

Modifications of Łukasiewicz’s intuitionistic fuzzy implication

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Abstract: In [6], G. Klir and B. Yuan named after J. Łukasiewicz the implication $p \rightarrow q = \min(1, p + q)$. In a series of papers, 198 different intuitionistic fuzzy implications have been introduced, and their basic properties have been studied. Here we introduce six new implications which are modifications of Łukasiewicz’s intuitionistic fuzzy implication, and we describe and prove some of their properties.

Keywords: Intuitionistic fuzzy implication, Intuitionistic fuzzy set, Łukasiewicz’s fuzzy implication.

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1 Introduction

In [1, 2], Łukasiewicz's intuitionistic fuzzy implication was introduced and some of its basic properties were studied. It is based on Łukasiewicz's fuzzy implication that has the following form (see, e.g., [6]):

$$p \rightarrow q = \min(1, p + q).$$

Here, following the scheme from [2, 4, 5], we will construct eight new implications, related to the Łukasiewicz's intuitionistic fuzzy implication but we will observe that two of them coincide and one is trivial as it permanently yields in result a constant. Therefore, of these eight six are the new implications which will present and study in details.

2 Preliminaries

In [3], K. Atanassov, E. Szmidt and J. Kacprzyk defined the object $\langle a, b \rangle$, with $a, b, a + b \in [0, 1]$, as Intuitionistic Fuzzy Pair (IFP).

Let everywhere below for IFPs x and y :

$$x = \langle a, b \rangle,$$

$$y = \langle c, d \rangle,$$

where $a, b, c, d \in [0, 1], a + b \leq 1, c + d \leq 1$. For the IFPs, in [3], different operations and relations have been defined. For our aims, we will remind the definitions of only three of these relations and one operation:

$$x \leq y \text{ if and only if } a \leq c \text{ and } b \geq d,$$

$$x \geq y \text{ if and only if } y \leq x,$$

$$x = y \text{ if and only if } x \leq y \text{ and } y \leq x,$$

$$\neg x = \langle b, a \rangle.$$

3 Main results

Łukasiewicz's intuitionistic fuzzy implication (see [1, 2]) is defined by

$$x \rightarrow_L y = \langle \min(1, b + c), \max(0, a + d - 1) \rangle.$$

The (standard) intuitionistic fuzzy modal operators over IFPs (see, e.g., [2]) are:

$$\Box x = \langle a, 1 - a \rangle,$$

$$\Diamond x = \langle 1 - b, b \rangle.$$

The intuitionistic fuzzy modal operators of second type over IFPs (see [2]) are:

$$\boxplus x = \left\langle \frac{a}{2}, \frac{b+1}{2} \right\rangle,$$

$$\boxtimes x = \left\langle \frac{a+1}{2}, \frac{b}{2} \right\rangle.$$

Now, using the idea from [4] and the logical scheme (see, e.g. [2, 5])

$$x \rightarrow_* y = M_1 x \rightarrow_L M_2 y, \quad (*)$$

where $M_1, M_2 \in \{\Box, \Diamond\}$, we define the following new implications:

$$\begin{aligned} x \rightarrow_1 y &= \Box x \rightarrow_L \Diamond y = \langle a, 1 - a \rangle \rightarrow \langle 1 - d, d \rangle \\ &= \langle \min(1, 2 - a - d), \max(0, a + d - 1) \rangle; \\ x \rightarrow_2 y &= \Box x \rightarrow_L \Box y = \langle a, 1 - a \rangle \rightarrow \langle c, 1 - c \rangle \\ &= \langle \min(1, 1 - a + c), \max(0, a - c) \rangle; \\ x \rightarrow_3 y &= \Diamond x \rightarrow_L \Diamond y = \langle 1 - b, b \rangle \rightarrow \langle 1 - d, d \rangle \\ &= \langle \min(1, 1 + b - d), \max(0, d - b) \rangle; \\ x \rightarrow_4 y &= \Diamond x \rightarrow_L \Box y = \langle 1 - b, b \rangle \rightarrow \langle c, 1 - c \rangle \\ &= \langle \min(1, b + c), \max(0, 1 - b - c) \rangle. \end{aligned}$$

For these four new implications we can prove the following assertions.

Theorem 1. *For every two IFPs x and y*

$$x \rightarrow_1 y \geq \left\{ \begin{array}{c} x \rightarrow_2 y \\ x \rightarrow_3 y \end{array} \right\} \geq x \rightarrow_4 y.$$

Proof. Let the two IFPs x and y be given. Then from the inequalities

$$\begin{aligned} \min(1, 2 - a - d) &\geq \min(1, 1 - a + c), \\ \min(1, 2 - a - d) &\geq \min(1, 1 + b - d), \\ \min(1, 1 - a + c) &\geq \min(1, b + c), \\ \min(1, 1 + b - d) &\geq \min(1, b + c), \\ \max(0, a + d - 1) &\leq \max(0, a - c), \\ \max(0, a + d - 1) &\leq \max(0, d - b), \\ \max(0, a - c) &\leq \max(0, 1 - b - c), \\ \max(0, d - b) &\leq \max(0, 1 - b - c). \end{aligned}$$

the validity of Theorem 1 follows. □

In contrast to [4], in the general case, the relations in the form

$$\neg x \rightarrow_i \neg y = \neg(x \rightarrow_j y)$$

are not valid for $1 \leq i, j \leq 4$.

Indeed, for $i = 1$ we obtain:

$$\begin{aligned}\neg x \rightarrow_1 \neg y &= \neg \langle a, b \rangle \rightarrow_1 \neg \langle c, d \rangle = \langle b, a \rangle \rightarrow_1 \langle d, c \rangle \\ &= \langle \min(1, 2 - b - c), \max(0, b + c - 1) \rangle \\ &= \neg \langle \max(0, b + c - 1), \min(1, 2 - b - c) \rangle,\end{aligned}$$

which is not equal to any $\neg(x \rightarrow_j y)$ with $1 \leq j \leq 4$.

Similarly to [4], if for the scheme $(*)$ we use $M_1, M_2 \in \{\boxplus, \boxtimes\}$, we will receive the following four new intuitionistic fuzzy implications:

$$\begin{aligned}x \rightarrow_5 y &= \boxplus x \rightarrow_L \boxplus y = \left\langle \frac{a}{2}, \frac{b+1}{2} \right\rangle \rightarrow_L \left\langle \frac{c}{2}, \frac{d+1}{2} \right\rangle \\ &= \left\langle \min\left(1, \frac{b+1}{2} + \frac{c}{2}\right), \max\left(0, \frac{a}{2} + \frac{d+1}{2} - 1\right) \right\rangle \\ &= \left\langle \min\left(1, \frac{b+c+1}{2}\right), \max\left(0, \frac{a+d-1}{2}\right) \right\rangle; \\ x \rightarrow_6 y &= \boxtimes x \rightarrow_L \boxplus y = \left\langle \frac{a+1}{2}, \frac{b}{2} \right\rangle \rightarrow_L \left\langle \frac{c}{2}, \frac{d+1}{2} \right\rangle \\ &= \left\langle \min\left(1, \frac{b}{2} + \frac{c}{2}\right), \max\left(0, \frac{a+1}{2} + \frac{d+1}{2} - 1\right) \right\rangle \\ &= \left\langle \min\left(1, \frac{b+c}{2}\right), \max\left(0, \frac{a+d}{2}\right) \right\rangle \\ &= \left\langle \frac{b+c}{2}, \frac{a+d}{2} \right\rangle; \\ x \rightarrow_7 y &= \boxplus x \rightarrow_L \boxtimes y = \left\langle \frac{a}{2}, \frac{b+1}{2} \right\rangle \rightarrow_L \left\langle \frac{c+1}{2}, \frac{d}{2} \right\rangle \\ &= \left\langle \min\left(1, \frac{b+1}{2} + \frac{c+1}{2}\right), \max\left(0, \frac{a}{2} + \frac{d}{2} - 1\right) \right\rangle \\ &= \left\langle \min\left(1, \frac{b+c+2}{2}\right), \max\left(0, \frac{a+d-2}{2}\right) \right\rangle \\ &= \langle 1, 0 \rangle; \\ x \rightarrow_8 y &= \boxtimes x \rightarrow_L \boxtimes y = \left\langle \frac{a+1}{2}, \frac{b}{2} \right\rangle \rightarrow_L \left\langle \frac{c+1}{2}, \frac{d}{2} \right\rangle \\ &= \left\langle \min\left(1, \frac{b}{2} + \frac{c+1}{2}\right), \max\left(0, \frac{a+1}{2} + \frac{d}{2} - 1\right) \right\rangle \\ &= \left\langle \min\left(1, \frac{b+c+1}{2}\right), \max\left(0, \frac{a+d-1}{2}\right) \right\rangle.\end{aligned}$$

Obviously, implication \rightarrow_7 is trivial, because its result is always a constant, while implications \rightarrow_5 and \rightarrow_8 coincide and hence, we can only work with the first of them, \rightarrow_5 .

Now, we formulate Theorem 2, which can be proved in the same manner as Theorem 1, hence the proof is skipped.

Theorem 2. *For every two IFPs x and y*

$$x \rightarrow_5 y \geq x \rightarrow_6 y.$$

We directly check the validity of the following equalities.

$$\langle 0, 1 \rangle \rightarrow_i \langle 0, 1 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2, 3, 4, 5 \\ \langle \frac{1}{2}, \frac{1}{2} \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 0, 1 \rangle \rightarrow_i \langle 0, 0 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2, 3, 4, 5 \\ \langle \frac{1}{2}, 0 \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 0, 1 \rangle \rightarrow_i \langle 1, 0 \rangle = \langle 1, 0 \rangle, \text{ for each } i = 1, \dots, 6;$$

$$\langle 0, 0 \rangle \rightarrow_i \langle 0, 1 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2 \\ \langle 0, 1 \rangle, & \text{for } i = 3, 4 \\ \langle \frac{1}{2}, 0 \rangle, & \text{for } i = 5 \\ \langle 0, 0 \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 0, 0 \rangle \rightarrow_i \langle 0, 0 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2, 3 \\ \langle 0, 1 \rangle, & \text{for } i = 4 \\ \langle \frac{1}{2}, 0 \rangle, & \text{for } i = 5 \\ \langle 0, 0 \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 0, 0 \rangle \rightarrow_i \langle 1, 0 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2, 3, 4, 5 \\ \langle \frac{1}{2}, 0 \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 1, 0 \rangle \rightarrow_i \langle 0, 1 \rangle = \begin{cases} \langle 0, 1 \rangle, & \text{for } i = 1, 2, 3, 4, 6 \\ \langle \frac{1}{2}, \frac{1}{2} \rangle, & \text{for } i = 5 \end{cases};$$

$$\langle 1, 0 \rangle \rightarrow_i \langle 0, 0 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 3 \\ \langle 0, 1 \rangle, & \text{for } i = 2, 4 \\ \langle \frac{1}{2}, 0 \rangle, & \text{for } i = 5 \\ \langle 0, \frac{1}{2} \rangle, & \text{for } i = 6 \end{cases};$$

$$\langle 1, 0 \rangle \rightarrow_i \langle 1, 0 \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{for } i = 1, 2, 3, 4, 5 \\ \langle \frac{1}{2}, \frac{1}{2} \rangle, & \text{for } i = 6 \end{cases}.$$

Following [2] and using, e.g. [7], we mention that

$$\neg x = x \rightarrow \langle 0, 1 \rangle.$$

Therefore, the new implications generate the following negations:

$$\begin{aligned}
\neg_1 x &= x \rightarrow_1 \langle 0, 1 \rangle = \langle 1 - a, a \rangle; \\
\neg_2 x &= x \rightarrow_2 \langle 0, 1 \rangle = \langle 1 - a, a \rangle; \\
\neg_3 x &= x \rightarrow_3 \langle 0, 1 \rangle = \langle b, 1 - b \rangle; \\
\neg_4 x &= x \rightarrow_4 \langle 0, 1 \rangle = \langle b, 1 - b \rangle; \\
\neg_5 x &= x \rightarrow_5 \langle 0, 1 \rangle = \left\langle \frac{b+1}{2}, \frac{a}{2} \right\rangle; \\
\neg_6 x &= x \rightarrow_6 \langle 0, 1 \rangle = \left\langle \frac{b}{2}, \frac{a+1}{2} \right\rangle.
\end{aligned}$$

We see immediately that for each IFP x

$$\neg_1 x = \neg_2 x \geq \neg_3 x = \neg_4 x$$

and

$$\neg_5 x > \neg_6 x.$$

It is important to mention that the constructed here implications are new ones, while negations \neg_1 and \neg_2 coincide with negation \neg_8 from [2], negations \neg_3 and \neg_4 – with negation \neg_4 from [2], negations \neg_6 – with negation \neg_{35} from [2] and only negation \neg_5 is a new one.

In [6], G. Klir and B. Yuan give nine axioms for fuzzy implication. In [2], K. Atanassov pre-formulated them for the case of intuitionistic fuzziness.

Let

$$\begin{aligned}
O^* &= \langle 0, 1 \rangle, \\
E^* &= \langle 1, 0 \rangle.
\end{aligned}$$

The IFP is a tautology iff $a = 1$ and $b = 0$, while it is an Intuitionistic Fuzzy Tautology (IFT) iff $a \geq b$.

Klir and Yuan's axioms are:

Axiom A1 $(\forall x, y)(\text{ if } x \leq y, \text{ then } (\forall z)(x \rightarrow z \geq y \rightarrow z))$,

Axiom A2 $(\forall x, y)(\text{ if } x \leq y, \text{ then } (\forall z)(z \rightarrow x \leq z \rightarrow y))$,

Axiom A3 $(\forall y)(O^* \rightarrow y = E^*)$,

Axiom A4 $(\forall y)(E^* \rightarrow y = y)$,

Axiom A5 $(\forall x)(x \rightarrow x = E^*)$,

Axiom A6 $(\forall x, y, z)(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z))$,

Axiom A7 $(\forall x, y)(x \rightarrow y = E^* \text{ iff } x \leq y)$,

Axiom A8 $(\forall x, y)(x \rightarrow y = \neg y \rightarrow \neg x)$,

Axiom A9 I is a continuous function.

In [2], having in mind the specific forms of the intuitionistic fuzzy implications, Atanassov modified five of these axioms, as follows.

Axiom A3* $(\forall y)(O^* \rightarrow y \text{ is an IFT})$,

Axiom A4* $(\forall y)(E^* \rightarrow y \leq y)$,

Axiom A5* $(\forall x)(x \rightarrow x \text{ is an IFT})$,

Axiom A7* $(\forall x, y)(\text{if } x \leq y, \text{ then } x \rightarrow y = E^*)$,

Axiom A8* $(\forall x, y)(x \rightarrow y = \neg(\neg(\neg y \rightarrow \neg x)))$.

The proofs of the following theorems are similar to that of Theorem 1.

Theorem 3. *For every three IFPs x, y and z*

- (a) *implications \rightarrow_1 and \rightarrow_3 satisfy Axioms A1, A2, A3, A3*, A5, A5*, A6, A7*;*
- (b) *implication \rightarrow_2 satisfies Axioms A1, A2, A3, A3*, A4*, A5, A5*, A6, A7*, A8, A8*;*
- (c) *implication \rightarrow_4 satisfies Axioms A1, A2, A3, A3*, A4*, A6;*
- (d) *implication \rightarrow_5 satisfies Axioms A1, A2, A3, A3*;*
- (e) *implication \rightarrow_6 satisfies only Axiom A4**

as tautologies.

Theorem 4. *For every three IFPs x, y and z*

- (a) *implications \rightarrow_1 and \rightarrow_3 satisfy Axioms A1, A2, A3, A3*, A5, A5*, A6, A7*;*
- (b) *implication \rightarrow_2 satisfies Axioms A1, A2, A3, A3*, A4*, A5, A5*, A6, A7*, A8, A8*;*
- (c) *implication \rightarrow_4 satisfies Axioms A1, A2, A3, A3*, A4*, A6;*
- (d) *implication \rightarrow_5 satisfies Axioms A1, A2, A3, A3*, A5*, A7*;*
- (e) *implication \rightarrow_6 satisfies Axioms A1, A2, A3*, A4*, A5**

as IFTs.

4 Conclusion

In the present paper, six new intuitionistic fuzzy implications and the negations generated by them are introduced and some of their properties are studied. In a next research, we will check which axioms of the Kolmogorov's, Łukasiewicz and Tarski's axioms for implications and which intuitionistic logic axioms are valid for the new implications.

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