Issues in IFSs and GNs, Vol. 14, 2018/19, 1–24

Basic Theorems from Extreme Value Theory for MV-algebras

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Abstract: In the paper the space of observables with respect to MV-algebras is considered. We prove the modification of the Fisher-Tippet-Gnedenko theorem and the Pickands-Balkema-de Haan theorem for sequence of independent observables in probability MV-algebra. We show that the results for MV-algebras can be applied for intuitionistic fuzzy sets and interval valued sets, too.
Keywords and phrases: MV-algebra, MV-state, Observable, Joint observable, Independence, Fisher-Tippet-Gnedenko theorem, Excess distribution, Maximum domain of attraction, Generalized Pareto distribution, Extreme value theory, Pickands-Balkema-de Haan theorem.
2000 Mathematics Subject Classification: 06D35, 60B10, 60B12, 62E17.

1 Introduction

The extreme value theory is a part of statistics, which deals with examination of probability of extreme and rare events with a large impact. The extreme value theory search endpoints of the distributions. The Fisher-Tippet-Gnedenko theorem says about convergence in probability distribution of maximums of independent, equally distributed random variables. An alternative to the maximal observation method is the method that models all observations that exceed any predefined boundary (i.e., threshold). This method is used in the Pickands-Balkema-de Haan theorem. In the paper we prove the modification of the Fisher-Tippet-Gnedenko theorem and the Pickands-Balkema-de Haan theorem for sequence of independent observables in probability MV-algebra. We show that the results for MV-algebras can be applied for intuitionistic fuzzy sets and interval valued sets, too.

Mention that the countable MV-algebras are a non-comutative generalization of countable Boolean algebras. Accordingly, the MV-algebraic probability theory developed in this survey can be thought of as a noncommutative generalization of classical Boolean algebraic probability theory (see [13]). Therefore in *Section 2* some basic information about MV-algebras, states and observables are presented (see [13]). Further in *Section 3* the independence of observables and the convergence in m in probability MV-algebra is studied. In *Section 4* the basic notions and theorems from extreme value theory are illustrated. In *Section 5* the Fisher-Tippet-Gnedenko theorem and the Pickands-Balkema-de Haan theorem are proved for MV-algebra case. In *Conclusion* we show a possibility of embedding the intuitionistic fuzzy sets introduced by K.T. Atanassov in [1, 2] and the interval valued sets introduced by L.A. Zadeh in [15] to suitable MV-algebras.

2 MV-algebras, States and Observables

By the Mundici theorem any MV-algebra can be defined by the help of an l-group (see [13]).

Definition 1. *By an ℓ-group we shall mean the structure $(G, +, \leq)$ such that the following properties are satisfied:*

(i) $(G, +)$ *is an Abelian group;*

(ii) $(G, \leq)$ *is a lattice;*

(iii) $a \leq b \Longrightarrow a + c \leq b + c$.

For each ℓ-group G, an element $u \in G$ is said to be a strong unit of G, if for all $a \in G$ there is an integer $n \geq 1$ such that $nu \geq a$ (nu is the sum $u + \ldots + u$ with n).

Example 1. *Consider* $G = R^2$,

$$\begin{aligned}(a,b)\hat{+}(c,d) &= (a+c, b+d-1),\\ (a,b) \leq (c,d) &\Longleftrightarrow a \leq c, b \geq d.\end{aligned}$$

Then $(R^2, \hat{+}, \leq)$ *is a lattice ordered group.*

Evidently the operation $\hat{+}$ *is commutative and associative,* $(0,1)$ *is the neutral element, since*

$$(0,1)\hat{+}(a,b) = (a+0, b+1-1) = (a,b),$$

and $(-a, 2-b)$ *is the inverse element, since*

$$(a,b)\hat{+}(-a, 2-b) = (0,1).$$

Further $\leq$ *is a partial order with*

$$\begin{aligned}(a,b) \vee (c,d) &= (\max(a,c), \min(b,d)),\\ (a,b) \wedge (c,d) &= (\min(a,c), \max(b,d)).\end{aligned}$$

Finally

$$(a,b) \leq (c,d) \Longrightarrow a \leq c, b \geq d,$$

hence

$$\begin{aligned}a+e &\leq c+e,\\ b+f-1 &\geq d+f-1,\\ (a,b)\hat{+}(e,f) = (a+e, b+f-1) &\leq (c+e, d+f-1) = (c,d)\hat{+}(e,f).\end{aligned}$$

Example 2. *Let* $(\Omega, \mathcal{S})$ *be a measurable space,* $\mathcal{S}$ *be a* σ*-algebra. Consider* $\mathcal{G} = \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \longrightarrow R \text{ are } \mathcal{S} - \text{measurable functions}\}$,

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega),\\ \mathbf{A} \leq \mathbf{B} &\Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.\end{aligned}$$

Then $(\mathcal{G}, +, \leq)$ *is an* ℓ*-group with the neutral element* $\mathbf{0} = (0_\Omega, 1_\Omega)$,

$$\mathbf{A} - \mathbf{B} = (\mu_A - \mu_B, \nu_A - \nu_B + 1_\Omega)$$

and the lattice operations

$$\begin{aligned}\mathbf{A} \vee \mathbf{B} &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),\\ \mathbf{A} \wedge \mathbf{B} &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B).\end{aligned}$$

Example 3. *Consider $G = R^2$,*

$$\begin{aligned}(a,b)\widetilde{+}(c,d) &= (a+c, b+d),\\ (a,b) \le (c,d) &\iff a \le c, b \le d.\end{aligned}$$

Then $(R^2, \widetilde{+}, \le)$ is a lattice ordered group.

Evidently the operation $\widetilde{+}$ is commutative and associative, $(0,0)$ is the neutral element, since

$$(0,0)\widetilde{+}(a,b) = (a+0, b+0) = (a,b),$$

and $(-a,-b)$ is the inverse element, since

$$(a,b)\widetilde{+}(-a,-b) = (0,0).$$

Further $\le$ is a partial order with

$$\begin{aligned}(a,b) \vee (c,d) &= \big(\max(a,c), \max(b,d)\big),\\ (a,b) \wedge (c,d) &= \big(\min(a,c), \min(b,d)\big).\end{aligned}$$

Finally

$$(a,b) \le (c,d) \Longrightarrow a \le c, b \le d,$$

hence

$$\begin{aligned} a+e &\le c+e,\\ b+f &\le d+f,\\ (a,b)\widetilde{+}(e,f) = (a+e, b+f) &\le (c+e, d+f) = (c,d)\widetilde{+}(e,f).\end{aligned}$$

Example 4. *Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ be a σ-algebra. Consider* $\mathcal{G} = \{\mathbf{C} = (\pi_C, \rho_C); \pi_C, \rho_C : \Omega \longrightarrow R \text{ are } \mathcal{S} - \text{measurable functions}\}$,

$$\begin{aligned}\mathbf{C} + \mathbf{D} &= (\pi_C + \pi_D, \rho_C + \rho_D),\\ \mathbf{C} \le \mathbf{D} &\iff \pi_C \le \pi_D, \rho_C \le \rho_D.\end{aligned}$$

Then $(\mathcal{G}, +, \le)$ is an ℓ-group with the neutral element $\mathbf{0} = (0_\Omega, 0_\Omega)$,

$$\mathbf{C} - \mathbf{D} = (\pi_C - \pi_D, \rho_C - \rho_D)$$

and the lattice operations

$$\begin{aligned}\mathbf{C} \vee \mathbf{D} &= (\pi_C \vee \pi_D, \rho_C \vee \rho_D),\\ \mathbf{C} \wedge \mathbf{D} &= (\pi_C \wedge \pi_D, \rho_C \wedge \rho_D).\end{aligned}$$

The following five definitions we can find in [13].

Definition 2. *An MV-algebra is an algebraic system* $(M, \oplus, \odot, \neg, 0, u)$*, where* $\oplus, \odot$ *are binary operations,* $\neg$ *is a unary operation,* $0, u$ *are fixed elements, which can be obtained by the following way: there exists a lattice group*

$$(G, +, \leq)$$

such that

$$M = \{x \in G; 0 \leq x \leq u\}$$

where 0 *is the neutral element of* G*,* u *is a strong unit of* G*, and*

$$\begin{aligned} a \oplus b &= (a+b) \wedge u = \min(a+b, u), \\ a \odot b &= (a+b-u) \vee 0 = \max(a+b-u, 0), \\ \neg a &= u - a. \end{aligned}$$

Here $\vee, \wedge$ *are the lattice operations with respect to the order and* $\neg a$ *is the opposite element of the element* a *with respect to the operation of the group.*

Example 5. *Let* $([0,1], \oplus, \odot, \neg, 0, 1)$ *be an* MV*-algebra, where* $a \oplus b = \min(a+b, 1), a \odot b = \max(a+b-1, 0), \neg a = 1-a$*. The corresponding group is* $(R, +, \leq)$ *where* $+$ *is usual sum, and* $\leq$ *is the usual ordering.*

Example 6. *Let* $([0,1]^2, \oplus, \odot, \neg, (0,1), (1,0))$ *be an* MV*-algebra, where*

$$\begin{aligned} (a,b) \oplus (c,d) &= (\min(a+c, 1), \max(b+d-1, 0)), \\ (a,b) \odot (c,d) &= (\max(a+c-1, 0), \min(b+d, 1)), \\ \neg(a,b) &= (1-a, 1-b). \end{aligned}$$

Here the corresponding group is $(R^2, \hat{+}, \leq)$ *considered in* Example 1.

Remark that the set $L_\Box = [0,1]^2$ (see Figure 1) with order relation $\leq_\Box$

$$(x_1, y_1) \leq_\Box (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \geq y_2$$

defined for all $(x_1, y_1), (x_2, y_2) \in L_\Box$ is a complete lattice. Since it is a product of the complete lattices $([0,1], \leq)$ and $([0,1], \geq)$ (see [5]).

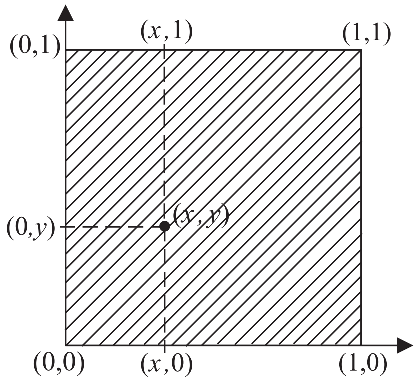

Figure 1: The shaded area constitutes the set $L_{\Box}$.

Example 7. *Let* $(\Omega, \mathcal{S})$ *be a measurable space,* $\mathcal{M}$ *the family of all pairs* $\mathbf{A} = (\mu_A, \nu_A)$*, where* $\mu_A, \nu_A : \Omega \to [0,1]$ *are* $\mathcal{S}$*-measurable functions,*

$$\begin{array}{rcl} \mathbf{A} \leq \mathbf{B} & \iff & \mu_A \leq \mu_B, \nu_A \geq \nu_B, \\ \mathbf{A} \oplus \mathbf{B} & = & \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} & = & ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\ \neg \mathbf{A} & = & (1_\Omega - \mu_A, 1_\Omega - \nu_A). \end{array}$$

Then the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ *is an* MV*-algebra. Here the corresponding group is* $(\mathcal{G}, +, \leq)$ *considered in* Example 2.

Example 8. *Let* $([0,1]^2, \oplus, \odot, \neg, (0,0), (1,1))$ *be an* MV*-algebra, where*

$$\begin{array}{rcl} (a,b) \oplus (c,d) & = & \big(\min(a+c,1), \min(b+d,1)\big), \\ (a,b) \odot (c,d) & = & \big(\max(a+c-1,0), \max(b+d-1,0)\big), \\ \neg(a,b) & = & (1-a, 1-b). \end{array}$$

Here the corresponding group is $(R^2, \widetilde{+}, \leq)$ *considered in* Example 3.

Example 9. *Let* $(\Omega, \mathcal{S})$ *be a measurable space,* $\mathcal{V}$ *the family of all pairs* $\mathbf{C} =$

(π_C, ρ_C)*, where* $\pi_C, \rho_C : \Omega \to [0,1]$ *are* $\mathcal{S}$*-measurable functions,*

$$\begin{aligned} \mathbf{C} \le \mathbf{D} &\Leftrightarrow \pi_C \le \pi_D, \rho_C \le \rho_D \\ \mathbf{C} \oplus \mathbf{D} &= \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big) \\ \mathbf{C} \odot \mathbf{D} &= \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega\big), \\ \neg\mathbf{C} &= (1_\Omega - \pi_C, 1_\Omega - \rho_C). \end{aligned}$$

Then the system $(\mathcal{V}, \oplus, \odot, \neg, (0_\Omega, 0_\Omega), (1_\Omega, 1_\Omega))$ *is an* MV*-algebra. Here the corresponding group is* $(\mathcal{G}, +, \le)$ *considered in* Example 4.

Definition 3. *An* MV*-algebra* M *is said to be* σ*-complete if its underlying lattice is* σ*-complete, i.e., every non-empty countable subset of* M *has a supremum in* M*.*

Every finite MV-algebra M is σ-complete - indeed, M is complete, in the sense that every non-empty subset of M has a supremum in M.

Definition 4. *Let* M *be a* σ*-complete* MV*-algebra. By a state on an* MV*-algebra* $(M, \oplus, \odot, \neg, 0, u)$ *is considered each mapping* $m : M \to [0,1]$ *satisfying the following conditions:*

(i) $m(u) = 1, m(0) = 0$*;*

(ii) $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$*;*

(iii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$*.*

We say that m *is faithful (also called, strictly positive) if* $m(x) \neq 0$ *whenever* $x \neq 0$*,* $x \in M$*.*

Definition 5. *A probability* MV*-algebra is a pair* (M, m)*, where* M *is* σ*-complete* MV*-algebra and* m *is a faithful state on* M*.*

Now we introduce a notion of n-dimensional observable of M. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \le x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 6. *Let M be a σ-complete MV-algebra. By an n-dimensional observable of MV-algebra $(M, \oplus, \odot, \neg, 0, u)$ we consider each mapping $x : \mathcal{B}(R^n) \to M$ satisfying the following conditions:*

(i) $x(R^n) = u$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = 0$ and $x(A \cup B) = x(A) \oplus x(B)$, where $A, B \in \mathcal{B}(R^n)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$ for $A_n, A \in \mathcal{B}(R^n)$.

When $n = 1$, then x is called an observable.

Having thus described the most general construction of an observable, let us now give a down-to-earth property of observables, when combined with states (see [13]):

Proposition 1. *Let M be a σ-complete MV-algebra with n-dimensional observable $x : \mathcal{B}(R^n) \to M$ and a state $m : M \to [0, 1]$. Define the composite map $m_x : \mathcal{B}(R^n) \to M$ by the stipulation that*

$$m_x(A) = (m \circ x)(A) = m(x(A)),$$

for all $A \in \mathcal{B}(R^n)$. Then m_x is a probability measure on $\mathcal{B}(R^n)$.

Proof. Let $B_n, B \in \mathcal{B}(R^n)$, $B_n \nearrow B$. Then

$$x(B_n) \nearrow x(B)$$

and therefore

$$m_x(B_n) = m(x(B_n)) \nearrow m(x(B)) = m_x(B).$$

Moreover

$$\begin{aligned} m_x(R^n) &= m(x(R^n)) = m(u) = 1, \\ m_x(\emptyset) &= m(x(\emptyset)) = m(0) = 0. \end{aligned}$$

Now we prove the additivity. Let $A, B \in \mathcal{B}(R^n), A \cap B = \emptyset$. Then

$$\begin{aligned} m_x(A \cup B) &= m\big(x(A \cup B)\big) = m\big(x(A) \oplus x(B)\big) = \\ &= m\big(x(A)\big) + m\big(x(B)\big) = m_x(A) + m_x(B). \end{aligned}$$

Hence m_x is a probability measure on $\mathcal{B}(R^n)$. □

The above map m_x has the same role as the probability distribution of a random variable in the classical Kolmogorov theory. One can for instance investigate conditions ensuring the existence of the moments of observables in probability MV-algebras, in the following sense (see [13]):

Definition 7. *Let (M, m) be a probability MV-algebra. Let $x : \mathcal{B}(R) \longrightarrow M$ be an observable of M. Then x is said to be integrable in (M, m), if the expectation (mean value)*

$$E(x) = \int_R t \, dm_x(t)$$

exists. We say that x is square integrable, if the dispersion (variance)

$$\sigma^2(x) = \int_R t^2 \, dm_x(t) - \big(E(x)\big)^2 = \int_R (t - E(x))^2 \, dm_x(t)$$

exists.

If we have an observable and an state in probability MV-algebra, then we can define the distribution function.

Definition 8. *Let (M, m) be a probability MV-algebra. If $x : \mathcal{B}(R) \to M$ is an observable of M and $m : M \to [0, 1]$ is an state of M, then the distribution function of x is the function $\overline{F} : R \to [0, 1]$ defined by the formula*

$$\overline{F}(t) = m\big(x((-\infty, t))\big)$$

for each $t \in R$.

Similarly as in the classical case the following theorem can be proved.

Theorem 1. *Let $\overline{F} : R \to [0, 1]$ be the distribution function of an observable $x : \mathcal{B}(R) \to M$ on MV-algebra M. Then $\overline{F}$ is non-decreasing on R, left continuous in each point $t \in R$ and*

$$\lim_{n \to -\infty} \overline{F}(t) = 0, \; \lim_{n \to \infty} \overline{F}(t) = 1.$$

Proof. If $r < s$, then

$$x\big((-\infty, s)\big) = x\big((-\infty, r)\big) \oplus x\big((r, s)\big) \geq x\big((-\infty, r)\big).$$

Hence

$$\overline{F}(s) = m\big(x\big((-\infty, s)\big)\big) \geq m\big(x\big((-\infty, r)\big)\big) = \overline{F}(r)$$

and $\overline{F}$ is non decreasing.

If $r_n \nearrow r$, then

$$x\big((-\infty, r_n)\big) \nearrow x\big((-\infty, r)\big).$$

Hence

$$\overline{F}(r_n) = m\big(x\big((-\infty, r_n)\big)\big) \nearrow m\big(x\big((-\infty, r)\big)\big) = \overline{F}(r)$$

and $\overline{F}$ is left continuous in any $u \in R$.

Similarly $r \nearrow \infty$ implies

$$x\big((-\infty, r)\big) \nearrow x\big((-\infty, \infty)\big) = u.$$

Therefore

$$\overline{F}(r) = m\big(x\big((-\infty, r)\big)\big) \nearrow m\big(u\big) = 1$$

for every $r \nearrow \infty$, hence $\lim_{r\to\infty} \overline{F}(r) = 1$.

Similarly we obtain

$$r \searrow -\infty \Longrightarrow -r \nearrow \infty,$$

hence

$$m\big(x\big((r, -r)\big)\big) \nearrow m\big(x\big((-\infty, \infty)\big)\big) = m\big(u\big) = 1.$$

Now

$$m\big(x\big((-\infty, -r)\big)\big) = m\big(x\big((-\infty, r)\big)\big) + m\big(x\big((r, -r)\big)\big),$$

hence

$$1 = \lim_{r\to-\infty} \overline{F}(-r) = \lim_{r\to-\infty} \overline{F}(r) + \lim_{r\to-\infty} m\big(x\big((r, -r)\big)\big) = \lim_{r\to-\infty} \overline{F}(r) + 1.$$

Therefore $\lim_{r\to-\infty} \overline{F}(r) = 0$. □

Theorem 2. *Let $\overline{F} : R \to [0, 1]$ be the distribution function of an observable $x : \mathcal{B}(R) \to M$ on MV-algebra M. Then*

$$\begin{aligned} E(x) &= \int_R t \, d\overline{F}(t), \\ \sigma^2(x) &= \int_R t^2 \, d\overline{F}(t) - \big(E(x)\big)^2 = \int_R (t - E(x))^2 \, d\overline{F}(t). \end{aligned}$$

Proof. Since $\overline{F}$ is the distribution function of the probability distribution m_x, we have

$$\lambda_{\overline{F}}\big([a,b)\big) = \overline{F}(b) - \overline{F}(a) = m_x\big([a,b)\big),$$

hence

$$\lambda_{\overline{F}} = m_x.$$

Therefore

$$\int_R t\, d\overline{F}(t) = \int_R t\, d\lambda_{\overline{F}}(t) = \int_R t\, dm_x(t) = E(x).$$

Similarly the other equality can be obtained. □

3 Independence

In the paper we shall work only with independent observables. First we define the notion of independence in probability MV-algebra (see [13]).

Definition 9. *Let (M, m) be a probability MV-algebra. We say that the observables $x_1, x_2, ...x_n : \mathcal{B}(R) \longrightarrow M$ are independent if there exists n-dimensional observable $h_n : \mathcal{B}(R^n) \longrightarrow M$ such that*

$$m\big(h_n(C_1 \times C_2 \times \ldots \times C_n)\big) = m\big(x_1(C_1)\big) \cdot m\big(x_2(C_2)\big) \cdot \ldots \cdot m\big(x_n(C_n)\big)$$

for each $C_1, C_2, ..., C_n \in \mathcal{B}(R)$.

The map h is called the joint observable of $x_1, \ldots, x_n$ (with respect to m) in (M, m).

Theorem 3. *Let R^N be the set of all sequences $(t_i)_i$ of real numbers. Let $(x_n)_n$ be a sequence of independent equally distributed observables in probability MV-algebra (M, m). Define for each $n \in N$ the mapping $\xi_n : R^N \longrightarrow R$ by the formula*

$$\xi_n((t_i)_i) = t_n.$$

Then $(\xi_n)_n$ is a sequence of independent random variables in a space $(R^N, \sigma(\mathcal{C}), P)$. If there exists $E(x_n)$, then $E(\xi_n) = E(x_n)$. If there exists $D^2(x_n)$, then $D^2(\xi_n) = D^2(x_n)$.

Proof. **Notation:** A set $C \subset R^N$ is called a cylinder, if there exists $n \in N$, and $D \in \mathcal{B}(R^n)$ such that

$$C = \{(t_i)_i : (t_1, ..., t_n) \in D\}.$$

By $\mathcal{C}$ we shall denote the family of all cylinders in R^N, by $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$.

Construction: Consider the measurable space $(R^N, \sigma(\mathcal{C}))$ a sequence $(x_n)_n$ of independent observables $x_n : \mathcal{B}(R) \longrightarrow M$ (i.e. $x_1, \ldots, x_n$ are independent for each $n \in N$) and the states $m_n : \mathcal{B}(R^n) \longrightarrow [0, 1]$ defined by

$$m_n(B) = m(h_n(B))$$

for each $B \in \mathcal{B}(R^n)$.

The states m_n are consisting, i.e.

$$\begin{aligned} m_{n+1}(B \times R) &= m\big(h_{n+1}(B \times R)\big) = (m \circ h_{n+1})(B \times R) = \\ &= (m_{x_1} \times \ldots \times m_{x_n} \times m_{x_{n+1}})(B \times R) = \\ &= m\big(h_n(B)\big) \cdot m\big(x(R)\big) = m\big(h_n(B)\big) \cdot 1 = m_n(B) \end{aligned}$$

for each $B \in \mathcal{B}(R^n)$.

Therefore by the Kolmogorov consistency theorem there exists the probability measure $P : \sigma(\mathcal{C}) \longrightarrow [0, 1]$ such that

$$P\big(\pi_n^{-1}(B)\big) = m_n(B) = m\big(h_n(B)\big)$$

for each $B \in \mathcal{C}$, where $\mathcal{C}$ is the family of all cylinders in R^N and $\pi_n : R^N \to R^n$ is a projection defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$.

Let $n \in N$, $A_1, ..., A_n \in \mathcal{B}(R)$. Then

$$\begin{aligned} P\big(\xi_1^{-1}(A_1) \cap \ldots \cap \xi_n^{-1}(A_n)\big) &= P\big(\{(t_i)_1^\infty : t_i \in A_i, i = 1, 2, \ldots, n\}\big) = \\ &= P\big(\pi_n^{-1}(A_1 \times \ldots \times A_n)\big) = \\ &= m\big(h_n(A_1 \times \ldots \times A_n)\big) = \\ &= m\big(x_1(A_1)\big) \cdot \ldots \cdot m\big(x_n(A_n)\big) = \\ &= P\big(\pi_{\{1\}}^{-1}(A_1)\big) \cdot \ldots \cdot P\big(\pi_{\{n\}}^{-1}(A_n)\big) = \\ &= P\big(\xi_1^{-1}(A_1)\big) \cdot \ldots \cdot P\big(\xi_n^{-1}(A_n)\big). \end{aligned}$$

Let $G : R \longrightarrow [0,1]$ be the distribution function of random variables ξ_n. Then

$$\begin{aligned} G(t) &= P\big(\xi_n^{-1}((-\infty,t))\big) = P\big(\pi_n^{-1}(R \times \ldots \times R \times (-\infty,t))\big) = \\ &= m\big(h_n(R \times ... \times R \times (-\infty,t))\big) = m\big(x_n((-\infty,t))\big) = m_{x_n}(t). \end{aligned}$$

If there exists mean value $E(x_n)$, then

$$E(x_n) = \int_R t\, dm_{x_n}(t) = \int_R t\, dG(t) = E(\xi_n).$$

Similarly the equality $D^2(\xi_n) = D^2(x_n)$ can be proved. □

We need the notion of functions of several observables yet (see [13]).

Definition 10. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to M$ be independent observables in a probability MV-algebra (M, m) and $g_n : R^n \to R$ be a Borel measurable function. Then the observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to M$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1}$$

where $h_n : \mathcal{B}(R^n) \to M$ is the n-dimensional observable (joint observable of $x_1, \ldots, x_n$).

Example 10. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to M$ be independent observables and $h_n : \mathcal{B}(R^n) \to M$ be their joint observable. Then*

1. *the observable $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} x_i - a\Big)$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} u_i - a\Big)$;

2. *the observable $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$;

3. *the observable* $y_n = \frac{1}{n}\sum\limits_{i=1}^{n}(x_i - E(x_i))$ *is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n}(u_i - E(x_i))$*;*

4. *the observable* $y_n = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ *is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big)$.

Definition 11. *Let* $(y_n)_n$ *be a sequence of observables in a probability MV-algebra* (M, m). *We say that* $(y_n)_n$ *converges in distribution to a function* $\Psi : R \longrightarrow [0, 1]$, *if for each* $t \in R$

$$\lim_{n\to\infty} m\big(y_n((-\infty, t))\big) = \Psi(t).$$

4 Basic Theorem from the Extreme Value Theory

The next notions of the extreme value theory on real numbers can be find in works [6], [7], [8] and [9].

Let $X_1, X_2, ...$ be independent, equally distributed random variables of real numbers with a distribution function $F : R \to R$ defined by

$$F(x) = P(X_i < x),\ (i = 1, 2, ...),$$

where $x \in R$. Denote M_n the maximum of n random variables

$$M_1 = X_1,\ M_n = \max(X_1, ..., X_n),$$

for $n \geq 2$.

Theorem 4. (Fisher-Tippett-Gnedenko) *Let* $X_1, X_2, ...$ *be a sequence of independent, equally distributed random variables. If there exists the sequences of real constant* $a_n > 0$, b_n *and a non-degenerate distribution function* H, *such that*

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} < x\right) = H(x),$$

then H *is the distribution function of one of the following three types of distributions:*

1. *Gumbel*

$$H_{\mu,\sigma}(x) = \exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right), \quad x \in R,$$

2. *Frechet*

$$H_{\mu,\sigma,\alpha}(x) = \begin{cases} 0, & \textit{for } x \leq \mu, \\ \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right), & \textit{for } x > \mu, \alpha > 0, \end{cases}$$

3. *Weibull*

$$H_{\mu,\sigma,\alpha}(x) = \begin{cases} \exp\left(-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right), & \textit{for } x \leq \mu, \alpha > 0, \\ 1, & \textit{for } x > \mu. \end{cases}$$

A parameter $\mu \in R$ is the **location parameter** and a parameter $\sigma > 0$ is the **scale parameter**.

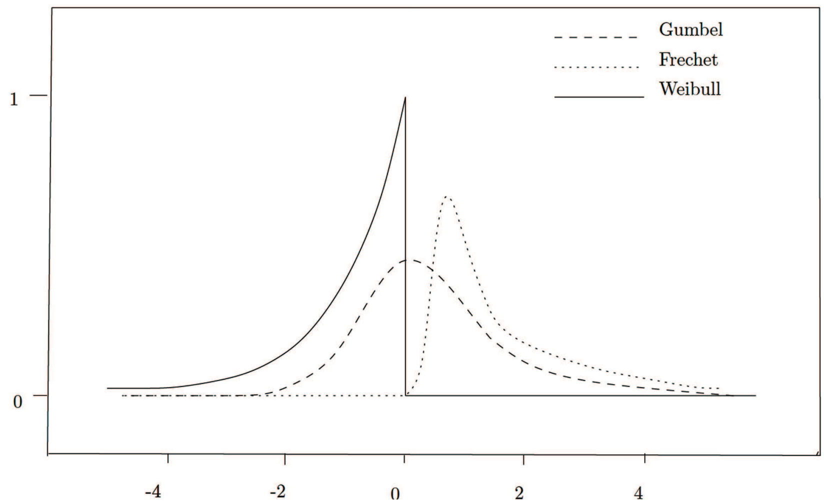

Figure 2: Gumbel, Frechet and Weibull distribution for $\alpha = 1$.

Gumbel, Frechet and Weibull distribution (see Figure 2) from *Theorem 2* can be described with using a **generalized distribution of extreme values - GEV:**

$$H_{\mu,\sigma,\varepsilon}(x) = \begin{cases} \exp\left[-\left(1+\varepsilon\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\varepsilon}}\right], & 1+\varepsilon\left(\frac{x-\mu}{\sigma}\right) > 0, \varepsilon \neq 0, \\ \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), & x \in R, \varepsilon = 0. \end{cases}$$

A parameter ε is called the **shape parameter**.

The Fisher-Tippet-Gnedenko theorem says about convergence in probability distribution of maximums of independent, equally distributed random variables. An alternative to the maximal observation method is the method that models all observations that exceed any predefined boundary (i.e. threshold).

Such the extremes occur "near" the upper end of distribution support, hence intuitively asymptotic behavior of M_n must be related to the distribution function F in its right tail near the right endpoint. We denote by

$$x_F = \sup\{x \in R : F(x) < 1\}$$

the right endpoint of F (see [6], [7], [8] and [9]).

Definition 12. (Maximum domain of attraction - MDA) *We say that the distribution function F of X_i belongs to the maximum domain of attraction of the extreme value distributions H if there exists constants $a_n > 0$, $b_n \in R$ such that*

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} < x\right) = H(x)$$

holds. We write $F \in MDA(H)$.

Definition 13. (Excess distribution function) *Let X be a random variable with distribution function F and right endpoint x_F. For fixed $u < x_F$, $u > 0$,*

$$F_u(x) = P(X - u \le x | X > u), x > 0,$$

is the excess distribution function of the random variable X (of the distribution function F) over the threshold u.

Remark 1. *The excess distribution function F_u can be expressed in the following form*

$$F_u(x) \quad = \quad P(X - u \le x | X > u) = \frac{P(u < X \le x + u)}{P(X > u)} = \frac{F(x+u) - F(u)}{1 - F(u)},$$

for $0 \le x \le x_F - u$.

Definition 14. (Generalized Pareto distribution - GPD) *Define the distribution function $G_{\varepsilon,\beta}$ by*

$$G_{\varepsilon,\beta}(x) = \begin{cases} 1 - \left(1 + \varepsilon \cdot \frac{x}{\beta}\right)^{-\frac{1}{\varepsilon}}, & \text{if } \varepsilon \neq 0, \\ 1 - e^{-\frac{x}{\beta}}, & \text{if } \varepsilon = 0, \end{cases}$$

where

$$\begin{aligned} x &\geq 0 \quad \textit{if } \varepsilon \geq 0, \\ 0 \leq \quad x &\leq -\frac{\beta}{\varepsilon} \quad \textit{if } \varepsilon < 0 \end{aligned}$$

and $\beta > 0$ is the scale parameter. $G_{\varepsilon,\beta}$ is called the generalised Pareto distribution. We can extend the family by adding a location parameter $\nu \in R$. Then we get the function $G_{\varepsilon,\nu,\beta}$ by replacing the argument x above by $x-\nu$ in $G_{\varepsilon,\beta}$. The support has to be adjusted accordingly.

Remark 2. *The GPD transforms into a number of other distributions depending on the value of ε. When $\varepsilon > 0$, it takes the form of the ordinary Pareto distribution. This case would be most relevant for financial time series data as it has a heavy tail. If $\varepsilon = 0$, the GPD corresponds to exponential distribution, and it is called a short-tailed, Pareto II type distribution for $\varepsilon < 0$.*

Theorem 5. **(Pickands-Balkema-de Haan)** *Let F be an excess distribution. For every $\varepsilon \in R$,*

$$F \in \mathrm{MDA}(H_\varepsilon) \iff \lim_{u \to x_F} \sup_{0<x<x_F-u} |F_u(x) - G_{\varepsilon,\beta(u)}(x)| = 0$$

for some positive function β.

Proof. See [7].

Remark 3. Theorem 5 *say that for some function β to be estimated from the data, the excess distribution F_u converges to the generalised Pareto distribution $G_{\varepsilon,\beta}$ for large u.*

Remark 4. *The GEV*

$$H_\varepsilon, \ \ \varepsilon \in R,$$

describes the limit distribution of normalised maxima.

The GPD

$$G_{\varepsilon,\beta}, \ \ \varepsilon \in R, \ \beta > 0,$$

appears as the limit distribution of scaled excesses over high thresholds.

5 Fisher-Tippett-Gnedenko Theorem and Pickands-Balkema-de Hann Theorem for MV-algebras

Now we return to the MV-algebra case. Let $x_1, x_2, ...$ be independent, equally distributed observables in a probability MV-algebra (M, m). Denote by $\overline{M}_n$ the maximum of n observables

$$\overline{M}_1 = x_1, \ \overline{M}_n = \max(x_1, ..., x_n),$$

for $n \geq 2$.

Theorem 6. (Fisher-Tippett-Gnedenko) *Let $x_1, x_2, ...$ be a sequence of independent, equally distributed observables in a probability MV-algebra (M, m) such that $D^2(x_n) = \sigma^2$, $E(x_n) = a, (n = 1, 2, \ldots)$. If there exists the sequences of real constant $a_n > 0$, b_n and a non-degenerate distribution function H, such that*

$$\lim_{n \to \infty} m\left(\frac{1}{a_n}\big(\overline{M}_n - b_n\big)\big((-\infty, t)\big)\right) = H(t),$$

then H is the distribution function one of the following three types of distributions:

1. *Gumbel*
$$H_{\mu,\sigma}(t) = \exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right), \quad t \in R,$$

2. *Frechet*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} 0, & \textit{for } t \leq \mu, \\ \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^{-\alpha}\right), & \textit{for } t > \mu, \alpha > 0, \end{cases}$$

3. *Weibull*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} \exp\left(-\left(-\left(\frac{t-\mu}{\sigma}\right)\right)^{\alpha}\right), & \textit{for } t \leq \mu, \alpha > 0, \\ 1, & \textit{for } t > \mu. \end{cases}$$

Where the parameter $\mu \in R$ is the location parameter and the parameter $\sigma > 0$ is the scale parameter.

Proof. For each $n = 1, 2, 3, \ldots$ let the Borel function $g_n : R^n \longrightarrow R$ be given by

$$g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big).$$

Let further the observable $y_n : \mathcal{B}(R) \longrightarrow M$ be given by stipulation

$$y_n = h_n \circ g_n^{-1} = g_n(x_1, \ldots, x_n) = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big).$$

Consider the measure space $(R^N, \sigma(\mathcal{C}), P)$ and the random variables

$$\xi_n((t_i)_i) = t_n, (n = 1, 2, ...).$$

Then by *Theorem 3* the random variables ξ_n are independent. Moreover,

$$E(\xi_n) = E(x_n) = a,\ D^2(\xi_n) = D^2(x_n) = \sigma^2.$$

Therefore by assumptions of classical Fisher-Tippett-Gnedenko theorem (*Theorem 4*) we have

$$\lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty;\ \frac{1}{a_n}\Big(\max\big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\big) - b_n\Big) < t\Big\}\Big)$$
$$= H(t).$$

Hence we have

$$\begin{aligned}
&\lim_{n\to\infty} m\Big(\frac{1}{a_n}\big(\overline{M}_n - b_n\big)\big((-\infty, t)\big)\Big) = \lim_{n\to\infty} m\big(y_n((-\infty, t))\big) = \\
&= \lim_{n\to\infty} m\big(h_n\big(g_n^{-1}((-\infty, t))\big)\big) = \lim_{n\to\infty} P\big(\pi_n^{-1}\big(g_n^{-1}((-\infty, t))\big)\big) = \\
&= \lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty; g_n\Big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\Big) \in (-\infty, t)\Big\}\Big) = \\
&= \lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty;\ \frac{1}{a_n}\Big(\max\big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\big) - b_n\Big) < t\Big\}\Big) \\
&= H(t),
\end{aligned}$$

where H is the distribution function of one of the mentioned distributions. □

Let x be an observable in MV-algebra M and $\overline{F}$ be an distribution function of x. We denote by

$$t_{\overline{F}} = \sup\{t \in R : \overline{F}(t) < 1\}$$

the right endpoint of distribution function $\overline{F}$.

Definition 15. (Maximum domain of attraction for MV-algebra) *We say that the distribution function $\overline{F}$ of observable x in MV-algebra M belongs to the maximum domain of attraction of the extreme value distributions H if there exists constants $a_n > 0$, $b_n \in R$ such that*

$$\lim_{n\to\infty} m\left(\frac{1}{a_n}\big(\overline{M}_n - b_n\big)\big((-\infty,t)\big)\right) = H(t),$$

holds. We write $\overline{F} \in \overline{MDA}(H)$.

Definition 16. (Excess distribution function) *Let $\overline{F}$ be an distribution function on MV-algebra M with right endpoint $t_{\overline{F}}$. For fixed $u < t_{\overline{F}}$, $u > 0$,*

$$\overline{F}_u(t) = \frac{\overline{F}(t+u) - \overline{F}(u)}{1 - \overline{F}(u)}, \ \ 0 \le t \le t_{\overline{F}} - u$$

is the excess distribution function of the observable x (of the distribution function $\overline{F}$) over the threshold u.

Theorem 7. (Pickands-Balkema-de Haan) *For every $\varepsilon \in R$,*

$$\overline{F} \in \overline{MDA}(H_\varepsilon) \iff \lim_{u\to t_{\overline{F}}} \sup_{0<t<t_{\overline{F}}-u} |\overline{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0$$

for some positive function β.

Proof. Let $(x_n)_n$ be a sequence of independent observables in probability MV-algebra (M, m) with the same distribution $\overline{F}$.

Consider the measure space $(R^N, \sigma(\mathcal{C}), P)$ and random variables

$$\xi_n((t_i)_i) = t_n, (n = 1, 2, ...).$$

Then by *Theorem 3* the random variables ξ_n are independent. Denote F the distribution function of random variable ξ_n.

We can see that $\overline{F} = F$ and $t_{\overline{F}} = t_F$, because

$$\begin{aligned} F(t) &= P\big(\xi_n^{-1}((-\infty,t))\big) = P\big(\pi_n^{-1}(R \times \ldots \times R \times (-\infty,t))\big) = \\ &= m\big(h_n(R \times ... \times R \times (-\infty,t))\big) = m\big(x_n((-\infty,t))\big) = \overline{F}(t). \end{aligned}$$

Hence $\overline{F}_u = F_u$.

For each $n = 1, 2, 3, \dots$ let the Borel function $g_n : R^n \longrightarrow R$ be given by

$$g_n(u_1, \dots, u_n) = \frac{1}{a_n}\big(\max(u_1, \dots, u_n) - b_n\big).$$

Let further the observable $y_n : \mathcal{B}(R) \longrightarrow M$ be given by stipulation

$$y_n = h_n \circ g_n^{-1} = g_n(x_1, \dots, x_n) = \frac{1}{a_n}\big(\max(x_1, \dots, x_n) - b_n\big).$$

Moreover

$$\begin{aligned}
&m\left(\frac{1}{a_n}\big(\overline{M}_n - b_n\big)\big((-\infty, t)\big)\right) = m\big(y_n((-\infty, t))\big) = \\
&= m\big(h_n\big(g_n^{-1}((-\infty, t))\big)\big) = P\big(\pi_n^{-1}\big(g_n^{-1}((-\infty, t))\big)\big) = \\
&= P\left(\left\{(u_i)_1^\infty; g_n\left(\xi_1((u_i)_1^\infty), \dots, \xi_n((u_i)_1^\infty)\right) \in (-\infty, t)\right\}\right) = \\
&= P\left(\left\{(u_i)_1^\infty ; \frac{1}{a_n}\left(\max\big(\xi_1((u_i)_1^\infty), \dots, \xi_n((u_i)_1^\infty)\big) - b_n\right) < t\right\}\right) = \\
&= P\left(\frac{1}{a_n}(M_n - b_n) < t\right).
\end{aligned}$$

Therefore we obtain for every $\varepsilon \in R$,

$$\overline{F} \in \overline{MDA}(H_\varepsilon) \Longleftrightarrow F \in \mathrm{MDA}(H_\varepsilon)$$

and

$$\lim_{u \to t_{\overline{F}}} \sup_{0 < t < t_{\overline{F}} - u} |\overline{F}_u(t) - G_{\varepsilon, \beta(u)}(t)| = \lim_{u \to t_F} \sup_{0 < t < t_F - u} |F_u(t) - G_{\varepsilon, \beta(u)}(t)| = 0$$

for some positive function β.

Finally from a classical Pickands-Balkema-de Haan theorem (*Theorem 5*) we have

$$F \in \mathrm{MDA}(H_\varepsilon) \quad \Longleftrightarrow \quad \lim_{u \to t_F} \sup_{0 < t < t_F - u} |F_u(t) - G_{\varepsilon, \beta(u)}(t)| = 0.$$

Hence

$$\overline{F} \in \overline{MDA}(H_\varepsilon) \quad \Longleftrightarrow \quad \lim_{u \to t_{\overline{F}}} \sup_{0 < t < t_{\overline{F}} - u} |\overline{F}_u(t) - G_{\varepsilon, \beta(u)}(t)| = 0.$$

□

Remark 5. Theorem 7 *say that for some function β to be estimated from the data, the excess distribution $\overline{F}_u$ on MV-algebra M converges to the generalised Pareto distribution $G_{\varepsilon,\beta}$ for large u.*

6 Conclusion

We have proved a very important assertion of mathematical statistics for observables in MV-algebras. Evidently the results can be applied also to fuzzy sets theory (see [14]). Since the family $\mathcal{F}$ of intuitionistic fuzzy events introduced by K.T. Atanassov in [1, 2]

$$\mathcal{F} = \{(\mu_A, \nu_A)\, ; \mu_A + \nu_A \leq 1_\Omega, \},$$

where $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable functions, can be embedded to a suitable MV-algebra $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ given in *Example 7* (see [12]), then the results for MV-algebras can be applied also for intuitionistic fuzzy sets theory. Thus we obtain another proof the same results given in [3,4]. Recall that the embedding the family $\mathcal{F}$ to the suitable MV-algebra $\mathcal{M}$ is given by the construction state $\overline{m} : \mathcal{M} \to [0, 1]$ such that

$$\overline{m}\big((\mu_A, \nu_A)\big) = \mathbf{m}\big((\mu_A, 0_\Omega)\big) - \mathbf{m}\big((0_\Omega, 1_\Omega - \nu_A)\big),$$

where $\mathbf{m} : \mathcal{F} \to [0, 1]$ is the intuitionistic fuzzy state and $\overline{m}|\mathcal{F} = \mathbf{m}$ (see [12]).

On the other hand the family $\mathcal{K}$ of interval valued events introduced by L.A. Zadeh in [15]

$$\mathcal{K} = \{(\pi_C, \rho_C)\, ; \pi_C \leq \rho_C\},$$

where $\pi_C, \rho_C : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable functions, can be embedded to a suitable MV-algebra $(\mathcal{V}, \oplus, \odot, \neg, (0_\Omega, 0_\Omega), (1_\Omega, 1_\Omega))$ given in *Example 9* (see [10, 11]). Therefore the results for MV-algebras can be applied also for interval valued sets theory. Recall that the embedding the family $\mathcal{K}$ to the suitable MV-algebra $\mathcal{V}$ is given by the construction state $\overline{k} : \mathcal{V} \to [0, 1]$ such that

$$\overline{k}\big((\pi_C, \rho_C)\big) = k(\pi_C, 1_\Omega) - k(0_\Omega, 1 - \rho_C),$$

where $k : \mathcal{K} \to [0, 1]$ is the interval valued state and $\overline{k}|\mathcal{K} = k$ (see [10, 11]).

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