Notes on Intuitionistic Fuzzy Sets
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Interval-valued intuitionistic fuzzy topological groups: Some properties

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Abstract: In this study, we introduce the concepts of interval-valued intuitionistic fuzzy group (abbreviated IVIFG), interval-valued intuitionistic fuzzy topological group (abbreviated IVIFTG), and interval-valued intuitionistic fuzzy product topology (abbreviated IVIFPT). We delve into the fundamental features of IVIFG, IVIFTG, and IVIFPT in further detail. The characterization of IVIFG and IVIFTG is examined using the concepts of interval-valued intuitionistic fuzzy continuous (abbreviated as IVIF-continuous) and interval-valued intuitionistic fuzzy relatively continuous (abbreviated as IVIF-relatively continuous).

Keywords: Interval-valued intuitionistic fuzzy sets, IVIF-topology, IVIF-group, IVIF-topological group, IVIF-product topology, IVIF-product topological group.
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1 Introduction

The components of a theory of fuzzy groups, including intuitionistic fuzzy (IF) groups, have been developed by numerous writers in recent years. To address complex uncertainty, the theory of interval-valued IF sets proves to be a useful tool in the development of decision-making models.

Zadeh [11] introduced the idea of a fuzzy set as a generalization of an ordinary set. Following that, several researchers [1, 5, 6, 8, 10] utilized topological group theory and algebras to apply the concept of fuzzy sets. Atanassov [2, 3] introduced the concept of IF sets, and furthermore, [4] developed the interval-valued IF set. Many researchers then employed IF sets to introduce the concepts of IF subgroups, subrings, and ideals. Kul Hur et al. [7] proposed the concepts of IF topological groups and IF quotient groups. Tapas Kumar Mondal and S.K. Samanta [9] presented the topology of IVIF sets and briefly studied its features. This paper introduces the generalized concept of a topological group under the IVIF set and defines it along with its basic property structure. Furthermore, the products of IVIF-topological groups are defined, and their properties are examined in an elementary manner.

2 Some basic definitions of interval-valued intuitionistic fuzzy groups

In this work, let's assume that $\mathcal{I}_{\mathcal{IV}}$ is a collection of all closed subintervals of $[0,1]$. Additionally, $\mathcal{I}_{\mathcal{IV}}^{X}$ and $\mathcal{I}_{\mathcal{IV}}^{Y}$ represent the notions characterizing the set of all closed subintervals of $[0,1]$ for X and Y, respectively. We denote the degree of lower and upper values of membership and non-membership intervals as $[\underline{\mu},\overline{\mu}]$ and $[\underline{\nu},\overline{\nu}]$. The interval-valued intuitionistic fuzzy set $\mathcal{I}_{\mathcal{IV}}$ is denoted by the IVIF-set $\mathcal{I}_{\mathcal{IV}}$.

Definition 2.1. *If $\mathcal{G}_{\mathcal{IV}} \in IVIFS(\mathcal{S})$ and is recognized as an interval-valued intuitionistic fuzzy group of $\mathcal{S}$, then the following conditions are satisfied:*

1. $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy) \geq ([\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)) \wedge ([\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y))$ *and*
   $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy) \leq ([\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)) \vee ([\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)), \forall x, y \in \mathcal{S}$
2. $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$ *and* $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x), \forall x \in \mathcal{S}$

Proposition 2.2. *$\mathcal{G}_{\mathcal{IV}}$ is an interval-valued intuitionistic fuzzy group of $\mathcal{S}$, if and only if $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$ for each $x, y \in \mathcal{S}$*

Proof. Let $\langle x, [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}, [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}} \rangle$, $\langle y, [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}, [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}} \rangle \in \mathcal{G}_{\mathcal{IV}}$ and $\mathcal{G}_{\mathcal{IV}}$ is in IVIF-group on $\mathcal{S}$. Consider,

$$[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$$

and

$$[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$$

Conversely, assume that $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$. By Definition of IVIF-group, $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$, implies that $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)$,
$[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y), \forall x, y \in \mathcal{S}$.
Hence $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-group on $\mathcal{S}$. $\square$

Proposition 2.3. *If $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-group on $\mathcal{S}$, then $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x^{-1}) = [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(e)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x^{-1}) = [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(e), \forall x \in \mathcal{S}$, where e is the unity of $\mathcal{S}$.*

Definition 2.4. *Let $\mathcal{G}_{\mathcal{IV}}$ be an IVIF-set on $\mathcal{S}$ have a subproperty-$*$, if for any $\mathcal{T} \subset \mathcal{S}$ then there exists a $p_0 \in \mathcal{T}$ such that $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(p_0) = \bigvee_{p\in\mathcal{T}}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(p_0) = \bigwedge_{p\in\mathcal{T}}[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$. This property gives the relationship between the membership and non-membership intervals of the topology $\mathcal{T}$ which defins under a group $\mathcal{S}$*

Definition 2.5. *Let $\mathcal{G}_{\mathcal{IV}}$ be an IVIF-set in $\mathcal{S}$ and $\mathfrak{f}$ is a function defined on $\mathcal{S}$. Let $\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})$ be defined on $\mathfrak{f}(\mathcal{S})$ by $[\underline{\mu},\overline{\mu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(y) = \bigvee_{x\in\mathfrak{f}^{-1}(y)}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$ and $[\underline{\nu},\overline{\nu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(y) = \bigwedge_{x\in\mathfrak{f}^{-1}(y)}[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$ for all $y \in \mathfrak{f}(\mathcal{S})$. Then $\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})$ in $\mathfrak{f}(\mathcal{S})$ is called the image of $\mathcal{G}_{\mathcal{IV}}$ under $\mathfrak{f}$.*

Proposition 2.6. *Let $\mathfrak{f}$ be an interval-valued intuitionistic fuzzy group homomorphism(IVIF-group homomorphism) from $\mathcal{I}^X_{\mathcal{IV}} \to \mathcal{I}^Y_{\mathcal{IV}}$ then*

1. *If $\mathcal{G}_{\mathcal{IV}} \in IVIFG(\mathcal{S})$ then the inverse image $\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})$ of $\mathcal{G}_{\mathcal{IV}}$ is in an IVIF-group of $\mathcal{I}^X_{\mathcal{IV}}$.*
2. *If $\mathcal{G}_{\mathcal{IV}} \in IVIFG(\mathcal{I}^X_{\mathcal{IV}})$ and $\mathcal{G}_{\mathcal{IV}}$ have the subproperty-$*$, then $\mathfrak{f}(\mathcal{G}_{\mathcal{IV}}) \in IVIFG(\mathcal{I}^Y_{\mathcal{IV}})$.*

Proof. For all membership and non-membership intervals of $x, y \in \mathcal{I}^X_{\mathcal{IV}}$ and let $\mathcal{G}_{\mathcal{IV}}$ be an IVIF-group from $\mathcal{I}^X_{\mathcal{IV}}$ into $\mathcal{I}^Y_{\mathcal{IV}}$ and defined by $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(\mathfrak{f}(x))$, and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(\mathfrak{f}(x))$. Consider,

$$\begin{aligned}
[\underline{\mu},\overline{\mu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(xy^{-1}) &= [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(\mathfrak{f}(xy^{-1})) \\
&\geq [\underline{\mu},\overline{\mu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(x) \wedge [\underline{\mu},\overline{\mu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(y)
\end{aligned}$$

$$\begin{aligned}
[\underline{\nu},\overline{\nu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(xy^{-1}) &= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(\mathfrak{f}(xy^{-1})) \\
&\leq [\underline{\nu},\overline{\nu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(x) \vee [\underline{\nu},\overline{\nu}]_{\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}})}(y)
\end{aligned}$$

Hence $\mathfrak{f}^{-1}(\mathcal{G}_{\mathcal{IV}}) \in IVIFG(\mathcal{I}^X_{\mathcal{IV}})$.

Let membership and non-membership intervals of $u, v \in \mathcal{I}^Y_{\mathcal{IV}}$. If $\mathfrak{f}^{-1}(u)$ or $\mathfrak{f}^{-1}(v)$ is empty, then the proof is trivial.
Suppose $\mathfrak{f}^{-1}(u)$ and $\mathfrak{f}^{-1}(v)$ are nonempty. If $u_0 \in \mathfrak{f}^{-1}(u)$ and $v_0 \in \mathfrak{f}^{-1}(v)$ then $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(u_0) = \bigvee_{p\in\mathfrak{f}^{-1}(u)}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$ and $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(v_0) = \bigvee_{p\in\mathfrak{f}^{-1}(v)}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$. Now,

$$\begin{aligned}
[\underline{\mu},\overline{\mu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(uv^{-1}) &= \bigvee_{\kappa\in\mathfrak{f}^{-1}(uv^{-1})} [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(\kappa) \\
&= \bigvee_{\kappa\in\mathfrak{f}^{-1}(uv^{-1})} [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(u_0 v_0) \\
&\geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(u) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(v)
\end{aligned}$$

Similarly, $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(u_0) = \bigwedge_{p\in\mathfrak{f}^{-1}(u)}[\underline{nu},\overline{nu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(v_0) = \bigwedge_{p\in\mathfrak{f}^{-1}(v)}[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(p)$.
Next consider,

$$\begin{aligned}
[\underline{\nu},\overline{\nu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(uv^{-1}) &= \bigwedge_{\kappa\in\mathfrak{f}^{-1}(uv^{-1})}[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(\kappa)\\
&= \bigwedge_{\kappa\in\mathfrak{f}^{-1}(uv^{-1})}[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(u_0v_0)\\
&\leq [\underline{\nu},\overline{\nu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(u) \vee [\underline{\nu},\overline{\nu}]_{\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})}(v)
\end{aligned}$$

Hence $\mathfrak{f}(\mathcal{G}_{\mathcal{IV}}) \in IVIFG(\mathcal{I}^Y_{\mathcal{IV}})$. □

Definition 2.7. *Let $\mathfrak{f} : \mathcal{I}^X_{\mathcal{IV}} \to \mathcal{I}^Y_{\mathcal{IV}}$ be an IVIF-group homomorphism and $\mathcal{G}_{\mathcal{IV}} \in IVIFG(\mathcal{I}^X_{\mathcal{IV}})$. If $\mathfrak{f}(x_1) = \mathfrak{f}(x_2)$ implies $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x_1) = [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x_2)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x_1) = [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x_2)$, for each membership and non-membership intervals of $x_1, x_2 \in \mathcal{I}^X_{\mathcal{IV}}$, then $\mathcal{G}_{\mathcal{IV}}$ is said to be an interval-valued intuitionistic fuzzy invariant (abbreviated IVIF-invariant).*

Remark 2.8. *The IVIF-homomorphic image of $\mathfrak{f}(\mathcal{G}_{\mathcal{IV}})$ on $\mathcal{S}$ is an IVIF-group if $\mathcal{G}_{\mathcal{IV}}$ is IVIF-invariant.*

Definition 2.9. *For every $a \in \mathcal{G}_{\mathcal{IV}}$, the mapping $\mathscr{I}_a : \mathcal{I}^X_{\mathcal{IV}} \to \mathcal{I}^X_{\mathcal{IV}}$ is said to be a RIVIF-translation on $\mathcal{I}^X_{\mathcal{IV}}$ into itself, defined by $\mathscr{I}_a = \left\langle [\underline{\mu},\overline{\mu}], [\underline{\nu},\overline{\nu}] \right\rangle (xa)$, for all intervals of $[\mathfrak{M},\mathfrak{N}](x) \in \mathcal{I}^X_{\mathcal{IV}}$. Also $\mathscr{I}_a{}^{-1} = \mathscr{I}_{a^{-1}}$.*

Definition 2.10. *For each $a \in \mathcal{G}_{\mathcal{IV}}$, the mapping ${}_a\mathscr{I} : \mathcal{I}^X_{\mathcal{IV}} \to \mathcal{I}^X_{\mathcal{IV}}$ is said to be a LIVIF-translation on $\mathcal{I}^X_{\mathcal{IV}}$ into itself, defined by ${}_a\mathscr{I} = \left\langle [\underline{\mu},\overline{\mu}], [\underline{\nu},\overline{\nu}] \right\rangle (ax)$, for all intervals of $[\mathfrak{M},\mathfrak{N}](x) \in \mathcal{I}^X_{\mathcal{IV}}$. Also ${}_a\mathscr{I}^{-1} ={}_{a^{-1}} \mathscr{I}$.*

Proposition 2.11. *If $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-group of $\mathcal{I}^X_{\mathcal{IV}}$, then $\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}}) = {}_a\mathscr{I}(\mathcal{G}_{\mathcal{IV}}) = \mathcal{G}_{\mathcal{IV}}$*

Proof. Let $a \in \mathcal{G}_{\mathcal{IV}}$ on $\mathcal{S}$ and membership and non-membership intervals of $x \in \mathcal{I}^X_{\mathcal{IV}}$.

$$\begin{aligned}
[\underline{\mu},\overline{\mu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) &= [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}\mathscr{I}_a(x)\\
&= \bigvee_{y\in\mathscr{I}_a{}^{-1}(x)}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)\\
&= \bigvee_{x=\mathscr{I}_a(y)=ya}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)\\
&= \bigvee_{y=xa^{-1}}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)\\
&= [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1})\\
&\geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)
\end{aligned}$$

This implies that $[\underline{\mu},\overline{\mu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) \geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$.
Now,

$$\begin{aligned}
[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) &= [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1}a)\\
&\geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1})\\
&= [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)\\
&= \bigvee_{y\in\mathscr{I}_a{}^{-1}(x)}[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)\\
&= [\underline{\mu},\overline{\mu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x)
\end{aligned}$$

This implies that $[\underline{\mu},\overline{\mu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) \leq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$.
Therefore, membership intervals of $\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})$ = membership intervals of $\mathcal{G}_{\mathcal{IV}}$.
Next, we have to prove,

$$\begin{aligned}
[\underline{\nu},\overline{\nu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) &= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}\mathscr{I}_a(x) \\
&= \bigwedge_{y\in \mathscr{I}_a{}^{-1}(x)} [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y) \\
&= \bigwedge_{x=\mathscr{I}_a(y)=ya} [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y) \\
&= \bigwedge_{y=xa^{-1}} [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y) \\
&= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(ya^{-1}) \\
&\leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)
\end{aligned}$$

This implies that $[\underline{\nu},\overline{\nu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) \leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$.

$$\begin{aligned}
[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) &= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1}a) \\
&\leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1}) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(a) \\
&\leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1}) \vee [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(e) \\
&= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xa^{-1}) \\
&= [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y) \\
&= \bigwedge_{y\in \mathscr{I}_a{}^{-1}(x)} [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y) \\
&= [\underline{\nu},\overline{\nu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x)
\end{aligned}$$

This implies that $[\underline{\nu},\overline{\nu}]_{\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}})}(x) \geq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$.
Therefore, intervals of non-membership of $\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}}) =$ intervals of non-membership of $\mathcal{G}_{\mathcal{IV}}(x)$.
Thus, $\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}}) = \mathcal{G}_{\mathcal{IV}}$.
Similarly, we can prove, ${}_a\mathscr{I}(\mathcal{G}_{\mathcal{IV}}) = \mathcal{G}_{\mathcal{IV}}$.
Hence $\mathscr{I}_a(\mathcal{G}_{\mathcal{IV}}) = {}_a\mathscr{I}(\mathcal{G}_{\mathcal{IV}}) = \mathcal{G}_{\mathcal{IV}}$. □

Proposition 2.12. *Let $\mathcal{G}_{\mathcal{IV}}$ be an interval-valued intuitionistic fuzzy group on $\mathcal{S}$. If $\mathcal{G}_{\mathcal{IV}} : \mathcal{G}_{\mathcal{IV}} \times \mathcal{G}_{\mathcal{IV}} \to \mathcal{G}_{\mathcal{IV}}$ is a mapping defined by $\mathcal{G}_{\mathcal{IV}}(u,v) = uv$ and $h : \mathcal{G}_{\mathcal{IV}} \to \mathcal{G}_{\mathcal{IV}}$ is a mapping defined by $h(u) = u^{-1}$, then the image of the product IVIF-set $\mathcal{G}_{\mathcal{IV}} \times \mathcal{G}_{\mathcal{IV}}$ is an IVIF-group on $\mathfrak{f}(\mathcal{S})$ and $h(\mathcal{G}_{\mathcal{IV}})$ is an IVIF-group on $\mathfrak{f}(\mathcal{S})$.*

Proof. Let $w \in \mathcal{G}_{\mathcal{IV}} \times \mathcal{G}_{\mathcal{IV}}$, implies

$$\begin{aligned}
[\underline{\mu},\overline{\mu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(w) &= \mathcal{G}_{\mathcal{IV}}[\underline{\mu},\overline{\mu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(u,v) \\
&= \bigvee_{(u,v)\in \mathcal{G}_{\mathcal{IV}}{}^{-1}(w)} [\underline{\mu},\overline{\mu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(uv) \\
&\geq [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(u) \wedge [\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(v)
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\underline{\nu},\overline{\nu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(w) &= \mathcal{G}_{\mathcal{IV}}[\underline{\nu},\overline{\nu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(u,v)\\
&= \bigwedge_{(u,v)\in\mathcal{G}_{\mathcal{IV}}{}^{-1}(w)} [\underline{\nu},\overline{\nu}]_{(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})}(uv)\\
&\leq [\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(u)\vee[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(v)
\end{aligned}$$

Therefore $\mathcal{G}_{\mathcal{IV}}(\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}})$ is in $\mathcal{G}_{\mathcal{IV}}$. Similarly we can prove, by the definition of IVIFG, $\beta(\mathcal{G}_{\mathcal{IV}})$ is in $\mathcal{G}_{\mathcal{IV}}$. □

3 Interval-valued intuitionistic fuzzy topological groups

In this section, we have introduced and discussed the fundamental characteristics of an IVIF-topological group and an IVIF-product topological group. An IVIF-topological group is a set equipped with both interval-valued intuitionistic fuzzy (IVIF) structure and a compatible topological structure that preserves the group operations. This combination allows us to incorporate the notions of uncertainty and vagueness into the study of topological groups.

Definition 3.1. *Let $(X,\mathcal{T}_{\mathcal{IV}})$ be an IVIF-topological space and $\mathcal{G}_{\mathcal{IV}}$ be an IVIF-group in $\mathcal{S}$. Then the collection $\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}}=\{\mathfrak{U}\cap\mathcal{G}_{\mathcal{IV}}\in\mathcal{I}^X_{IV}:\mathfrak{U}\in\mathcal{T}\}$ is called an induced interval-valued intuitionistic fuzzy topology (abbreviated induced IVIFT) on $\mathcal{G}_{\mathcal{IV}}$.*

Remark 3.2. *The pair $(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ is called an IVIF-subspaces of $(X,\mathcal{T}_{\mathcal{IV}})$.*

Definition 3.3. *Let $\mathcal{T}_{\mathcal{IV}}$ be an IVIF-topology on $\mathcal{I}^X_{\mathcal{IV}}$. Let $\mathcal{G}_{\mathcal{IV}}\in IVIFG(\mathcal{I}^X_{\mathcal{IV}})$ and $(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ be the IVIF-subspaces of $(X,\mathcal{T}_{\mathcal{IV}})$. Then $\mathcal{G}_{\mathcal{IV}}$ is called an interval-valued IF topological group (in short IVIFTG) on $\mathcal{I}^X_{\mathcal{IV}}$ if*

1. *The $\mathcal{G}_{\mathcal{IV}}:(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})\times(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})\to(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ is relatively IVIF-continuous mapping.*
2. *The $h:(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})\to(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ is relatively IVIF-continuous mapping.*

Proposition 3.4. *Let $\mathcal{T}_{\mathcal{IV}}$ be an IVIFT on X and $\mathcal{G}_{\mathcal{IV}}\in IVIFTG(X)$, if and only if $\mathfrak{f}:(u,v)\to uv^{-1}$ is relatively IVIF-continuous and $\mathfrak{f}$ is defined from $(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})\times(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ into $(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$.*

Proof. Assume that, the mapping $\mathfrak{f}=\mathcal{G}_{\mathcal{IV}}\circ h$ of $(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})\times(\mathcal{G}_{\mathcal{IV}},\mathcal{T}_{\mathcal{G}_{\mathcal{IV}}})$ into itself is relatively IVIF-continuous. This implies that the composition $\mathcal{G}_{\mathcal{IV}}\circ h\to(u,v)\to(u,v^{-1})\to(uv^{-1})$ is relatively IVIF-continuous.

Conversely, assume that $[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(e)\geq[\underline{\mu},\overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$ and $[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(e)\leq[\underline{\nu},\overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x)$, for each membership and non-membership intervals of $x\in\mathcal{I}^X_{\mathcal{IV}}$. Consider the constant mapping $i:\mathcal{G}_{\mathcal{IV}}\to\mathcal{G}_{\mathcal{IV}}\times\mathcal{G}_{\mathcal{IV}}$ is relatively IVIF-continuous and defined by $i(y)=(e,y)$ for each membership and non-membership of $x\in\mathcal{I}^X_{\mathcal{IV}}$. The composition $\mathfrak{f}\circ i=h$ is relatively IVIF-continuous since h is relatively IVIF-continuous. And the mapping $\mathcal{G}_{\mathcal{IV}}=\mathfrak{f}\circ h$ is relatively IVIF-continuous. Hence $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-topological group in $\mathcal{T}$. □

Definition 3.5. *Let $\mathcal{G}_{\mathcal{IV}}$ be an interval-valued intuitionistic fuzzy product group of a finite family of IVIFGs. If $\mathcal{S} = \prod_{j=1}^{n} \mathcal{S}_j$ and $\mathcal{G}_{\mathcal{IV}} = \prod_{j=1}^{n} \mathcal{G}_{j\mathcal{IV}}$ then the mapping $[\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is defined by $[\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1) \wedge \ldots \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n)$ and the mapping $[\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ defined by $[\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) = [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1) \vee ... \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n)$*

Proposition 3.6. *If $\mathcal{G}_{\mathcal{IV}}$ is a product of IVIF-groups in $\mathcal{T}$ then $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-group.*

Proof. If $\mathcal{G}_{\mathcal{IV}} \in IVIFS(\mathcal{I}_{\mathcal{IV}}^{X})$ and $\mathcal{G}_{\mathcal{IV}} = \prod_{j=1}^{n} \mathcal{G}_{j\mathcal{IV}}; x = (x_1, x_2, ..., x_n); y = (y_1, y_2, ..., y_n)$. By Proposition 2.2,

$$
\begin{aligned}
[\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) &= [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}((x_1 y_1^{-1}, x_2 y_2^{-1}, ..., x_n y_n^{-1})) \\
&= [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1 y_1^{-1}) \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{2_{\mathcal{IV}}}}(x_2 y_2^{-1}) \wedge ... \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n {y_n}^{-1}) \\
&\geq ([\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1) \wedge ... \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n)) \wedge ([\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(y_1^{-1}) \wedge ... \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(y_n^{-1})) \\
&= [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \wedge [\underline{\mu}, \overline{\mu}]_{\mathcal{G}_{\mathcal{IV}}}(y)
\end{aligned}
$$

$$
\begin{aligned}
[\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(xy^{-1}) &= [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}((x_1 y_1^{-1}, x_2 y_2^{-1}, ..., x_n y_n^{-1})) \\
&= [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1 y_1^{-1}) \wedge [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{2_{\mathcal{IV}}}}(x_2 y_2^{-1}) \vee ... \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n y_n^{-1}) \\
&\leq [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}(x_1) \vee ... \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}(x_n)) \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{1_{\mathcal{IV}}}}({y_1}^{-1}) \vee ... \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{n_{\mathcal{IV}}}}({y_n}^{-1}) \\
&= [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(x) \vee [\underline{\nu}, \overline{\nu}]_{\mathcal{G}_{\mathcal{IV}}}(y)
\end{aligned}
$$

Hence $\mathcal{G}_{\mathcal{IV}}$ is an IVIF-group in $\mathcal{T}$. □

4 Conclusion

In this chapter, we have established the properties of the IVIF-topological group and the IVIF-product topological group. Our approach involves extending many of the results from the realm of fuzzy groups and fuzzy topological groups. We put forth definitions for IVIF-group and IVIF-topological group by leveraging the concept of interval-valued intuitionistic fuzzy sets. This extension allows us to incorporate uncertainty and vagueness into the study of group structures, providing a more flexible framework for analysis and application.

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