Individual Ergodic Theorem for this case of intuitionistic fuzzy observables.
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1 Introduction and basic notions

Let (X, σ, P) be a probability space, $T : X \to X$ be a measure preserving transformation, $\xi : X \to R$ be an integrable random variable. By the Individual Ergodic Theorem then there exists an integrable random variable ξ^* such that the following conditions are satisfied (see [14]):

(i) $E(\xi) = E(\xi^*)$,

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\xi \circ T^i) = \xi^*$ P-almost everywhere,

(iii) $\xi^* = \xi^* \circ T$ P-almost everywhere.

By a measure preserving transformation we understand a mapping $T : X \to X$ such that $A \in \sigma$ implies $T^{-1}(A) \in \sigma$ and $P\big(T^{-1}(A)\big) = P(A)$).

In the paper [7], we generalized the first two properties (i) and (ii) of the Individual Ergodic Theorem with respect to an intuitionistic fuzzy state, see the theorem below.

Theorem 1. (Individual Ergodic Theorem) *Let $(\mathcal{F}, *)$ be a family of IF-events with product, $\mathbf{m}$ be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Then there exists an integrable IF-observable $x^\star$ such that*

(i) $\mathbf{E}(x) = \mathbf{E}(x^\star)$,

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^\star$ $\mathbf{m}$*-almost everywhere.*

Now we explain the basic notions, which we used in previous theorem. By **the family of intuitionistic fuzzy events** on $(\Omega, \mathcal{S})$ we understand a set

$$\mathcal{F} = \{\mathbf{A} = (\mu_A, \nu_A);\ \mu_A, \nu_A : \Omega \to [0,1] \textit{ are } \mathcal{S}\text{-measurable and } \mu_A + \nu_A \le 1_\Omega\},$$

where $\Omega \ne \varnothing$ and $\mathcal{S}$ is a σ-algebra of subsets of Ω (see [2, 3, 12]). Recall that the notion of intuitionistic fuzzy sets was introduced by K. T. Atanassov in 1983 in paper [1].

Next we will work with Łukasiewicz binary operations $\oplus$ and $\odot$ on $\mathcal{F}$, which are given by these equalities

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),\\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))\end{aligned}$$

for $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$. The partial ordering $\le$ on $\mathcal{F}$ is defined as follows:

$$\mathbf{A} \le \mathbf{B} \Longleftrightarrow \mu_A \le \mu_B, \nu_A \ge \nu_B$$

and $\max - \min$ connectives are given by $\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$, see [1–3].

A product operation $*$ on a family of intuitionistic fuzzy events $\mathcal{F}$ is defined by

$$\mathbf{A} * \mathbf{B} = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big) = \big(\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B\big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ and $\cdot$ is a multiplication, see [11]. The product $*$ is satisfying the following four properties for each $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A}_n, \mathbf{B}_n \in \mathcal{F}$, $n \in N$, see [11]:

(i) $(1_\Omega, 0_\Omega) * \mathbf{A} = \mathbf{A}$;

(ii) the operation $*$ is commutative and associative;

(iii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\mathbf{C} * (\mathbf{A} \oplus \mathbf{B}) = (\mathbf{C} * \mathbf{A}) \oplus (\mathbf{C} * \mathbf{B})$ and $(\mathbf{C} * \mathbf{A}) \odot (\mathbf{C} * \mathbf{B}) = (0_\Omega, 1_\Omega)$;

(iv) if $\mathbf{A}_n \searrow (0_\Omega, 1_\Omega)$, $\mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$, then $\mathbf{A}_n * \mathbf{B}_n \searrow (0_\Omega, 1_\Omega)$.

In Theorem 1, we discussed an integrable intuitionistic fuzzy observable and an intuitionistic fuzzy state $\mathbf{m}$. Recall that the notion of intuitionistic fuzzy observable in the intuitionistic fuzzy space has the same role as the notion of random variable in the classical probability space. Similarly, the notion of intuitionistic fuzzy state corresponds with the notion of probability.

By **an intuitionistic fuzzy state** we mean a mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ that satisfies the following three conditions for each $\mathbf{A}, \mathbf{B}, \mathbf{A}_n \in \mathcal{F}$, $n \in N$ (see [13]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e., $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Recall that by **intuitionistic fuzzy observable** we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ that satisfies the following conditions:

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\varnothing) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \varnothing$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$

for each $A, B, A_n \in \mathcal{B}(R)$, $n \in N$, see [13]. There $\mathcal{B}(R)$ is the σ-algebra of the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$, see [14].

We say that an intuitionistic fuzzy observable x is an integrable, if the integral $\int_R t \, d\mathbf{m}_x(t)$ exists. In this case we define **intuitionistic fuzzy mean value** by

$$\mathbf{E}(x) = \int_R t \, d\mathbf{m}_x(t),$$

see [6].

In the modification of Individual ergodic theorem for the intuitionistic fuzzy case we work with $\mathbf{m}$-preserving transformation τ, which is a mapping $\tau : \mathcal{F} \to \mathcal{F}$ with following four conditions:

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ and $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$;

(iv) $\mathbf{m}\big(\tau(\mathbf{A}) * \tau(\mathbf{B})\big) = \mathbf{m}(\mathbf{A} * \mathbf{B})$

for each $\mathbf{A}_n, \mathbf{A}, \mathbf{B} \in \mathcal{F}$, $n \in N$, see [7].

Last notion from Theorem 1 is an almost everywhere convergence with respect an intuitionistic fuzzy state $\mathbf{m}$. In paper [4] we defined $\mathbf{m}$**-almost everywhere convergence** with help of limes inferior and limes superior as follows:

A sequence $(x_n)_n$ of intuitionistic fuzzy observables converges $\mathbf{m}$-almost everywhere to an intuitionistic fuzzy observable x, if there exist intuitionistic fuzzy observables $\overline{x}$, $\underline{x} : \mathcal{B}(R) \to \mathcal{F}$ such that $\mathbf{m}\big(\overline{x}((-\infty, t))\big) = \mathbf{m}\big(\underline{x}((-\infty, t))\big) = \mathbf{m}\big(x((-\infty, t))\big)$ for every $t \in R$.

There $\overline{x} = \limsup_{n\to\infty} x_n$ and $\underline{x} = \liminf_{n\to\infty} x_n$ are given by

$$\overline{x}((-\infty,t)) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),\ \underline{x}((-\infty,t)) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)$$

for every $t \in R$.

It is very important to note that for a function of several intuitionistic fuzzy observables created by the composition of their joint intuitionistic fuzzy observable and a Borel measurable function, almost everywhere convergence is related to the almost everywhere convergence of random variables in the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$, see [5]. The following theorem discuss about this.

Theorem 2. *Let $(x_n)_n$ be a sequence of intuitionistic fuzzy observables, $(\xi_n)_n$ be the sequence of corresponding projections, $(g_n)_n$ be a sequence of Borel measurable functions $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1,\ldots,\xi_n)\big)_n$ converges P-almost everywhere, then the sequence $\big(g_n(x_1,\ldots,x_n)\big)_n$ converges $\mathbf{m}$-almost everywhere and*

$$\mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big) = \mathbf{m}\Big(\liminf_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$. Moreover

$$P\Big(\{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u),\ldots,\xi_n(u)\big) < t\}\Big) = \mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$.

Remark that $\mathcal{C}$ is the family of all sets of the form $\{(t_i)_{i=1}^{\infty} : t_1 \in A_1,\ldots,t_n \in A_n\}$ and P is the probability measure given by the equality

$$P\big(\{(t_i)_{i=1}^{\infty} : t_1 \in A_1,\ldots,t_n \in A_n\}\big) = \mathbf{m}\big(x_1(A_1) * \cdots * x_n(A_n)\big).$$

The corresponding projections $\xi_n : R^N \to R$ are formulated as $\xi_n\big((t_i)_{i=1}^{\infty}\big) = t_n$ for $n \in N$, see [5, 7].

In Theorem 2, the intuitionistic fuzzy observable $g_n(x_1,\ldots,x_n) : \mathcal{B}(R) \to \mathcal{F}$ is defined by the formula $g_n(x_1,\ldots,x_n)(A) = h_n\big(g_n^{-1}(A)\big)$ for each $A \in \mathcal{B}(R)$. Then $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is an n-dimensional intuitionistic fuzzy observable with property

$$h(A_1 \times \cdots \times A_n) = x_1(A_1) * \ldots * x_n(A_n)$$

for each $A_1,\ldots,A_n \in \mathcal{B}(R)$ and it is called **the joint intuitionistic fuzzy observable** of intuitionistic fuzzy observables $x_1,\ldots,x_n : \mathcal{B}(R) \to \mathcal{F}$, see [12].

In the next section, we will generalize the property (iii) from the classical Individual Ergodic Theorem for an intuitionistic fuzzy space $(\mathcal{F}, \mathbf{m})$. We will use a notation "IF" as an abbreviation for "intuitionistic fuzzy" below.

2 Invariant intuitionistic fuzzy observables

In this section we will study the sequence of IF-observables $\big(\tau^n \circ x\big)_1^\infty$ and the sequence of Cesaro means

$$\left(\frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x\right)_1^\infty$$

in IF-space $(\mathcal{F}, \mathbf{m})$. There $x : \mathcal{B}(R) \to \mathcal{F}$ is an integrable IF-observable and $\tau : \mathcal{F} \to \mathcal{F}$ is an $\mathbf{m}$-preserving transformation defined as follows:

Definition 1. *Let $(\mathcal{F}, *)$ be an IF-space with product, $\mathbf{m}$ be an IF-state. The $\mathbf{m}$-preserving transformation is a mapping $\tau : \mathcal{F} \to \mathcal{F}$, which it is satisfying the following properties:*

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) *if* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, *then* $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ *and* $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;

(iii) *if* $\mathbf{A}_n \nearrow \mathbf{A}$, *then* $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$;

(iv) $\tau(\mathbf{A}) * \tau(\mathbf{B}) = \tau(\mathbf{A} * \mathbf{B})$

(v) $\tau(\mathbf{A} \wedge \mathbf{B}) = \tau(\mathbf{A}) \wedge \tau(\mathbf{B})$

(vi) $\mathbf{m}(\tau(\mathbf{A})) = \mathbf{m}(\mathbf{A})$

for each $\mathbf{A}, \mathbf{B}, \mathbf{A}_n \in \mathcal{F}$, $n \in N$.

One can see that this definition of $\mathbf{m}$-preserving transformation must satisfy additional conditions compared to the preserving transformation used in Theorem 1. So the transformation from Definition 1 is weaker. In this section, we will work with the $\mathbf{m}$-preserving transformation formulated in Definition 1.

The convergence of Cesaro means for observables in MV-algebras and in fuzzy quantum spaces was studied by B. Riečan and M. Jurečková in the papers [9, 10]. They showed that in both spaces the limit of the sequence of Cesaro means is invariantly observable. In this section we will extend this result for the IF-space $(\mathcal{F}, *)$.

Theorem 3. *Let $(\mathcal{F}, *)$ be an IF-space with product, $\mathbf{m}$ be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Put for $n \in N$*

$$\xi_n = \frac{1}{n}\sum_{i=0}^{n-1} \xi_1 \circ T^i \text{ and } y_n = \frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x.$$

The sequence $(y_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost everywhere to an integrable IF-observable $x^\star$ and

$$P\big((\xi^\star)^{-1}\big((-\infty, t)\big)\big) = \mathbf{m}\big(x^\star\big((-\infty, t)\big)\big)$$

for each $t \in R$. There $\xi^\star = \lim_{n\to\infty} \xi_n$ P-almost everywhere and $T : R^N \to R^N$ is the shift given by $T((t_n)_n) = (s_n)_n$, $s_n = t_{n+1}$, $n \in N$.

Proof. By Theorem 1 we have that the sequence of IF-observables $(y_n)_1^\infty$ converges $\mathbf{m}$-almost everywhere to the integrable IF-observable $x^\star$. Moreover from Theorem 2 we obtain that

$$P\big((\xi^\star)^{-1}\big((-\infty,t)\big)\big) = \mathbf{m}\big(x^\star\big((-\infty,t)\big)$$

for each $t \in R$. Then, from the classical Individual Ergodic Theorem, the sequence of random variables

$$(\xi_n)_1^\infty = \Big(\frac{1}{n}\sum_{i=0}^{n-1}\xi_1 \circ T^i\Big)_1^\infty$$

converges P-almost everywhere to the integrable random variable $\xi^\star$ and

$$(y_n)_1^\infty = \big(g_n(x_1,\dots,x_n)\big)_n^\infty = \Big(\frac{1}{n}\sum_{i=0}^{n-1}\tau^i \circ x\Big)_1^\infty, \;\; x_i = \tau^{i-1}\circ x$$

for $i = 1,\dots,n$, where $x^\star = \limsup_{n\to\infty} y_n = \limsup_{n\to\infty} g_n(x_1,\dots,x_n)$ and $g_n(t_1,\dots,t_n)$ $= \frac{1}{n}\sum_{i=1}^n t_i$ is a Borel measurable function. □

Now we return to the sequence of Cesaro means of IF-observables

$$\Big(\frac{1}{n}\sum_{i=1}^{n}\tau^i \circ x\Big)_1^\infty.$$

Lemma 1. *Let $l_n : \mathcal{B}(R^n) \to \mathcal{F}$ be the joint IF-observable of IF-observables $\tau \circ x, \tau^2 \circ x, \dots,$ $\tau^n \circ x : \mathcal{B}(R) \to \mathcal{F}$. Define the IF-observable z_n by*

$$z_n = \frac{1}{n}\sum_{i=1}^{n}\tau^i \circ x, n \in N.$$

Then $l_n = \tau \circ h_n$ and $z_n = \tau \circ y_n$, where h_n is the joint IF-observable of IF-observables $x, \tau \circ x, \dots, \tau^{n-1} \circ x : \mathcal{B}(R) \to \mathcal{F}$ and $y_n = \frac{1}{n}\sum_{i=0}^{n-1}\tau^i \circ x$.

Proof. Consider the Borel measurable function $g_n(t_1,\dots,t_n) = \frac{1}{n}\sum_{i=1}^n t_i$. Then from the definition of the function of several IF-observables we have

$$z_n = \frac{1}{n}\sum_{i=1}^{n}\tau^i \circ x = l_n \circ g_n^{-1}, \;\; y_n = \frac{1}{n}\sum_{i=0}^{n-1}\tau^i \circ x = h_n \circ g_n^{-1}.$$

Since l_n is the joint IF-observable of IF-observables $\tau \circ x, \tau^2 \circ x, \dots, \tau^n \circ x$ and h_n is the joint IF-observable of IF-observables $x, \tau \circ x, \dots, \tau^{n-1} \circ x$, then using the property (iv) of the $\mathbf{m}$-preserving transformation τ we obtain

$$\begin{aligned} l_n(A_1 \times A_2 \times \cdots \times A_n) &= \tau \circ x(A_1) * \tau^2 \circ x(A_2) * \cdots * \tau^n \circ x(A_n) \\ &= \tau\big(x(A_1)\big) * \tau\big(\tau \circ x(A_2)\big) * \cdots * \tau\big(\tau^{n-1} \circ x(A_n)\big) \\ &= \tau\big(x(A_1) * \tau \circ x(A_2) * \cdots * \tau^{n-1} \circ x(A_n)\big) \\ &= \tau\big(h_n(A_1 \times A_2 \times \cdots \times A_n)\big) \end{aligned}$$

for each $A_1 \times A_2 \times \cdots \times A_n \in \mathcal{B}(R^n)$. Hence $l_n = \tau \circ h_n$.

Moreover we obtain

$$z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x = l_n \circ g_n^{-1} = \tau \circ h_n \circ g_n^{-1} = \tau \circ y_n. \qquad \square$$

Lemma 2. *Let $g_n(t_1, \ldots, t_n) = \frac{1}{n}\sum_{i=1}^{n} t_i$ be a Borel measurable function, x be an integrable IF-observable. Put*

$$k_{n+1}(t_1, t_2, \ldots, t_{n+1}) = \frac{1}{n}\sum_{i=2}^{n+1} t_i = g_n(t_2, \ldots, t_{n+1}) \text{ and } z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x,$$

where τ is an $\mathbf{m}$-preserving transformation. Then $z_n = h_{n+1} \circ k_{n+1}^{-1}$, where h_{n+1} is a joint IF-observable of IF-observables $x_i = \tau^{i-1} \circ x$ for $i = 1, \ldots, n+1$.

Proof. Consider a projection $\pi_n : R^{n+1} \to R^n$ given by $\pi_n(t_1, t_2, \ldots, t_{n+1}) = (t_2, \ldots, t_{n+1})$. Then we obtain that $k_{n+1} = g_n \circ \pi_n$ and, therefore,

$$h_{n+1} \circ k_{n+1}^{-1} = h_{n+1} \circ (g_n \circ \pi_n)^{-1} = h_{n+1} \circ \pi_n^{-1} \circ g_n^{-1}. \tag{1}$$

But

$$\begin{aligned} h_{n+1} \circ \pi_n^{-1}(A_1 \times \cdots \times A_n) &= h_{n+1}(R \times A_1 \times \cdots \times A_n) \\ &= x_1(R) * x_2(A_1) * \cdots * x_{n+1}(A_n) \\ &= (1_\Omega, 0_\Omega) * x_2(A_1) * \cdots * x_{n+1}(A_n) \\ &= x_2(A_1) * \cdots * x_{n+1}(A_n) \\ &= \tau \circ x(A_1) * \cdots * \tau^n \circ x(A_n) \\ &= l_n(A_1 \times \cdots \times A_n), \end{aligned} \tag{2}$$

where l_n is the joint IF-observable of IF-observables $\tau^i \circ x$, $i = 1, \ldots, n$. Hence, using (1) and (2), we have

$$z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x = l_n \circ g_n^{-1} = h_{n+1} \circ \pi_n^{-1} \circ g_n^{-1} = h_{n+1} \circ k_{n+1}^{-1}. \qquad \square$$

Theorem 4. *Let $(\mathcal{F}, *)$ be an IF-space with product, $\mathbf{m}$ be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Put for $n \in N$*

$$\eta_n = \frac{1}{n}\sum_{i=1}^{n} \xi_1 \circ T^i \text{ and } z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x.$$

The sequence $(z_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost everywhere to an integrable IF-observable $z^\star$ and

$$P\big((\eta^\star)^{-1}\big((-\infty, t)\big)\big) = \mathbf{m}\big(z^\star\big((-\infty, t)\big)$$

for each $t \in R$ and $z^\star = \tau \circ x^\star$. There $\eta^\star = \lim_{n\to\infty} \eta_n$ P-almost everywhere, $x^\star = \lim_{n\to\infty} y_n$ $\mathbf{m}$-almost everywhere, $y_n = \frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x$ and $T : R^N \to R^N$ is the shift given by $T((t_n)_n) = (s_n)_n$, $s_n = t_{n+1}$, $n \in N$.

Proof. Denote for $n \in N$

$$\xi_n = \frac{1}{n}\sum_{i=0}^{n-1} \xi_1 \circ T^i, \;\; y_n = \frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x, \; \eta_n = \frac{1}{n}\sum_{i=1}^{n} \xi_1 \circ T^i, \;\; z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x.$$

It is easy to see that

$$\eta_n = \frac{1}{n}\sum_{i=1}^{n} \xi_1 \circ T^i = \frac{1}{n}\sum_{i=1}^{n} \xi_1 \circ T^{i-1} \circ T = \frac{1}{n}\sum_{j=0}^{n-1} \xi_1 \circ T^j \circ T = \xi_n \circ T. \tag{3}$$

By Lemma 1 we have that

$$z_n = \tau \circ y_n. \tag{4}$$

Let x be an integrable IF-observable. Then the first coordinate function ξ_1 is an integrable random variable and by the classical Individual Ergodic Theorem there exists an integrable random variable $\xi^\star$ such that P-almost everywhere

$$\xi^\star = \lim_{n\to\infty} \xi_n \text{ and } \xi^\star = \xi^\star \circ T, \tag{5}$$

where $(R^N, \sigma(\mathcal{C}), P)$ is the corresponding Kolmogorov probability space. Hence, using (3) and (5), we obtain that P-almost everywhere

$$\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \xi_n \circ T = \xi^\star \circ T = \xi^\star, \tag{6}$$

i.e., there exists an integrable random variable $\eta^\star = \xi^\star \circ T = \xi^\star$.

From Theorem 3 we have that the sequence of IF-observables $(y_n)_1^\infty$ converges $\mathbf{m}$-almost everywhere to the integrable IF-observable $x^\star$, i.e.,

$$\lim_{n\to\infty} y_n = x^\star \tag{7}$$

and for each $t \in R$

$$P\big((\xi^\star)^{-1}\big((-\infty, t)\big)\big) = \mathbf{m}\big(x^\star\big((-\infty, t)\big)\big). \tag{8}$$

Therefore, using (4), (7) and the property (iii) from Definition 1, we obtain

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \tau \circ y_n = \tau\Big(\lim_{n\to\infty} y_n\Big) = \tau \circ x^\star, \tag{9}$$

i.e., the sequence of IF-observables $(z_n)_1^\infty$ converges $\mathbf{m}$-almost everywhere to an integrable IF-observable $z^\star = \tau \circ x^\star$.

Finally, by (9), (8), (6) and the property (vi) from Definition 1, we have

$$\begin{aligned}\mathbf{m}\big(z^\star\big((-\infty, t)\big)\big) &= \mathbf{m}\big(\tau \circ x^\star\big((-\infty, t)\big)\big) = \mathbf{m}\big(x^\star\big((-\infty, t)\big)\big) = P\big((\xi^\star)^{-1}\big((-\infty, t)\big)\big) \\ &= P\big((\eta^\star)^{-1}\big((-\infty, t)\big)\big)\end{aligned}$$

for each $t \in R$. □

In paper [8] we defined an almost everywhere coincidence of IF-observables with using IF-state $\mathbf{m}$. We used two approaches: with and without using an joint IF-observable, see Definition 2 and Theorem 5.

Definition 2. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathbf{m} : \mathcal{F} \to [0,1]$ be an IF-state. We say that the IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathbf{m}$-almost everywhere if and only if*

$$\mathbf{m}\Big(h\big(\{(u,v) \in R^2\,; u = v\}\big)\Big) = 1,$$

where $h : \mathcal{B}(R) \to \mathcal{F}$ is the joint IF-observable of IF-observables x, y. We write that $x = y$ $\mathbf{m}$-almost everywhere.

Theorem 5. *Let $(\mathcal{F}, *)$ be the IF-space with product $*$ and $\mathbf{m} : \mathcal{F} \to [0,1]$ be an IF-state. Two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ coincide $\mathbf{m}$-almost everywhere if and only if*

$$\mathbf{m}\Big(x\big((-\infty,u)\big) * y\big((u,\infty)\big)\Big) = 0 \text{ and } \mathbf{m}\Big(x\big((u,\infty)\big) * y\big((-\infty,u)\big)\Big) = 0$$

for each $u \in R$.

In what follows, we will define the notion of a P-intuitionistic fuzzy observable and will show that the limit of a sequence of a Cesaro means is an invariant IF-observable for this case of IF-observables.

Definition 3. *An IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ is called P-intuitionistic fuzzy observable, if $x(C \cap D) \leq x(C) * x(D)$ for each $C, D \in \mathcal{B}(R)$.*

Lemma 3. *Let x be a P-intuitionistic fuzzy observable and h_n be the joint IF-observable of IF-observables $x_i = \tau^{i-1} \circ x$ for $i = 1, \ldots, n$. Then $h_n(C \cap D) \leq h_n(C) * h_n(D)$ for each $C, D \in \mathcal{B}(R^n)$.*

Proof. Let x be a P-intuitionistic fuzzy observable. Then by Definition 3 we have

$$x(C \cap D) \leq x(C) * x(D)$$

for each $C, D \in \mathcal{B}(R)$.

Denote $x_i = \tau^{i-1} \circ x$ for $i = 1, \ldots, n$. Now we show that $x_2 = \tau \circ x$ is a P-intuitionistic fuzzy observable. Really, using the property (iv) from Definition 1, we obtain

$$\begin{aligned}\tau \circ x(C \cap D) &= \tau\big(x(C \cap D)\big) \leq \tau\big(x(C) * x(D)\big) = \tau\big(x(C)\big) * \tau\big(x(D)\big) \\ &= \tau \circ x(C) * \tau \circ x(D).\end{aligned}$$

Such $x_i = \tau \circ x_{i-1}$ and x_{i-1} is a P-intuitionistic fuzzy observable, then by mathematical induction we have that $x_i = \tau^{i-1} \circ x$ are P-intuitionistic fuzzy observables for $i = 1, \ldots, n$.

Put $C = C_1 \times \cdots \times C_n$ and $D = D_1 \times \cdots \times D_n$, then $C \cap D = C_1 \cap D_1 \times \cdots \times C_n \cap D_n$. Let h_n be the joint IF-observable of IF-observables $x_i = \tau^{i-1} \circ x$ for $i = 1, \ldots, n$. Since $x_i = \tau^{i-1} \circ x$ are P-intuitionistic fuzzy observables for $i = 1, \ldots, n$, so for each $C, D \in \mathcal{B}(R^n)$

$$\begin{aligned}h_n(C \cap D) &= h_n(C_1 \cap D_1 \times C_2 \cap D_2 \times \cdots \times C_n \cap D_n) \\ &= x_1(C_1 \cap D_1) * x_2(C_2 \cap D_2) * \cdots * x_n(C_n \cap D_n) \\ &= x(C_1 \cap D_1) * \tau \circ x(C_2 \cap D_2) * \cdots * \tau^{n-1} \circ x(C_n \cap D_n) \\ &\leq x(C_1) * x(D_1) * \tau \circ x(C_2) * \tau \circ x(D_2) * \cdots * \tau^{n-1} \circ x(C_n) * \tau^{n-1} \circ x(D_n) \\ &= x(C_1) * \tau \circ x(C_2) * \cdots * \tau^{n-1} \circ x(C_n) * x(D_1) * \tau \circ x(D_2) * \cdots * \tau^{n-1} \circ x(D_n) \\ &= x_1(C_1) * x_2(C_2) * \cdots * x_n(C_n) * x_1(D_1) * x_2(D_2) * \cdots * x_n(D_n) \\ &= h_n(C_1 \times \cdots \times C_n) * h_n(D_1 \times \cdots \times D_n) = h_n(C) * h_n(D),\end{aligned}$$

i.e., h_n is a P-intuitionistic fuzzy observable. □

Theorem 6. *Let* $(\mathcal{F}, *, \mathbf{m})$ *be a IF-space with product and with an IF-state* $\mathbf{m}$*. Let* x *be an integrable P-intuitionistic fuzzy observable and* τ *be an* $\mathbf{m}$*-preserving transformation. Let* $x^\star$*,* $z^\star$ *be the IF-observables such that* $z^\star = \tau \circ x^\star$*,* $x^\star = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x$ $\mathbf{m}$*-almost everywhere,* $z^\star = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x$ $\mathbf{m}$*-almost everywhere. Then for each* $t \in R$ *it holds*

$$\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((t,\infty)\big)\Big) = 0 \ \textit{and} \ \mathbf{m}\Big(x^\star\big((t,\infty)\big) * z^\star\big((-\infty,t)\big)\Big) = 0$$

i.e., $x^\star = \tau \circ x^\star$ $\mathbf{m}$*-almost everywhere.*

Proof. First we will prove that $\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((t,\infty)\big)\Big) = 0$.
It is easy to see that $z^\star\big((t,\infty)\big) = \bigvee_{n=1}^{\infty} z^\star\big([t+\frac{1}{n},\infty)\big)$, hence

$$\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((t,\infty)\big)\Big) = \mathbf{m}\left(x^\star\big((-\infty,t)\big) * \bigvee_{n=1}^{\infty} z^\star\left(\left[t+\frac{1}{n},\infty\right)\right)\right).$$

Denote $t + \frac{1}{n} = s$. We will prove that $\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) = 0$ for $t < s$. Such $(-\infty,s) \cup [s,\infty) = R$ and $(-\infty,s) \cap [s,\infty) = \varnothing$, then using property (ii) of IF-observables and (iii) property of IF-product we obtain

$$\begin{aligned}
x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\cup[s,\infty)\big) &= x^\star\big((-\infty,t)\big) * z^\star(R)\\
x^\star\big((-\infty,t)\big) * \Big(z^\star\big((-\infty,s)\big) \oplus z^\star\big([s,\infty)\big)\Big) &= x^\star\big((-\infty,t)\big) * (0_\Omega, 1_\Omega)\\
\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big) \oplus \Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) &= x^\star\big((-\infty,t)\big)
\end{aligned}$$

and

$$\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big) \odot \Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) = (0_\Omega, 1_\Omega).$$

Hence, from property (ii) of the IF-state we have

$$\begin{aligned}
\mathbf{m}\bigg(\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big) \oplus \Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big)\bigg) &= \mathbf{m}\Big(x^\star\big((-\infty,t)\big)\Big),\\
\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big) + \mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) &= \mathbf{m}\Big(x^\star\big((-\infty,t)\big)\Big).
\end{aligned}$$

Therefore

$$\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) = \mathbf{m}\Big(x^\star\big((-\infty,t)\big)\Big) - \mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big).$$

Let x be an integrable P-intuitionistic fuzzy observable. Then the first coordinate function ξ_1 is an integrable random variable and by the classical Individual Ergodic Theorem there exist integrable random variables $\xi^\star$, $\eta^\star$ such that P-almost everywhere

$$\xi^\star = \lim_{n\to\infty} \xi_n, \ \ \eta^\star = \lim_{n\to\infty} \eta_n, \ \ \xi^\star = \xi^\star \circ T = \eta^\star,$$

where $\xi_n = \frac{1}{n}\sum_{i=0}^{n-1} \xi_1 \circ T^i$, $\eta_n = \frac{1}{n}\sum_{i=1}^{n} \xi_1 \circ T^i$ and $(R^N, \sigma(\mathcal{C}), P)$ is the corresponding Kolmogorov probability space. For this reason

$$P\Big((\xi^\star)^{-1}\big((-\infty,t)\big) \cap (\eta^\star)^{-1}\big([s,\infty)\big)\Big) = 0. \tag{10}$$

From Theorem 3 and Theorem 4 we have that

$$\mathbf{m}\big(x^\star\big((-\infty,t)\big)\big)=P\big((\xi^\star)^{-1}\big((-\infty,t)\big)\big) \tag{11}$$

and $\mathbf{m}$-almost everywhere

$$x^\star=\lim_{n\to\infty}\xi_n,\ \ \xi_n=\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x,\ \ z^\star=\lim_{n\to\infty}z_n,\ \ z_n=\frac{1}{n}\sum_{i=1}^{n}\tau^i\circ x.$$

But

$$\begin{aligned} x^\star\big((-\infty,t)\big) &= \overline{y}((-\infty,t))=\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}y_n\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big),\\ z^\star\big((-\infty,s)\big) &= \overline{z}((-\infty,s))=\bigvee_{q=1}^{\infty}\bigvee_{l=1}^{\infty}\bigwedge_{m=l}^{\infty}z_m\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big),\end{aligned}$$

hence

$$\begin{aligned} &\mathbf{m}\Big(x^\star\big((-\infty,t)\big)*z^\star\big((-\infty,s)\big)\Big)\\ =\ &\mathbf{m}\left(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}y_n\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)*\bigvee_{q=1}^{\infty}\bigvee_{l=1}^{\infty}\bigwedge_{m=l}^{\infty}z_m\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)\right)\\ =\ &\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\lim_{q\to\infty}\lim_{l\to\infty}\lim_{j\to\infty}\mathbf{m}\left(\bigwedge_{n=k}^{k+i}y_n\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)*\bigwedge_{m=l}^{l+j}z_m\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)\right).\end{aligned}$$

From Lemma 2 we have

$$\begin{aligned} &\mathbf{m}\left(\bigwedge_{n=k}^{k+i}y_n\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)*\bigwedge_{m=l}^{l+j}z_m\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)\right)\\ =\ &\mathbf{m}\left(\bigwedge_{n=k}^{k+i}h_n\circ g_n^{-1}\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)*\bigwedge_{m=l}^{l+j}h_{m+1}\circ k_{m+1}^{-1}\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)\right).\end{aligned}$$

Denote $A_n=\pi_{\omega,n}^{-1}\circ g_n^{-1}\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)$, $B_m=\pi_{\omega,m+1}^{-1}\circ g_m^{-1}\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)$, where $\omega\geq k+i$, $\omega\geq l+j$ and t,s,p,q are constants. Therefore,

$$\mathbf{m}\left(\bigwedge_{n=k}^{k+i}y_n\Big(\Big(-\infty,t-\frac{1}{p}\Big)\Big)*\bigwedge_{m=l}^{l+j}z_m\Big(\Big(-\infty,s-\frac{1}{q}\Big)\Big)\right)=\mathbf{m}\left(\bigwedge_{n=k}^{k+i}h_\omega(A_n)*\bigwedge_{m=l}^{l+j}h_\omega(B_m)\right).$$

Using the monotonicity of the joint IF-observable h_ω we obtain for $n=k,\ldots,k+i$ and for $m=l,\ldots,l+j$

$$h_\omega(A_n)\geq h_\omega\left(\bigcap_{n=k}^{k+i}A_n\right),\ \ h_\omega(B_m)\geq h_\omega\left(\bigcap_{m=l}^{l+j}B_m\right),$$

hence

$$\bigwedge_{n=k}^{k+i}h_\omega(A_n)\geq h_\omega\left(\bigcap_{n=k}^{k+i}A_n\right),\ \ \bigwedge_{m=l}^{l+j}h_\omega(B_m)\geq h_\omega\left(\bigcap_{m=l}^{l+j}B_m\right). \tag{12}$$

From (12) and Lemma 3 it follows that

$$\begin{aligned}\bigwedge_{n=k}^{k+i} h_\omega(A_n) * \bigwedge_{m=l}^{l+j} h_\omega(B_m) &\geq h_\omega\Big(\bigcap_{n=k}^{k+i} A_n\Big) * h_\omega\Big(\bigcap_{m=l}^{l+j} B_m\Big)\\ &\geq h_\omega\Big(\Big(\bigcap_{n=k}^{k+i} A_n\Big) \cap \Big(\bigcap_{m=l}^{l+j} B_m\Big)\Big)\end{aligned}$$

and therefore

$$\begin{aligned}&\mathbf{m}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big) * \bigwedge_{m=l}^{l+j} z_m\Big(\Big(-\infty, s-\frac{1}{q}\Big)\Big)\Big)\\ =\ &\mathbf{m}\Big(\bigwedge_{n=k}^{k+i} h_\omega(A_n) * \bigwedge_{m=l}^{l+j} h_\omega(B_m)\Big) \geq \mathbf{m}\Big(h_\omega\Big(\Big(\bigcap_{n=k}^{k+i} A_n\Big) \cap \Big(\bigcap_{m=l}^{l+j} B_m\Big)\Big)\Big)\\ =\ &P\Big(\pi_\omega^{-1}\Big(\Big(\bigcap_{n=k}^{k+i} A_n\Big) \cap \Big(\bigcap_{m=l}^{l+j} B_m\Big)\Big)\Big)\\ =\ &P\Big(\bigcap_{n=k}^{k+i} \xi_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big) \cap \bigcap_{m=l}^{l+j} \eta_m^{-1}\Big(\Big(-\infty, s-\frac{1}{q}\Big)\Big)\Big).\end{aligned}$$

For this reason, we obtain

$$\begin{aligned}&\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big)\\ =\ &\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\lim_{q\to\infty}\lim_{l\to\infty}\lim_{j\to\infty} \mathbf{m}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big) * \bigwedge_{m=l}^{l+j} z_m\Big(\Big(-\infty, s-\frac{1}{q}\Big)\Big)\Big)\\ \geq\ &\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\lim_{q\to\infty}\lim_{l\to\infty}\lim_{j\to\infty} P\Big(\bigcap_{n=k}^{k+i} \xi_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big) \cap \bigcap_{m=l}^{l+j} \eta_m^{-1}\Big(\Big(-\infty, s-\frac{1}{q}\Big)\Big)\Big)\\ =\ &P\Big(\bigcup_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} \xi_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big) \cap \bigcup_{q=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{m=l}^{\infty} \eta_m^{-1}\Big(\Big(-\infty, s-\frac{1}{q}\Big)\Big)\Big)\\ =\ &P\Big((\xi^\star)^{-1}\big((-\infty,t)\big) \cap (\eta^\star)^{-1}\big((-\infty,s)\big)\Big). \qquad (13)\end{aligned}$$

Finally, using (11), (13) and (10), we have

$$\begin{aligned}\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big([s,\infty)\big)\Big) &= \mathbf{m}\Big(x^\star\big((-\infty,t)\big)\Big) - \mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((-\infty,s)\big)\Big)\\ &\leq P\big((\xi^\star)^{-1}\big((-\infty,t)\big)\big) - P\Big((\xi^\star)^{-1}\big((-\infty,t)\big) \cap (\eta^\star)^{-1}\big((-\infty,s)\big)\Big)\\ &= P\Big((\xi^\star)^{-1}\big((-\infty,t)\big) \cap (\eta^\star)^{-1}\big([s,\infty)\big)\Big) = 0\end{aligned}$$

and hence

$$\mathbf{m}\Big(x^\star\big((-\infty,t)\big) * z^\star\big((t,\infty)\big)\Big) = \mathbf{m}\Big(x^\star\big((-\infty,t)\big) * \bigvee_{n=1}^{\infty} z^\star\Big(\Big[t+\frac{1}{n},\infty\Big)\Big)\Big) = 0.$$

Similarly we can prove that $\mathbf{m}\Big(x^\star\big((t,\infty)\big) * z^\star\big((-\infty,t)\big)\Big) = 0$. Therefore $x^\star = z^\star = \tau \circ x^\star$ $\mathbf{m}$-almost everywhere. □

3 Conclusion

The article deals with the invariance of an intuitionistic fuzzy observable, which is the limit of Cesaro means of intuitionistic fuzzy observables called P-intuitionistic fuzzy observables. We can thus formulate a modification of the Individual Ergodic Theorem for intuitionistic fuzzy space with product and with intuitionistic fuzzy state $\mathbf{m}$ as follows:

Theorem 7. (Modified Individual Ergodic Theorem). *Let $(\mathcal{F}, *)$ be a IF-space with product, $\mathbf{m}$ be an IF-state. Let x be an integrable P-intuitionistic fuzzy observable and τ be an $\mathbf{m}$-preserving transformation. Then there exists an integrable P-intuitionistic fuzzy observable $x^\star$ such that*

(i) $\mathbf{E}(x) = \mathbf{E}(x^\star)$,

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^\star$ $\mathbf{m}$-*almost everywhere,*

(iii) $x^\star = \tau \circ x^\star$ $\mathbf{m}$-*almost everywhere.*

The proof of Theorem 7 follows from Theorem 1 and Theorem 6.

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