Notes on Intuitionistic Fuzzy Sets
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New results on the InterCriteria Analysis

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Abstract: The Intercriteria Analysis (ICA) is an decision support approach based on the apparatus of intuitionistic fuzzy sets and the relations between the evaluations of a set of given objects evaluated against a set of fixed criteria. In the present paper we discuss three new procedures for evaluation of ICA when used over objects containing elements of uncertainty.

Keywords: Data, InterCriteria Analysis, Index matrix, Intuitionistic fuzzy pair, Interval-valued intuitionistic fuzzy pair, Uncertainty.
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1 Introduction

The concept of InterCriteria Analysis (ICA) was introduced in [8]. It is based on the apparatus of the Index Matrices (IMs, see [1, 3, 12]) and of Intuitionistic Fuzzy Sets (IFSs, see, e.g., [2]). During last years a lot of papers over the theory and applications of ICA were published (see [11]).

Here, for the first time we discuss the idea to compare the differences of the evaluation of the objects with a fixed threshold. In a result we will obtain Intuitionistic Fuzzy Pairs (IFP, see [4, 10]), determining the nearness between the criteria.

2 Short notes on intuitionistic fuzzy and interval-valued intuitionistic fuzzy pairs

The Intuitionistic Fuzzy Pair (IFP, see [4, 10]) is an object in the form $\langle a, b\rangle$, where $a = a(x), b = b(x) \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and which components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

The number $c(x) = 1-a-b$ is called a degree of uncertainty or indeterminacy. The particular case, when $c(x) = 0$, the IFP x can be called a fuzzy pair.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. We define the relations

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x > y & \text{iff} & a > c \text{ and } b < d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x = y & \text{iff} & a = c \text{ and } b = d \end{array} .$$

The Interval-Valued IFP (IVIFP, see [5,9]) is an object in the form $\langle A, B\rangle$, where $A, B \subseteq [0, 1]$ and $\sup A + \sup B \leq 1$. Now, intervals A and B are interpreted as the intervals where the degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc., are placed.

Here, for the first time, we define two new operators over IVIFS, which will be used below for IVIFIMs.

Let us define for a fixed IVIFP x

$$\begin{array}{ll} \iota(x) & = \sup A(x) - \inf A(x) + \sup B(x) - \inf B(x), \\ \omega(x) & = 1 - \sup A(x) - \sup B(x). \end{array}$$

We call operator ι (*"iota"*) an interior degree of uncertainty, and we call operator ω (*"omega"*) an outside degree of uncertainty. We can see directly that

$$\iota(x) = 0 \text{ if and only if the IVIFP } x \text{ is an IFP}$$

and

$$\iota(x) = \omega(x) = 0 \text{ if and only if the IVIFP } x \text{ is a fuzzy pair.}$$

3 Short remarks on index matrices

The concept of Index Matrix (IM) was discussed in a series of papers collected in [1, 3].

Let I be a fixed set of indices and $\mathcal{R}$ be the set of the real numbers. By IM with index sets K and L $(K, L \subset I)$, we denote the object:

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \cdots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \cdots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \cdots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \le i \le m$, and $1 \le j \le n : a_{k_i,l_j} \in \mathcal{R}$.

In [1, 3], different operations, relations and operators are defined over IMs. For the needs of the present research, we will introduce the definitions of some of them.

When elements a_{k_i,l_j} are some variables, propositions or formulas, we obtain an extended IM with elements from the respective type. Then, we can define the evaluation function V that juxtaposes to this IM a new one with elements – IFPs $\langle \mu, \nu \rangle$, where $\mu, \nu, \mu + \nu \in [0, 1]$. The new IM, called Intuitionistic Fuzzy IM (IFIM), contains the evaluations of the variables, propositions, etc., i.e., it has the form

$$V([K, L, \{a_{k_i,l_j}\}]) = [K, L, \{V(a_{k_i,l_j})\}] = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$= \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \cdots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \cdots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \cdots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \cdots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \cdots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \cdots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$: $V(a_{k_i,l_j}) = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ and $0 \le \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \le 1$.

When IFPs $\langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle$ are changed with the IVIFPs $\langle M_{k_m,l_j}, N_{k_m,l_j} \rangle$, the matrix is called an Interval-Valued IFIM (IVIFIM, see [5]).

4 Three new versions of InterCriteria Analysis

Following and modifying [8], we describe the first new version of the ICA.

Let us have the set of objects $O = \{O_1, O_2, ..., O_n\}$ that must be evaluated by criteria from the set $C = \{C_1, C_2, ..., C_m\}$.

Let us have an IM

$$A = \begin{array}{c|ccccccc} & O_1 & \cdots & O_i & \cdots & O_j & \cdots & O_n \\ \hline C_1 & a_{C_1,O_1} & \cdots & a_{C_1,O_i} & \cdots & a_{C_1,O_j} & \cdots & a_{C_1,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_k & a_{C_k,O_1} & \cdots & a_{C_k,O_i} & \cdots & a_{C_k,O_j} & \cdots & a_{C_k,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_l & a_{C_l,O_1} & \cdots & a_{C_l,O_i} & \cdots & a_{C_l,O_j} & \cdots & a_{C_l,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_m & a_{C_m,O_1} & \cdots & a_{C_m,O_i} & \cdots & a_{C_m,O_j} & \cdots & a_{C_m,O_n} \end{array}, \qquad (*)$$

where for every p, q $(1 \leq p \leq m\,,\ 1 \leq q \leq n)$:

(1) C_p is a criterion, taking part in the evaluation,

(2) O_q is an object being evaluated.

(3) a_{C_p,O_q} is a real number that represents the evaluations of the q-th object by the p-th criterion,

(4) ε_p is a fixed threshold for the p-th criterion.

For example, ε_p can have the form

$$\varepsilon_p = \omega(\max_{1 \leq q \leq m} a_{C_p,O_q} - \min_{1 \leq q \leq m} a_{C_p,O_q}),$$

where ω can be equal for some of $\varepsilon_1, \ldots, \varepsilon_m$ (e.g., the maximum or the minimum).

Let $S^{\mu}_{k,l}$ be the number of cases in which

$$|a_{C_k,O_i} - a_{C_k,O_j}| < \varepsilon_k$$

and

$$|a_{C_l,O_i} - a_{C_l,O_j}| < \varepsilon_l.$$

Let $S^{\nu}_{k,l}$ be the number of cases in which

$$|a_{C_k,O_i} - a_{C_k,O_j}| > \varepsilon_k$$

or

$$|a_{C_l,O_i} - a_{C_l,O_j}| > \varepsilon_l.$$

Let $S^{\pi}_{k,l}$ be the number of cases in which

$$|a_{C_k,O_i} - a_{C_k,O_j}| = \varepsilon_k$$

or

$$|a_{C_l,O_i} - a_{C_l,O_j}| = \varepsilon_l.$$

Obviously,

$$S^{\mu}_{k,l} + S^{\nu}_{k,l} + S^{\pi}_{k,l} = \frac{n(n-1)}{2}.$$

Now, for every k, l, such that $1 \leq k, l \leq m$ and for $n \geq 2$, we define

$$\mu_{C_k,C_l} = 2\frac{S^{\mu}_{k,l}}{n(n-1)}, \quad \nu_{C_k,C_l} = 2\frac{S^{\nu}_{k,l}}{n(n-1)}.$$

Hence,

$$\mu_{C_k,C_l} + \nu_{C_k,C_l} = 2\frac{S^{\mu}_{k,l}}{n(n-1)} + 2\frac{S^{\nu}_{k,l}}{n(n-1)} \leq 1.$$

Therefore, $\langle \mu_{C_k,C_l}, \nu_{C_k,C_l} \rangle$ is an IFP. Now, we can construct the IM

$$\begin{array}{c|ccc} & C_1 & \cdots & C_m \\ \hline C_1 & \langle \mu_{C_1,C_1}, \nu_{C_1,C_1} \rangle & \cdots & \langle \mu_{C_1,C_m}, \nu_{C_1,C_m} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ C_m & \langle \mu_{C_m,C_1}, \nu_{C_m,C_1} \rangle & \cdots & \langle \mu_{C_m,C_m}, \nu_{C_m,C_m} \rangle \end{array},$$

that determines the degrees of correspondence between criteria $C_1, \ldots, C_m$.

Now, following the idea from [6], we can show the geometrical interpretation of the elements of the above IM.

Let $\alpha, \beta, \gamma, \delta, \varphi \in [0, 1]$ and

$$\alpha + \beta \leq 1,$$

$$\gamma + \delta \leq 1,$$

$$\varphi \leq \min(\alpha, \delta).$$

These numbers (thresholds) determine the criteria that are in:

- strong positive consonance – if $\langle \mu_{C_r,C_s}, \nu_{C_r,C_s} \rangle > \langle \alpha, \beta \rangle$,
- positive consonance – if $\langle \mu_{C_r,C_s}, \nu_{C_r,C_s} \rangle \geq \langle \alpha, \beta \rangle$,
- strong negative consonance – if $\langle \mu_{C_r,C_s}, \nu_{C_r,C_s} \rangle < \langle \gamma, \delta \rangle$,
- negative consonance – if $\langle \mu_{C_r,C_s}, \nu_{C_r,C_s} \rangle \leq \langle \gamma, \delta \rangle$,
- dissonance – if $\mu_{C_r,C_s} < \alpha, \nu_{C_r,C_s} < \delta$ and $\mu_{C_r,C_s} + \nu_{C_r,C_s} \geq \varphi$,
- uncertainty – if $\mu_{C_r,C_s} + \nu_{C_r,C_s} < \varphi$

(see Figure 1).

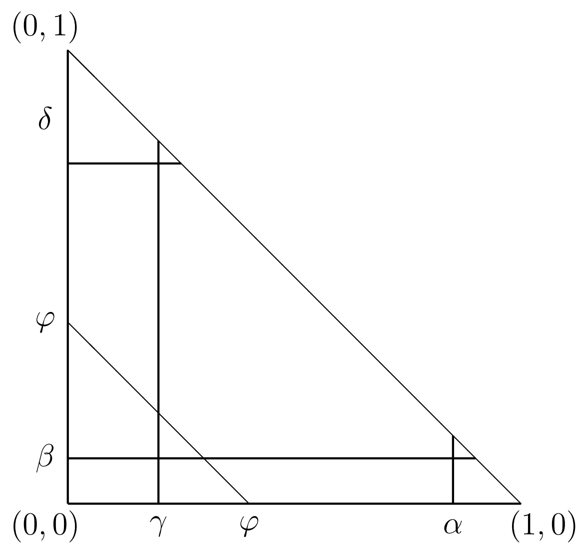

Figure 1.

For $\alpha, \beta, \gamma, \delta$ we can use, e.g.

$$\alpha = \delta = 1 - \omega, \;\; \beta = \gamma = \omega,$$

or

$$\alpha = \delta = \frac{2}{3}, \;\; \beta = \gamma = \frac{1}{3},$$

or

$$\alpha = \delta = \frac{3}{4}, \;\; \beta = \gamma = \frac{1}{4}.$$

Second, following the idea of the algorithm from [5,7], we will modify the above construction as follows.

Let in the IM A from $(*)$, $a_{C_i,O_k} = [M_{C_i,O_k}, N_{C_i,O_k}]$ (for brevity, we will call it as a-object), where $M_{C_i,O_k}, N_{C_i,O_k} \subseteq [0,1]$ and $\sup M_{C_i,O_k} + \sup N_{C_i,O_k} \leq 1$. Therefore, we can juxtapose to this IVIFP the real numbers ι_{C_i,O_k} and can repeat the above procedure, using thresholds ε_k and ε_l, or using the standard procedure from [8]. The same two cases can be realized using real numbers ω_{C_i,O_k}.

Third, let relations $R_1, R_2, \ldots, R_8 \in \{<, =, >\}$. Let

- $S^{U,\inf}_{C_k,C_l}$ be the number of cases in which the relations in the expressions

$$R_1(\inf M_{C_k,O_i}, \inf M_{C_k,O_j}) \text{ and } R_2(\inf M_{C_l,O_i}, \inf M_{C_l,O_j}),$$
$$R_3(\sup M_{C_k,O_i}, \sup M_{C_k,O_j}) \text{ and } R_4(\sup M_{C_l,O_i}, \sup M_{C_l,O_j}),$$
$$R_5(\inf N_{C_k,O_i}, \inf N_{C_k,O_j}) \text{ and } R_6(\inf N_{C_l,O_i}, \inf N_{C_l,O_j}),$$
$$R_7(\sup N_{C_k,O_i}, \sup N_{C_k,O_j}) \text{ and } R_8(\sup N_{C_l,O_i}, \sup N_{C_l,O_j})$$

  coincide and they are elements of set $\{<, >\}$;

- $S^{U,*}_{C_k,C_l}$ be the number of cases in which the relations in the expressions

$$R_1(\inf M_{C_k,O_i}, \inf M_{C_k,O_j}) \text{ and } R_2(\inf M_{C_l,O_i}, \inf M_{C_l,O_j}),$$
$$R_3(\sup M_{C_k,O_i}, \sup M_{C_k,O_j}) \text{ and } R_4(\sup M_{C_l,O_i}, \sup M_{C_l,O_j}),$$
$$R_5(\inf N_{C_k,O_i}, \inf N_{C_k,O_j}) \text{ and } R_6(\inf N_{C_l,O_i}, \inf N_{C_l,O_j}),$$
$$R_7(\sup N_{C_k,O_i}, \sup N_{C_k,O_j}) \text{ and } R_8(\sup N_{C_l,O_i}, \sup N_{C_l,O_j})$$

  coincide and a part of them are elements of set $\{<, >\}$, but between them there are relations "$=$";

- $S^{V,\inf}_{C_k,C_l}$ be the number of cases in which the relations in the expressions

$$R_1(\inf M_{C_k,O_i}, \inf M_{C_k,O_j}) \text{ and } R_2(\inf M_{C_l,O_i}, \inf M_{C_l,O_j}),$$
$$R_3(\sup M_{C_k,O_i}, \sup M_{C_k,O_j}) \text{ and } R_4(\sup M_{C_l,O_i}, \sup M_{C_l,O_j}),$$
$$R_5(\inf N_{C_k,O_i}, \inf N_{C_k,O_j}) \text{ and } R_6(\inf N_{C_l,O_i}, \inf N_{C_l,O_j}),$$
$$R_7(\sup N_{C_k,O_i}, \sup N_{C_k,O_j}) \text{ and } R_8(\sup N_{C_l,O_i}, \sup N_{C_l,O_j})$$

  are elements of set $\{<, >\}$, but in each pair they are opposite;

- $S^{V,*}_{C_k,C_l}$ be the number of cases in which a part of the relations in the expressions

$$R_1(\inf M_{C_k,O_i}, \inf M_{C_k,O_j}) \text{ and } R_2(\inf M_{C_l,O_i}, \inf M_{C_l,O_j}),$$

$$R_3(\sup M_{C_k,O_i}, \sup M_{C_k,O_j}) \text{ and } R_4(\sup M_{C_l,O_i}, \sup M_{C_l,O_j}),$$

$$R_5(\inf N_{C_k,O_i}, \inf N_{C_k,O_j}) \text{ and } R_6(\inf N_{C_l,O_i}, \inf N_{C_l,O_j}),$$

$$R_7(\sup N_{C_k,O_i}, \sup N_{C_k,O_j}) \text{ and } R_8(\sup N_{C_l,O_i}, \sup N_{C_l,O_j})$$

are elements of set $\{<,>\}$ and they are opposite in each pair, but between them there are relations "=" that can be observed in only one of the relations in each pair;

- $S^{W}_{C_k,C_l}$ be the number of the remaining cases.

Let

$$N = \frac{n(n-1)}{2}.$$

Obviously,

$$S^{U,\inf}_{C_k,C_l} + S^{U,*}_{C_k,C_l} + S^{V,\inf}_{C_k,C_l} + S^{V,*}_{C_k,C_l} = N.$$

Now, we define

$$\begin{aligned}
\inf M_{C_k,C_l} &= \tfrac{S^{U,\inf}_{C_k,C_l}}{N},\\
\sup M_{C_k,C_l} &= \tfrac{S^{U,\inf}_{C_k,C_l}+S^{U,*}_{C_k,C_l}}{N},\\
\inf N_{C_k,C_l} &= \tfrac{S^{V,\inf}_{C_k,C_l}}{N},\\
\sup N_{C_k,C_l} &= \tfrac{S^{V,\inf}_{C_k,C_l}+S^{V,*}_{C_k,C_l}}{N}.
\end{aligned}$$

Hence, we can construct the intervals

$$M_{C_k,C_l} = [\inf M_{C_k,C_l}, \sup M_{C_k,C_l}]$$

and

$$N_{C_k,C_l} = [\inf N_{C_k,C_l}, \sup N_{C_k,C_l}],$$

so that

$$\sup M_{C_k,C_l} + \sup N_{C_k,C_l} = \frac{S^{U,\inf}_{C_k,C_l} + S^{U,*}_{C_k,C_l} + S^{V,\inf}_{C_k,C_l} + S^{V,*}_{C_k,C_l}}{N} \leq 1.$$

Using the above values for pairs $\langle M_{C_k,C_l}, N_{C_k,C_l}\rangle$, we can construct the final form of the IM that determines the degrees of correspondence between criteria $C_1, ..., C_m$:

$$\begin{array}{c|ccc}
 & C_1 & \cdots & C_m \\
\hline
C_1 & \langle M_{C_1,C_1}, N_{C_1,C_1}\rangle & \cdots & \langle M_{C_1,C_m}, N_{C_1,C_m}\rangle \\
\vdots & \vdots & \ddots & \vdots \\
C_m & \langle M_{C_m,C_1}, N_{C_m,C_1}\rangle & \cdots & \langle M_{C_m,C_m}, N_{C_m,C_m}\rangle
\end{array}.$$

If we know which criteria are more complex, or whose measurement or evaluation is a matter of more time, cost or resources, then we can omit these criteria keeping the simpler, cheaper or faster ones.

Now, we discuss a procedure for simplifying the IM that determines the degrees of correspondence between criteria.

Let $\gamma, \delta \in [0, 1]$ be given, so that $\gamma + \delta \leq 1$. We say that criteria C_k and C_l are in

- strong (γ, δ)-positive consonance, if $\inf M_{C_k,C_l} > \gamma$ and $\sup N_{C_k,C_l} < \delta$;
- weak (γ, δ)-positive consonance, if $\sup M_{C_k,C_l} > \gamma$ and $\inf N_{C_k,C_l} < \delta$;
- strong (γ, δ)-negative consonance, if $\sup M_{C_k,C_l} < \gamma$ and $\inf N_{C_k,C_l} > \delta$;
- weak (γ, δ)-negative consonance, if $\inf M_{C_k,C_l} < \gamma$ and $\sup N_{C_k,C_l} > \delta$;
- (γ, δ)-dissonance, otherwise.

Analogically, we can compare the objects, determining which of them are in strong (γ, δ)-positive, weak (γ, δ)-positive, strong (γ, δ)-negative, weak (γ, δ)-negative consonance, or in (γ, δ)-dissonance.

5 Conclusion

The ICA was generated ten years ago and for this period it has shown and proven its usefulness and effectiveness. It has been modified in various directions and has found applications in a different areas of the theory and practice. Its toolbox contains a lot of procedures for evaluation when the ICA is applied to different objects – real numbers, sentences and predicates, intuitionistic fuzzy pairs, interval-valued intuitionistic fuzzy pairs, etc. With the present paper, we extend this toolbox with three new procedures that can be used over objects containing elements of uncertainty.

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