Eighth Int. Conf. on IFSs, Varna, 20-21 June 2004
NIFS Vol. 10 (2004), 4, 56-60

Intuitionistic Fuzzy Voronoi Diagrams – Definition and Properties

Lyudmila Todorova, Anton Antonov and Stefan Hadjitodorov

Centre of Biomedical Engineering, Bulgarian Acadmey of Sciences
Bl. 105 Acad. G. Bontchev Str., 1113 Sofia, Bulgaria
E-mails: *{lpt, a.antonov}@clbme.bas.bg, sthadj@argo.bas.bg*

In memory of Prof. Ivan Daskalov

Abstract: It is defined modification of Voronoi diagrams named intuitionistic fuzzy Voronoi diagrams (IFVD). Properties of IFVD are examined. The base of geometrical properties of IFVD are placed.
Keywords: Voronoi Diagrams, intuitionistic fuzzy Voronoi Diagrams, intuitionistic fuzzy sets.

Voronoi Diagrams [1] are widely discussed because they are applied in different scopes of science in order to decide many and diversity geometrical problems. Aurenhammer [2] present the wide spectrum of science scopes where Voronoi diagrams are applied successfully and determine their basic application:

1. Modeling natural phenomenon;
2. Examine their mathematical, private, geometrical, combinatorial and stochastic problems;
3. Their computer algorithms and images.

Basic idea in Voronoi diagrams is reduced to transforming part of space in convex districts. Every district of Voronoi diagrams consist of all points p_i in space whose distance to point q (q belong to this district) is smaller than their distance to the other points p_j from fixed set, where is created Voronoi diagram. Hence :

$$dist(q, p_i) < dist(q, p_j),$$

where dist(r , s) is Euclidean distance between points r and s.

This is the reason because Voronoi diagrams are widely used in algorithms for the nearest neighbours ant the remotest neigbours [3-6].

In [7] is proposed modification of Voronoi diagrams named intuitionistic fuzzy Voronoi diagrams (IFVD) because they are intuitionistic fuzzy modification over Voronoi diagrams.

Intuitionistic fuzzy set A is defined in Atanassov[8] on this way:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E \},$$

where E is fixed set, functions $\mu_A(x): E \to [0,1]$ and $\nu_A(x): E \to [0,1]$ give us degree of membership and non-membership of the element $x \in E$ to set A. Set A is subset to E and $\forall x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1$$

Function $\pi_A(x)$ is defined on this way:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

and give us the degree of non-determinacy of the element $x \in E$ to the set A.

Euclidean distance between two points p and q with plane coordinates (p_x, p_y) and (q_x, q_y) is equal to :

$$dist(p,q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2} \ .$$

In the aspect of upper formula we will examine properties over IFVD beginning with their definition.

We will define IFVD over random set of points P in plain like part of the plane, placed in n cells, every point of which is unique for place of p_i from P and with properties that point q belong to cell in which is placed p_i. If

$$dist(q, p_i) < \sqrt{dist(q, p_j)^2 + 2\mu_i - 1}$$

$\forall p_j$ of P and $j \neq i$, where μ, ν and π are degree of membership, non-membership and non-determinacy. These values are determined by one of proposed ways in [6].

Intuitionistic fuzzy Voronoi diagrams over P will note with $IFVor(P)$. For cell corresponding to point p_i we will use symbol $IFVor(P)$ and will call it intuitionistic fuzzy cell (region, district, area) of Voronoi to p_i.

Borderline between fuzzy regions of point p_i and p_j will note with $IFB(p_i, p_j)$. The difference between Voronoi diagrams and intuitionistic fuzzy Voronoi diagrams is that borderlines aren't only part of lines (segment) or half-lines but also streaks and half-streaks which will be the zone of non-determinacy to IFVD. Then it is clear that vertexes in IFVD could be points from plane analogy to Voronoi diagrams and will appear like intersection of streaks. We will use vertex of given district of point p_i in IFVD symbol $IFA(p_i)$.

Properties of intuitionistic fuzzy Voronoi diagrams :

1. For set P of n points on plane IFVD divide the plane to n regions. Every region is consisted of points whose degree of membership is bigger than its degree of membership in another regions.
2. In case $n = 1$ IFVD doesn't contain vertices and edges, therefore take the whole plain.
3. In case $n = 2$ IFVD possess 1 infinite edge and doesn't contain any vertices. The edge divide the plane into half-planes and these half-planes are unlimited(fig.1);

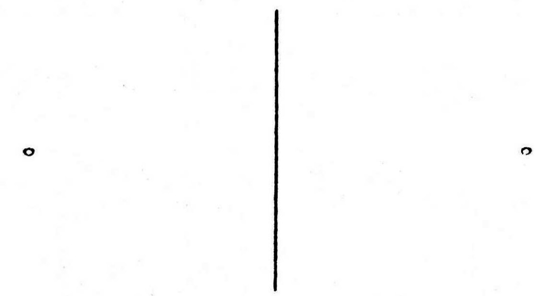

fig.1

4. In case when P is set of n collinear points on plane, IFVD doesn't possess vertices and it is consisted of $n-1$ infinite edges and these edges are parallel. Then IFVD has n unlimited cells(fig.2). In these case IFVD is unconnected.

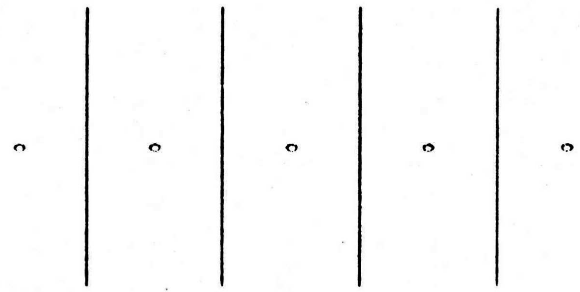

fig.2

5. Every cell of IFVD is non-empty convex polygon.
6. Intutionistic fuzzy cells of Voronoi diagrams are mutually excluded except for borderlines(edges of IFVD).
7. Cell of Voronoi from IFVD is infinite if and only if when correspond with point and that point is placed on convex envelope of set over which are created IFVD. Every point from the given set P that is at the smallest distance of given point q from the same set generate edge or streak of cell for point q in IFVD over P.

Note : Under convex envelope we will understand border of convex set. Set P is convex if its border is polygon with the following properties :

1) Every point of set P belong to plane of polygon.
2) For every two points of set P polygon plane's contain the segment connecting these two points.

Lemma 1. Point p_i from set P lies on the convex envelope of P if its cell $IFV(p_i)$ in IFVD is infinite.

Proofs: Let p_j and p_j have adjacent to p_i districts in IFVD over P and edges $IFB(p_i, p_j)$, $IFB(p_i, p_k)$ are borders between them and these two edges are endless. Let $IFB(p_i, p_j)$ and $IFB(p_i, p_k)$ are crossed in point O. Therefore $\sphericalangle JOK < 90^\circ$. From quadrangle $p_i JOK$ ($\sphericalangle p_i JO = \sphericalangle OKp_i = 90^\circ$) follow that $\sphericalangle Jp_i K > 90^\circ$. Therefore point p_i from set P on convex envelope to P.

The proof is valid in case when $IFB(p_i, p_j)$ and $IFB(p_i, p_k)$ are infinite streaks.

Theorem 1. Intuitionistic fuzzy Voronoi diagram for set of n points on the plane where $n \geq 3$ possess $2n-5$ vertices and $3n-6$ edges.

Proof: If all points are collinear then proof will follow from next theorem that is proved in theory of Voronoi diagrams:

Let P is set from n different points in the plane. If all points are collinear then Voronoi diagram will consist of $n-1$ parallel lines. In opposite case Voronoi diagram will contain segments from straight lines or/and half-lines.

In case when points are not collinear we will use classical approach from Voronoi diagrams namely we will add extraordinary vertex "at infinity". We will use Euler's formula [9] in accordance with the formula for connected planar graph the following relation holds:

$$n_v - n_e + n_f = 2,$$

n_v is number of nodes, n_e is number of arcs and n_f is number of faces.

Euler's theorem could not be applied on that way for offset Voronoi diagram because that diagram like Voronoi diagram has half-infinite edges therefore have cells that are not real polygons. By reason of we add to set of vertices one additional vertex v_∞ "at infinity" and connect all half-infinite edges with the additional vertex see fig.3.

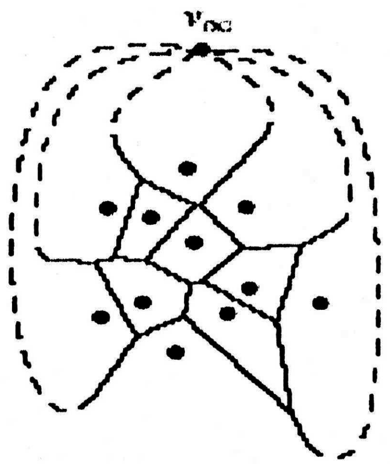

fig. 3

Now number of vertices is $n_v + 1$ and number of faces is n. In such a way over created polygons on the plane can be applied Euler's formula. We get the following relation among number n_v- vertexes of offset Voronoi diagram, n_e- number of edges and n - number of districts :

$$(n_v + 1) - n_e + n = 2. \quad (1)$$

Moreover, every edge in the argumented graph has exactly two vertices, so if we sum the degrees of all vertices v_i we get twice the number of edges:

$$\sum_{i=1}^{n_v} \deg(v_i) = 2.n_e.$$

Because every vertex, including v_∞, has degree at least three i.e. $(\forall v_i \in V)(\deg(v_i) \geq 3)$, where V is set of edges of IFVD hence:

$$2.n_e \geq 3.(n_v + 1). \quad (2)$$

Therefore with (1) and (2) when $n \geq 3$ we get that $n_v \leq 2n - 5$ and $n_e \leq 3n - 6$.

Propositions from this work will be used for base of further examination of geometrical properties to IFVD as well as development of whole theory of IFVD.

References

1. Voronoi, G.M.. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. deuxième Mémoire: Recherches sur les parallélloèdres primitifs. *J. Reine Angew. Math.*, 134:198–287, 1908.
2. Aurenhammer, F. Voronoi diagrams a survey of a fundamental geometric data structure. *ACM Comput. Surv.*, 23(3):345-405, Sept. 1991.
3. McAllister, M., D. Kirkpatrick, and J. Snoeyink, A compact piecewise-linear Voronoi diagram for convex sites in the plane, *Discrete & Computational Geometry*, 15 (1996), 73--105.
4. Barequet, G., M. Dickerson, and M.T. Goodrich, Voronoi Diagrams for Polygon-Offset Distance Functions, Proc. 5th Workshop on Algorithms and Data Structures, Halifax, Nova Scotia, Canada, *Lecture Notes in Computer Science*, 1272, Springer-Verlag, pp 200-209, 1997.
5. Lee, D.T., On *k*-nearest neighbors Voronoi diagrams in the plane. *IEEE Transactions on Computers*, C-31: pp 478-487, 1982.
6. Kuncheva, L., L. Todorova. Prototype Selection for an RBF Network by a Genetic Algorithm. Proceedings of the international ICSC Symposia on Intelligent Industrial Automation (IIA'96) and Soft Computing (SOCO'96), (P. Anderson and K. Warwick, Eds.), Reading, U.K., March 26-28, pp. B100 – B106, 1996.
7. Todorova, L. An intuitionistic fuzzy version of the nearest prototype classification method, based on a Voronoi's diagrams.
8. Atanassov K. *Intuitionistic Fuzzy Sets. Heidelberg*: Physica-Verlag, 1999.
9. Gibbons, A.. *Algorithmic Graph Theory.* Cambridge Universite Press, Cambridge, 1985.