Notes on Intuitionistic Fuzzy Sets
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Comparison study based on the divergence measures between intuitionistic fuzzy sets and some applications

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Abstract: Many authors investigated possibilities how two fuzzy sets can be compared. The basic study of fuzzy sets theory was introduced by Lotfi Zadeh in 1965. The previous approach to the dissimilarities is too restrictive, because it assumes the inclusion relation between fuzzy sets and many pairs of fuzzy sets are incomparable to each other with respect to this relation. Therefore we need new concept for measuring - divergence measures. We discuss the divergences defined on more general objects, namely intuitionistic fuzzy sets (IFSs). We have focused on some applications of this concept to pattern recognition and to decision making. In both cases, we present an illustrative example.

Keywords: Intuitionistic fuzzy set, Divergence measure, Applications, Pattern recognition, Decision making.
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1 Introduction

Intuitionistic fuzzy sets (IFSs) have been introduced by K. Atanassov in 1983 [2]. For each point in the universe X a degree of membership and a degree of non-membership are assigned. More formally, Atanassov defined an **intuitionistic fuzzy set** (IF-set) as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where μ_A and ν_A are membership (non-membership) functions $\mu_A, \nu_A : X \to [0,1]$, such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$ and $\mu_A(x), \nu_A(x)$ are membership and non-membership degrees, respectively, of the element $x \in X$ to the set A. The family of all intuitionistic fuzzy sets defined on the universe X will be denoted by symbol $IFS(X)$.

The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation index. The lack of knowledge on the membership of an element $x \in X$ to the intuitionistic fuzzy set A is expressed by function π.

Clearly, IF-sets are one possible generalizations of the fuzzy sets. Each fuzzy set can be considered as a special case of an IF-set, such that $\nu_A(x) = 1 - \mu_A(x)$ and $\pi_A(x) = 0$. Moreover, each IF-set can be presented as an interval-valued fuzzy set since for each element $x \in X$ the following interval $[\mu_A(x), 1 - \nu_A(x)]$ can be associated.

Triangular norms have been introduced into the mathematical literature by Karl Menger in 1942. Triangular norms and conorms are operations which generalize the conjunction and disjunction in fuzzy logic. They were originally used to generalize the triangle inequality from classical metric spaces to probabilistic metric spaces. In the original axioms for triangular norms no associativity was required. Theory of continuous t-norms has two rather independent roots, namely, the field of functional equations and the theory of topological semigroups. The full characterization of continuous Archimedean t-norms by means of additive generators has been done after 1960 by Ling and Schweizer and Sklar.

Triangular norms will be mentioned in this section. These functions are useful for modeling a conjunction in fuzzy logic and intersection of fuzzy sets.

The triangular norm (t-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(a,b) = T(b,a)$, for all $a, b \in [0,1]$ (commutativity),

(T2) $T(T(a,b),c) = T(a,T(b,c))$, for all $a, b, c \in [0,1]$ (associativity),

(T3) $b \leq c \Rightarrow T(a,b) \leq T(a,c)$, for all $a, b, c \in [0,1]$ (monotonicity),

(T4) $T(a,1) = a$, for all $a \in [0,1]$ (boundary condition).

Therefore, the function T is a monotone, associative and commutative operation defined on $[0,1] \times [0,1]$ with neutral element 1. Some important examples of t-norms, so-called basic t-norms, are the following:

- Minimum t-norm: $T_M(a,b) = \min(a,b)$, for all $a, b \in [0,1]$,
- Product t-norm: $T_P(a,b) = a \cdot b$, for all $a, b \in [0,1]$,

- Łukasiewicz t-norm: $T_L(a,b) = \max(a+b-1, 0)$, for all $a, b \in [0,1]$,
- Drastic t-norm:

$$T_D(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1, \\ 0, & \text{otherwise .} \end{cases}$$

For these basic t-norms, it holds that $T_D \leq T_L \leq T_P \leq T_M$. In fact, for any t-norm T, it is fulfilled that $T_D \leq T \leq T_M$.

Changing the neutral element from 1 to 0, we obtain the triangular conorm (t-conorm), a function used for modeling a disjunction in fuzzy logic and union of fuzzy sets.

The t-norm T and t-conorm S are dual if and only if for each $a, b \in [0,1]$ the equation $T(a,b) = 1 - S(1-a, 1-b)$ is fulfilled.

For each previous example of basic t-norm we can consider its dual basic t-conorm as follows:

Let $A, B \in \mathcal{F}(X)$. Given a t-norm T and a t-conorm S,

- the intersection of A and B with respect to T is defined as the fuzzy set whose membership function is $A \cap_T B(x) = T(A(x), B(x))$, for all $x \in X$;
- the union of A and B with respect to S is defined as the fuzzy set whose membership function is $A \cup_S B(x) = S(A(x), B(x))$, for all $x \in X$.

Thus, we can denote by (X, T, S) the triple formed by the universe with the t-norm and the t-conorm defining the intersection and the union, respectively.

Usually we write shortly $\cup, \cap$ instead of $\cup_S, \cap_T$ if it is clear what triple (X, T, S) is considered.

Let A, B be IF-sets and $T(S)$ be the triangular norm (conorm). Then their union, intersection and complement will be defined in the following way:

(i) union of A and B:

$$A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\},$$

where $\mu_{A\cup B}(x) = S(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = T(\nu_A(x), \nu_B(x))$.

(ii) intersection of A and B:

$$A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cup B}(x)\rangle \mid x \in X\},$$

where $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cup B}(x) = S(\nu_A(x), \nu_B(x))$.

(iii) complement of A:

$$A^c = \{\langle x, \mu_{A^c}(x), \nu_{A^c}(x)\rangle \mid x \in X\},$$

where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$.

The possible orderings of two IF-sets A and B can be introduced in the following way:

$$A \leq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \text{ for all } x \in X,$$

$$A \preceq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \text{ for all } x \in X,$$

$$A = B \text{ if and only if } A \leq B \text{ and } B \leq A.$$

2 Divergence measures

In the framework of fuzzy set theory, we can find in the literature several measures of comparison between fuzzy sets. In 1996, Bouchon-Meunier ([3]) tried to define a general measure of comparison for fuzzy sets. Since more measures for comparing fuzzy sets have been introduced (see, among many others, [1, 14, 15]). A nice study about that can be found in [4]. Among all them, the most usual measures of comparison are dissimilarities [7].

There are several examples of dissimilarities. The restriction associated to this definition is only given for sets such that $A \subseteq B \subseteq C$, but there are a lot of sets which are not comparable with respect to $\subseteq$ and therefore, nothing is required for them.

Thus, we need a concept where the restriction about "proximity" are given for any set. In order to overcome this problem, another measure of comparison between fuzzy sets was proposed in [8], the divergence measure, which satisfies the following natural properties: it becomes zero when the two sets coincide, it is a non-negative and symmetric function and it decreases when the two subsets become "more similar" in some sense, i.e. if we add (in the sense of union) a subset C to both fuzzy subsets A, B, we obtain two subsets which are closer to each other; the same for the intersection. So we propose the following (see [6, 9]):

Definition 1. *Let (X, T, S) be a triple with X a universe and T and S any t-norm and t-conorm, respectively. A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a **divergence measure** with respect to (X, T, S) if and only if for all $A, B \in \mathcal{F}(X)$, D satisfies the following conditions:*

(D1) $D(A, A) = 0$;

(D2) $D(A, B) = D(B, A)$;

(D3) $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$, *for all $C \in \mathcal{F}(X)$, where the union and intersection are defined by means of S and T, respectively.*

The basic study related to the topic can be found in [10, 11].

Distance is a measure of the difference between two objects. For the case of IFSs the axiomatic definitions of a distance (metric) are described as follows:

Definition 2. *A distance (metric) d in an intuitionistic fuzzy set A in a universe of discourse X is a real function $d : A \times A \to \mathbb{R}$, which satisfies the following conditions for $x, y, z \in A$:*

(d1) $d(x, y) \geq 0$ *(non-negativity),*

(d2) $d(x, y) = 0 \Leftrightarrow x = y$ *(coincidence),*

(d3) $d(x, y) = d(y, x)$ *(symmetry),*

(d4) $d(x, z) + d(z, y) \geq d(x, y)$ *(triangle inequality).*

In the literature (see [10]) different measures can be found, namely the Type 1-Distance measures (based on the Hamming distance, normalized Hamming distance, Euclidean distance, normalized Euclidean distance) and Type 2-Distance measures (based on fuzzy implications). These approaches have been deeply developed by K. Atanassov (see [2]).

Analogously, for the similarity measure we have another approach:

Definition 3. *A similarity measure S in an intuitionistic fuzzy set A in a universe of discourse X is a real function $S : A \times A \to \mathbb{R}$, which satisfies the following conditions for $x, y, z \in A$:*

(S1) $0 \leq S(x, y) \leq 1$,

(S2) $S(x, y) = 1$ *if and only if* $x = y$,

(S3) $S(x, y) = S(y, x)$,

(S4) if $x \subseteq y \subseteq z$, *then* $S(x, z) \leq S(x, y)$ *and* $S(x, z) \leq S(y, z)$.

Also in this case many formulas in order to compute the similarity between IFSs have been appeared. For our purposes, we do not follow it, but we present the another concept of divergence measure originally based for fuzzy sets. Some generalizations of divergence measure between two intuitionistic fuzzy sets was presented in a similar way (see more in [5]).

Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is the finite universe and (X, T_M, S_M) is the triple. We present some examples of IF-divergence measures based on Hamming (D_{HM}) and Hausdorff distance (D_{HD}), respectively:

$$D_{HM}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \right),$$

$$D_{HD}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \max \left\{ |\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)| \right\}.$$

Additional theoretical approach was described in [5]. In the next text, we give some computational study for two applications (namely to pattern recognition and to decision making) in order to see how our result can differ depending on the triple (X, T, S) and the weighted vector α be used. The results are presented in the next two sections.

3 Applications to pattern recognition

Let X be a finite universe, let assume that the patterns $A_1, A_2, \ldots, A_m$ are represented by intuitionistic fuzzy sets and let B be a sample represented also by an intuitionistic fuzzy set.

As we can measure the difference between B and A_i for $i \in \{1, \ldots, m\}$, we obtain the finite set of divergences: $D(A_1, B), \ldots, D(A_m, B)$.

Finally, the sample B will be associated to the pattern A_j whenever

$$D(A_j, B) = \min_{i=1,\ldots,m} D(A_i, B).$$

That means, the sample B is classified into the pattern from which it differs least.

The following example is based on the one proposed in [12].

Example 1. Let us consider five kinds of mineral fields, each of them featured by the content of six minerals and containing one kind of typical hybrid mineral. Those five kinds of typical hybrid mineral are represented by intuitionistic fuzzy sets A_1, A_2, A_3, A_4 and A_5 in $X = \{x_1, \ldots, x_6\}$, respectively. Let us assume that there is another kind of hybrid mineral B, and we want to classify it into one of the aforementioned mineral fields. The minerals are described by means of the intuitionistic fuzzy sets defined in Table 1, where the pair $A_i(x_j) = \langle \mu_A(x), \nu_A(x) \rangle$; $i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 6\}$ represents values of the membership and the non-membership degrees, respectively.

X	x_1	x_2	x_3	x_4	x_5	x_6
A_1	$\langle 0.74, 0.19 \rangle$	$\langle 0.03, 0.68 \rangle$	$\langle 0.19, 0.73 \rangle$	$\langle 0.49, 0.32 \rangle$	$\langle 0.02, 0.88 \rangle$	$\langle 0.74, 0.17 \rangle$
A_2	$\langle 0.12, 0.71 \rangle$	$\langle 0.03, 0.94 \rangle$	$\langle 0.05, 0.52 \rangle$	$\langle 0.14, 0.62 \rangle$	$\langle 0.02, 0.91 \rangle$	$\langle 0.39, 0.52 \rangle$
A_3	$\langle 0.45, 0.49 \rangle$	$\langle 0.66, 0.28 \rangle$	$\langle 1.00, 0.00 \rangle$	$\langle 1.00, 0.00 \rangle$	$\langle 1.00, 0.00 \rangle$	$\langle 1.00, 0.00 \rangle$
A_4	$\langle 0.28, 0.61 \rangle$	$\langle 0.52, 0.44 \rangle$	$\langle 0.47, 0.50 \rangle$	$\langle 0.30, 0.39 \rangle$	$\langle 0.19, 0.64 \rangle$	$\langle 0.74, 0.22 \rangle$
A_5	$\langle 0.33, 0.57 \rangle$	$\langle 1.00, 0.00 \rangle$	$\langle 0.18, 0.62 \rangle$	$\langle 0.16, 0.84 \rangle$	$\langle 0.05, 0.72 \rangle$	$\langle 0.68, 0.23 \rangle$
B	$\langle 0.63, 0.29 \rangle$	$\langle 0.52, 0.41 \rangle$	$\langle 0.21, 0.73 \rangle$	$\langle 0.22, 0.48 \rangle$	$\langle 0.07, 0.92 \rangle$	$\langle 0.66, 0.30 \rangle$

Table 1. The kinds of hybrid minerals represented by intuitionistic fuzzy sets.

We consider the weighted vector assigned to the experts and established as follows:

$$\alpha = \{0.2, 0.3, 0.125, 0.125, 0.125, 0.125\}.$$

The values for the differences between membership degrees, i.e.,

$$\alpha_j |\mu_{A_i}(x_j) - \mu_B(x_j)|$$

where $i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 6\}$ are scheduled in the following Table 2.

	x_1	x_2	x_3	x_4	x_5	x_6
$\alpha_j \|\mu_{A_1}(x_j) - \mu_B(x_j)\|$	0.022	0.147	0.003	0.034	0.006	0.010
$\alpha_j \|\mu_{A_2}(x_j) - \mu_B(x_j)\|$	0.102	0.147	0.020	0.010	0.006	0.034
$\alpha_j \|\mu_{A_3}(x_j) - \mu_B(x_j)\|$	0.036	0.042	0.099	0.098	0.116	0.043
$\alpha_j \|\mu_{A_4}(x_j) - \mu_B(x_j)\|$	0.070	0	0.033	0.010	0.015	0.010
$\alpha_j \|\mu_{A_5}(x_j) - \mu_B(x_j)\|$	0.060	0.144	0.004	0.008	0.003	0.003

Table 2. Calculation for membership degrees

Analogously, we present the values for the differences between non-membership degrees, i.e.

$$\alpha_j |\nu_{A_i}(x_j) - \nu_B(x_j)|,$$

where $i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 6\}$, see Table 3.

	x_1	x_2	x_3	x_4	x_5	x_6
$\alpha_j \left\|\nu_{A_1}(x_j) - \nu_B(x_j)\right\|$	0.020	0.081	0	0.020	0.005	0.016
$\alpha_j \left\|\nu_{A_2}(x_j) - \nu_B(x_j)\right\|$	0.084	0.159	0.026	0.018	0.001	0.028
$\alpha_j \left\|\nu_{A_3}(x_j) - \nu_B(x_j)\right\|$	0.040	0.039	0.091	0.060	0.115	0.038
$\alpha_j \left\|\nu_{A_4}(x_j) - \nu_B(x_j)\right\|$	0.064	0.009	0.029	0.011	0.035	0.010
$\alpha_j \left\|\nu_{A_5}(x_j) - \nu_B(x_j)\right\|$	0.056	0.123	0.014	0.045	0.025	0.009

Table 3. Calculation for non-membership degrees

We use the Hamming distance (denoted by d_{HM}) to compute the differences between $A_i(x)$ and $B(x)$ for $i \in \{1, \ldots, 5\}$ and some element $x \in \{x_1, \ldots, x_6\}$ as follows:

$$d_{HM}\left(A_i(x), B(x)\right) = \frac{1}{2}\left[\left|\mu_{A_i}(x) - \mu_B(x)\right| + \left|\nu_{A_i}(x) - \nu_B(x)\right|\right].$$

	x_1	x_2	x_3	x_4	x_5	x_6
$d_{HM}(A_1, B)$	0.021	0.114	0.001	0.027	0.006	0.013
$d_{HM}(A_2, B)$	0.093	0.153	0.023	0.014	0.004	0.031
$d_{HM}(A_3, B)$	0.038	0.041	0.095	0.079	0.116	0.040
$d_{HM}(A_4, B)$	0.067	0.005	0.031	0.011	0.025	0.010
$d_{HM}(A_5, B)$	0.058	0.134	0.009	0.026	0.014	0.006

Table 4. Calculation for Hamming distance

In similar way, the Hausdorff distance (denoted by d_{HD}) to compute the differences between $A_i(x)$ and $B(x)$ for $i \in \{1, \ldots, 5\}$ and some element $x \in \{x_1, \ldots, x_6\}$ can be used:

$$d_{HD}\left(A_i(x), B(x)\right) = \max\left[\left|\mu_{A_i}(x) - \mu_B(x)\right|, \left|\nu_{A_i}(x) - \nu_B(x)\right|\right].$$

	x_1	x_2	x_3	x_4	x_5	x_6
$d_{HD}(A_1, B)$	0.022	0.147	0.003	0.034	0.006	0.016
$d_{HD}(A_2, B)$	0.102	0.159	0.026	0.018	0.006	0.034
$d_{HD}(A_3, B)$	0.040	0.042	0.099	0.098	0.116	0.043
$d_{HD}(A_4, B)$	0.070	0.009	0.033	0.011	0.035	0.010
$d_{HD}(A_5, B)$	0.060	0.144	0.014	0.045	0.025	0.009

Table 5. Calculation for Hausdorff distance

We will use our method to classify B. We consider the divergence measure proposed as we have mentioned previously. Let us suppose that $X = \{x_1, \ldots, x_6\}$ and for weighted vector $\alpha_x \geq 0$ for any $x \in X$ and $\sum_{x \in X} \alpha_x = 1$.

The divergence measure based on the Hamming distance

$$D(A_i, B) = \underset{x \in X}{S}\, d_{HM}\left(A_i(x), B(x)\right),$$

where $i \in \{1, \ldots, 5\}$ and $S \in \{S_M, S_P, S_L, S_D\}$ represents the set of all basic triangular conorms. We recall that:

(S_M) the maximum t-conorm:

$$D(A_i, B) = \max\left\{d_{HM}\left(A_i(x_1), B(x_1)\right), \ldots, d_{HM}\left(A_i(x_6), B(x_6)\right)\right\},$$

(S_P) the probabilistic sum:

$$D(A_i, B) = 1 - \prod_{j=1}^{6}\left(1 - d_{HM}\left(A_i(x_j), B(x_j)\right)\right),$$

(S_L) the Łukasiewicz t-conorm:

$$D(A_i, B) = \min\left\{1, \sum_{j=1}^{6} d_{HM}\left(A_i(x_j), B(x_j)\right)\right\},$$

(S_D) drastic t-conorm:
$D(A_i, B) = 1$ since there exists $j \in \{1, \ldots, 6\}$ such that $d_{HM}\left(A_i(x_j), B(x_j)\right) > 0$.

The results for divergences are scheduled in the following Table 6.

	S_M	S_P	S_L	S_D
$D(A_1, B)$	0.114	0.173	0.182	1
$D(A_2, B)$	0.153	0.285	0.317	1
$D(A_3, B)$	0.116	0.347	0.408	1
$D(A_4, B)$	0.067	0.140	0.148	1
$D(A_5, B)$	0.134	0.227	0.246	1

Table 6. The S-local divergences based on the Hamming distance obtained for $S \in \{S_M, S_P, S_L, S_D\}$.

We can conclude that for the Hamming distance:

- for S_M: $D(A_4, B) < D(A_1, B) < D(A_3, B) < D(A_5, B) < D(A_2, B)$,
- for S_P: $D(A_4, B) < D(A_1, B) < D(A_5, B) < D(A_2, B) < D(A_3, B)$,
- for S_L: $D(A_4, B) < D(A_1, B) < D(A_5, B) < D(A_2, B) < D(A_3, B)$,
- for S_D: $D(A_4, B) = D(A_1, B) = D(A_3, B) = D(A_5, B) = D(A_2, B)$.

The divergence measure based on the Hausdorff distance

$$D(A_i, B) = \underset{x \in X}{S}\, d_{HD}\left(A_i(x), B(x)\right),$$

where $i \in \{1, \ldots, 5\}$ and $S \in \{S_M, S_P, S_L, S_D\}$. The results for divergences are scheduled in the following Table 7.

	S_M	S_P	S_L	S_D
$D(A_1, B)$	0.147	0.214	0.228	1
$D(A_2, B)$	0.159	0.306	0.345	1
$D(A_3, B)$	0.116	0.367	0.437	1
$D(A_4, B)$	0.070	0.158	0.168	1
$D(A_5, B)$	0.144	0.268	0.230	1

Table 7. The S-local divergences based on the Hausdorff distance obtained for $S \in \{S_M, S_P, S_L, S_D\}$.

We can conclude that for the Hausdorff distance:

- for S_M: $D(A_4, B) < D(A_3, B) < D(A_5, B) < D(A_1, B) < D(A_2, B)$,
- for S_P: $D(A_4, B) < D(A_1, B) < D(A_5, B) < D(A_2, B) < D(A_3, B)$,
- for S_L: $D(A_4, B) < D(A_1, B) < D(A_5, B) < D(A_2, B) < D(A_3, B)$,
- for S_D: $D(A_4, B) = D(A_1, B) = D(A_3, B) = D(A_5, B) = D(A_2, B)$.

We can see that in all the cases we should classify B into the hybrid mineral A_4 (in the case of the drastic t-conorm it is just an option). However, the behaviour of any divergence is different. Thus, for instance, for the maximum t-conorm, a different rank for divergences between A_1, A_3, A_5 occured. In other case, for S_P and S_L the same results have been obtained, in both cases since the Hamming and the Hausdorff distance, respectively, has been considered. In general, all the points in the referential X are essential to obtain the value of the divergence. Of course, in the case of S_D, the information given by the divergence is insignificant.

Apart from the t-conorm used to define the S-divergence measure, the weights can also play an interesting role. For further work we could consider the weight vector

$$\alpha = \{k, 0.5 - k, 0.125, 0.125, 0.125, 0.125\}$$

for $k \in \{0, 0.1, \ldots, 0.5\}$.

4 Applications to decision making

Now, we will apply previous theoretical results in the multiple attribute decision making.

First we present the following notation: let $A = \{A_1, \ldots, A_m\}$ denote a set of m alternatives; let $X = \{x_1, \ldots, x_n\}$ be a set of n attributes; and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be its associated weight vector, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

Each alternative A_i will be expressed by an intuitionistic fuzzy set with the elements x_j, where $A_i(x_j)$ represents the degree in which alternative A_i agrees with attribute x_j. We create the new alternatives A^+ and A^- defined by

$$A^+ = \bigcup_{i=1}^{m} A_i, \qquad \text{and} \qquad A^- = \bigcap_{i=1}^{m} A_i.$$

The alternatives A^+ and A^- can be interpreted as the "optimal" and the "least optimal", respectively. In this sense, the preferred alternative A would be more similar to A^+ and more different from A^-, simultaneously.

Finally, we consider the quotient k_i defined as:

$$k_i = \frac{D(A_i, A^+)}{D(A_i, A^+) + D(A_i, A^-)}.$$

It means that if some alternative A_j has a quotient k_j for which $k_j < k_i$ for all $i \neq j$, then the alternative A_j is better as A_i in the sense previous described. Thus, the optimal is the alternative A_i whose k_i is the minimum.

The previous procedure will now be explained by the means of an example based on the one proposed in [13].

Example 2. The government has to decide among five different energy strategies: $A_1 - A_5$. Each of them is assessing four attributes: economic (x_{EC}), technological (x_T), environmental (x_{EN}) and socio-political (x_P). The following weight vector of these attributes $(\alpha_{EC}, \alpha_T, \alpha_{EN}, \alpha_P) = (0.4, 0.2, 0.3, 0.1)$ will be considered.

Let us assume that alternatives A_i are defined by the intuitionistic fuzzy sets given in Table 8.

X	x_{EC}	x_T	x_{EN}	x_P
A_1	$\langle 0.2, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.4\rangle$
A_2	$\langle 0.4, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.6, 0.3\rangle$
A_3	$\langle 0.5, 0.5\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.7, 0.2\rangle$
A_4	$\langle 0.3, 0.4\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.3\rangle$
A_5	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.3, 0.6\rangle$

Table 8. Definition of five energy strategies.

We will consider the triple (X, T, S), where S is used to define the union and T to define the intersection five alternatives represented by intuitionistic fuzzy sets. The corresponding intuitionistic fuzzy sets A^+ and A^- defined in Table 9 will be computed. For all basic t-norms and t-conorms, the obtained results will be compared. The results are illustrated in the following four cases:

(1) for the triple (X, T_M, S_M) and S_M-local divergence

$$A^+ = \bigcup_{i=1}^{m} A_i = \langle S_M(\mu_{A_1}, \ldots, \mu_{A_5}), T_M(\nu_{A_1}, \ldots, \nu_{A_5})\rangle,$$

$$A^- = \bigcap_{i=1}^{m} A_i = \langle T_M(\mu_{A_1}, \ldots, \mu_{A_5}), S_M(\nu_{A_1}, \ldots, \nu_{A_5})\rangle.$$

X	x_{EC}	x_T	x_{EN}	x_P
A^+	$\langle 0.8, 0.1\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.7, 0.2\rangle$
A^-	$\langle 0.2, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.3, 0.6\rangle$

Table 9. Definition of the most optimal and the least optimal alternatives for (X, T_M, S_M).

In the next step for any element $x \in \{x_{EC}, x_T, x_{EN}, x_P\}$ and each $i \in \{1, \ldots, 5\}$ we have computed the differences

$$\alpha_x \left|\mu_{A_i}(x) - \mu_{A^+}(x)\right|, \alpha_x \left|\nu_{A_i}(x) - \nu_{A^+}(x)\right|,$$

$$\alpha_x \left|\mu_{A_i}(x) - \mu_{A^-}(x)\right|, \alpha_x \left|\nu_{A_i}(x) - \nu_{A^-}(x)\right|,$$

since alpha vector $\alpha = (\alpha_{EC}, \alpha_T, \alpha_{EN}, \alpha_P) = (0.4, 0.2, 0.3, 0.1)$ is considered.

For the Hamming distance d_{HM} for $i \in \{1, \ldots, 5\}$ we obtain:

$$d_{HM}\left(A_i(x), A^+(x)\right) = \frac{1}{2}\left[\alpha_x \left|\mu_{A_i}(x) - \mu_{A^+}(x)\right| + \alpha_x \left|\nu_{A_i}(x) - \nu_{A^+}(x)\right|\right]$$

and similarly:

$$d_{HM}\left(A_i(x), A^-(x)\right) = \frac{1}{2}\left[\alpha_x \left|\mu_{A_i}(x) - \mu_{A^-}(x)\right| + \alpha_x \left|\nu_{A_i}(x) - \nu_{A^-}(x)\right|\right]$$

Analogously, for the Hausdorff distance d_{HD} for $i \in \{1, \ldots, 5\}$ we obtain:

$$d_{HD}\left(A_i(x), A^+(x)\right) = \max\left[\alpha_x \left|\mu_{A_i}(x) - \mu_{A^+}(x)\right|, \alpha_x \left|\nu_{A_i}(x) - \nu_{A^+}(x)\right|\right]$$

and similarly:

$$d_{HD}\left(A_i(x), A^-(x)\right) = \max\left[\alpha_x \left|\mu_{A_i}(x) - \mu_{A^-}(x)\right|, \alpha_x \left|\nu_{A_i}(x) - \nu_{A^-}(x)\right|\right]$$

For $i \in \{1, \ldots, 5\}$ we consider the S_M-local divergence measure proposed in the previous example with the maximum t-conorm, that is,

$$D_{HM}(A_i, A^+) = \underset{x \in X}{S_M}\left(d_{HM}\left(A_i(x), A^+(x)\right)\right),$$

$$D_{HM}(A_i, A^-) = \underset{x \in X}{S_M}\left(d_{HM}\left(A_i(x), A^-(x)\right)\right),$$

$$D_{HD}(A_i, A^+) = \underset{x \in X}{S_M}\left(d_{HD}\left(A_i(x), A^+(x)\right)\right),$$

$$D_{HD}(A_i, A^-) = \underset{x \in X}{S_M}\left(d_{HD}\left(A_i(x), A^-(x)\right)\right),$$

then the following divergences will be obtained.

Finally, we have computed the coefficients for each $i \in \{1, \ldots, 5\}$:

$$k_i = \frac{D_{HM}(A_i, A^+)}{D_{HM}(A_i, A^+) + D_{HM}(A_i, A^-)} \quad \text{and} \quad k_i = \frac{D_{HD}(A_i, A^+)}{D_{HD}(A_i, A^+) + D_{HD}(A_i, A^-)}$$

The results obtained for divergences based on the Hamming distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.18	0.14	0.57
2	0.16	0.18	0.47
3	0.14	0.21	0.40
4	0.16	0.17	0.49
5	0.21	0.20	0.51

Table 10. Comparison of five alternatives optimality for S_M-local divergence.

The results obtained for divergences based on the Hausdorff distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.24	0.15	0.62
2	0.16	0.21	0.43
3	0.16	0.24	0.40
4	0.20	0.18	0.53
5	0.24	0.24	0.50

Table 11. Comparison of five alternatives optimality for S_M-local divergence.

We see that:

- $k_3 < k_2 < k_4 < k_5 < k_1$ for Hamming distance,
- $k_3 < k_2 < k_5 < k_4 < k_1$ for Hausdorff distance,

and conclude that in accordance with the considered criteria the most optimal alternative is A_3 in both cases, but for alternatives A_4 and A_5 different results can be obtained.

(2) for the triple (X, T_P, S_P) and S_P-local divergence

X	x_{EC}	x_T	x_{EN}	x_P
A^+	$\langle 0.966, 0.004\rangle$	$\langle 0.996, 0\rangle$	$\langle 0.998, 0.001\rangle$	$\langle 0.979, 0.004\rangle$
A^-	$\langle 0.010, 0.919\rangle$	$\langle 0.118, 0.682\rangle$	$\langle 0.030, 0.879\rangle$	$\langle 0.032, 0.906\rangle$

Table 12. Definition of the most optimal and the least optimal alternatives for (X, T_P, S_P)

Now, we consider the S_P-local divergence measure with the probabilistic sum, that is,

$$D(A, B) = \underset{x\in X}{S_P}\, \alpha_x \cdot |A(x) - B(x)|,$$

then the following results will be obtained.

The results obtained for divergences based on the Hamming distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.369	0.403	0.478
2	0.342	0.430	0.443
3	0.289	0.477	0.377
4	0.321	0.448	0.417
5	0.359	0.434	0.453

Table 13. Comparison of five alternatives optimality for S_P-local divergence.

The results obtained for divergences based on the Hausdorff distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.452	0.451	0.501
2	0.369	0.445	0.453
3	0.304	0.501	0.378
4	0.388	0.489	0.442
5	0.401	0.455	0.468

Table 14. Comparison of five alternatives optimality for S_P-local divergence.

We conclude that $k_3 < k_4 < k_2 < k_5 < k_1$ for Hamming distance as well as for Hausdorff distance, i.e. the most optimal alternative is A_3, too.

(3) for the triple (X, T_L, S_L) and S_L-local divergence

X	x_{EC}	x_T	x_{EN}	x_P
A^+	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$
A^-	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$

Table 15. Definition of the most optimal and the least optimal alternatives for (X, T_L, S_L).

The results obtained for divergences based on the Hamming distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.44	0.58	0.431
2	0.40	0.61	0.396
3	0.33	0.68	0.323
4	0.37	0.64	0.366
5	0.42	0.59	0.416

Table 16. Comparison of five alternatives optimality for S_L-local divergence.

The results obtained for divergences based on the Hausdorff distance:

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	0.55	0.69	0.444
2	0.44	0.65	0.404
3	0.34	0.69	0.330
4	0.46	0.73	0.387
5	0.48	0.65	0.425

Table 17. Comparison of five alternatives optimality for S_L-local divergence.

For the Łukasiewicz t-conorm the same result can be concluded, i.e.,

$$k_3 < k_4 < k_2 < k_5 < k_1$$

for the Hamming and also the Hausdorff distance, i.e., the most optimal alternative is again A_3.

(4) for the triple (X, T_D, S_D) and S_D-local divergence

X	x_{EC}	x_T	x_{EN}	x_P
A^+	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$
A^-	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$

Table 18. Definition of the most optimal and the least optimal alternatives for (X, T_D, S_D).

i	$D(A_i, A^+)$	$D(A_i, A^-)$	k_i
1	1	1	0.5
2	1	1	0.5
3	1	1	0.5
4	1	1	0.5
5	1	1	0.5

Table 19. Comparison of five alternatives optimality for S_D-local divergence.

Since $k_i = 0.5$ for all $i \in \{1, \ldots, 5\}$, one can not make decision based on the obtained information.

We could consider not only different t-conorms, but also different weights for any particular problem and again different results could be obtained.

5 Conclusions

We have extended the results for divergence measures into the more general objects as intuitionistic fuzzy sets. Some examples of possible applications of this approach to pattern recognitions and to decision making are presented in this work. In fact, the results depend on the triple (X, T, S), weighted vector α as well as on the kind of metric used to define the divergence measure based on the distance (in our case only the Hamming or Hausdorff distance). In the future work, we intend to continue with deeper study in field of possible applications.

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