Notes on Intuitionistic Fuzzy Sets
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Research on intuitionistic fuzzy implications. Part 3

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Abstract: Continuing the research from [1, 2], here we give the lists of the intuitionistic fuzzy implications, introduced in [2], that satisfy at least one of two forms of *Modus Ponens (MP)* – a standard and a new one, called "mixed", forms. We show the relationship between every two of these implications that satisfy the mixed form of MP.
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1 Introduction

As was mentioned in [2], *in Part 1 of the present research [1], giving short remarks on the results related to the intuitionistic fuzzy implications, for a first time in fuzzy sets theory, we introduced the relations between separate implications.* In Part 2 (see, [2]), we selected those implications that satisfy a greater number of the basic properties of implications, discussed by different authors.

Here, we will check which intuitionistic fuzzy implications chosen in [2] satisfy the *Modus Ponens* law in its standard and its new one, called "mixed", forms.

All necessary notations are given in Parts 1 and 2, but below, we will repeat a part of them, wherever this is deemed suitable.

2 Main results

It is well-known that *Modus Ponens* (MP) has the form

$$\frac{A, \;\; A \to B}{B},$$

where A and B are some logical forms (variables, terms, predicates, etc.).

In intuitionistic fuzzy calculus (see, e.g., [3]), the Intuitionistic Fuzzy Pair (IFP) $\langle a, b\rangle$, i.e., pair for which $a, b, a+b \in [0,1]$, is called a *tautological IFP (TIFP)* if and only if $a = 1, b = 0$ and *intuitionistic fuzzy tautological IFP (IFTIFP)* if and only if $a \geq b$.

In [4], the MP with IFPs was discussed in its standard form, i.e., when A and $A \to B$ are TIFPs.

First, we will collect the implications from [2] that satisfy the standard MP-form and after this we will introduce a new MP-form.

Theorem 1. *For* $i = 1, 2, 3, 4, 5, 9, 11, 13, 14, 17, 18, 24, 28, 29, 61, 71, 76, 77, 79, 81, 100, 109, 110, 112, 125, 127, 166, 186$ *implication* $\to_i$ *satisfies the standard MP.*

Proof. We will give three examples of checks, from which the first two are checks for implications for the list satisfying the standard MP, while the third one is exemplary of why implications not included in the list do not satisfy it.

For instance, implication $\to_1$ has the form

$$\langle a, b\rangle \to_1 \langle c, d\rangle = \langle \max(b, \min(a, c)), \min(a, d)\rangle.$$

When

$$\begin{aligned} a &= 1, \\ b &= 0, \\ \max(b, \min(a, c)) &= 1, \\ \min(a, d) &= 0 \end{aligned}$$

we obtain

$$1 = \max(0, \min(1, c)) = \max(0, c) = c,$$
$$0 = \min(1, d) = d,$$

i.e., $\langle c, d\rangle$ is a TIFP.

A more complex is the check of implication $\rightarrow_{14}$, hence it deserves our attention. Implication $\rightarrow_{14}$ has the form

$$\langle a, b\rangle \rightarrow_{14} \langle c, d\rangle = \langle 1-(1-c).\,\mathrm{sg}(a-c) - d.\,\overline{\mathrm{sg}}(a-c).\,\mathrm{sg}(d-b), d.\,\mathrm{sg}(d-b)\rangle.$$

When

$$\begin{aligned} a &= 1,\\ b &= 0,\\ 1-(1-c).\,\mathrm{sg}(a-c) - d.\,\overline{\mathrm{sg}}(a-c).\,\mathrm{sg}(d-b) &= 1,\\ d.\,\mathrm{sg}(d-b) &= 0, \end{aligned}$$

where for each real number x:

$$\mathrm{sg}(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1, & \text{if } x > 0 \end{cases}$$

and

$$\overline{\mathrm{sg}}(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 0, & \text{if } x > 0 \end{cases},$$

we obtain

$$1 = 1-(1-c).\,\mathrm{sg}(1-c) - d.\,\overline{\mathrm{sg}}(1-c).\,\mathrm{sg}(d-0),$$

i.e.,

$$(1-c).\,\mathrm{sg}(1-c) + d.\,\overline{\mathrm{sg}}(1-c).\,\mathrm{sg}(d) = 0.$$

Let us assume that $d > 0$. Therefore, $c < 1$. Then the equality obtains the form

$$\begin{aligned} 0 &= (1-c).\,\mathrm{sg}(1-c) + d.\,\overline{\mathrm{sg}}(1-c)\\ &= 1-c, \end{aligned}$$

i.e., $c = 1$, i.e. contradiction. On the other hand, we directly see from the equality

$$0 = d.\,\mathrm{sg}(d)$$

that $d = 0$ (this follows from the fact that $\langle c, d\rangle$ is an IFP, too).

Therefore, both of the checked implications satisfy the standard MP.

Now, we will show why some implications from the list in [2] do not satisfy the standard MP. For example, $\rightarrow_{101}$ has the form

$$\langle a, b\rangle \rightarrow_{101} \langle c, d\rangle = \langle \max(b.\,\mathrm{sg}(a), c.\,\mathrm{sg}(d)), \min(a.\,\mathrm{sg}(b), d.\,\mathrm{sg}(c))\rangle.$$

When

$$\begin{aligned} a &= 1,\\ b &= 0,\\ \max(b.\,\mathrm{sg}(a), c.\,\mathrm{sg}(d)) &= 1,\\ \min(a.\,\mathrm{sg}(b), d.\,\mathrm{sg}(c)) &= 0, \end{aligned}$$

we obtain

$$1 = \max(0, c.\,\mathrm{sg}(d)) = c.\,\mathrm{sg}(d),$$

but this can be valid only if $c = 1$ and $\mathrm{sg}(d) = 1$, i.e., $d > 0$, that is impossible. Therefore, implication $\rightarrow_{101}$ does not satisfy the standard MP. The rest of the implications that are not included in the list of the Theorem are checked analogously. $\square$

Now, we will introduce a second — mixed — form of the MP. In it, A and B are IFTIFPs while $A \to B$ is a TIFP.

Theorem 2. *For* $i = 1, 2, 3, 4, 5, 9, 11, 13, 17, 18, 24, 28, 29, 61, 71, 77, 81, 100, 109, 110, 112, 125, 127, 166, 186$ *implication* $\to_i$ *satisfies the mixed MP.*

Proof. We will give as an example the check for implication $\to_{24}$ that has the form

$$\langle a, b\rangle \to_1 \langle c, d\rangle = \langle \overline{\text{sg}}(a-c).\, \overline{\text{sg}}(d-b), \text{sg}(a-c).\, \text{sg}(d-b)\rangle.$$

When

$$\begin{aligned} a &\geq b, \\ \overline{\text{sg}}(a-c).\, \overline{\text{sg}}(d-b) &= 1, \\ \text{sg}(a-c).\, \text{sg}(d-b) &= 0 \end{aligned}$$

we obtain that

$$\overline{\text{sg}}(a-c) = 1,$$

$$\overline{\text{sg}}(d-b) = 0,$$

i.e., $a \leq c$ and $d \leq b$. But $a \geq b$. Therefore,

$$c \geq a \geq b \geq d$$

and hence $c \geq d$, i.e., the IFP $\langle c, d\rangle$ is an IFTIFP. □

The following 25 implications in Table 1 satisfy the mixed MP. All of them satisfy the standard MP as well, while $\to_{14}$, $\to_{76}$ and $\to_{79}$ satisfy only the standard MP.

Table 1. List of the intuitionistic fuzzy implications
that satisfy both the mixed *Modus Ponens*

$\to_1$	$\langle \max(b, \min(a, c)), \min(a, d)\rangle$
$\to_2$	$\langle \overline{\text{sg}}(a-c), d.\, \text{sg}(a-c)\rangle$
$\to_3$	$\langle 1-(1-c).\, \text{sg}(a-c), d.\, \text{sg}(a-c)\rangle$
$\to_4$	$\langle \max(b, c), \min(a, d)\rangle$
$\to_5$	$\langle \min(1, b+c), \max(0, a+d-1)\rangle$
$\to_9$	$\langle b + a^2c, ab + a^2d\rangle$
$\to_{11}$	$\langle 1-(1-c).\, \text{sg}(a-c), d.\, \text{sg}(a-c).\, \text{sg}(d-b)\rangle$
$\to_{13}$	$\langle b+c-bc, ad\rangle$
$\to_{17}$	$\langle \max(b, c), \min(ab+a^2, d)\rangle$
$\to_{18}$	$\langle \max(b, c), \min(1-b, d)\rangle$
$\to_{24}$	$\langle \overline{\text{sg}}(a-c).\, \overline{\text{sg}}(d-b), \text{sg}(a-c).\, \text{sg}(d-b)\rangle$
$\to_{28}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(a, d)\rangle$
$\to_{29}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(a, \overline{\text{sg}}(1-d))\rangle$

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Table 1 (Continued from previous page)

$\to_{61}$	$\langle \max(c, \min(b, d)), \min(a, d)\rangle$
$\to_{71}$	$\langle \max(b, c), \min(cd + d^2, a)\rangle$
$\to_{77}$	$\langle (1 - \min(\overline{\mathrm{sg}}(1 - a), \mathrm{sg}(1 - c))), \min(\overline{\mathrm{sg}}(1 - a), \mathrm{sg}(1 - c))\rangle$
$\to_{81}$	$\langle \max(\overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(1 - c)), \min(d, \overline{\mathrm{sg}}(1 - a))\rangle$
$\to_{100}$	$\langle \max(b.\,\mathrm{sg}(a), c), \min(a.\,\mathrm{sg}(b), d)\rangle$
$\to_{109}$	$\langle b + \min(\overline{\mathrm{sg}}(1 - a), c), ab + \min(\overline{\mathrm{sg}}(1 - a), d))\rangle$
$\to_{110}$	$\langle \max(b, c), \min(ab + \overline{\mathrm{sg}}(1 - a), d)\rangle$
$\to_{112}$	$\langle b + c - bc, ab + \overline{\mathrm{sg}}(1 - a).d\rangle$
$\to_{125}$	$\langle \max(b, c), \min(cd + \overline{\mathrm{sg}}(1 - d), a)\rangle$
$\to_{127}$	$\langle b + c - bc, (cd + \overline{\mathrm{sg}}(1 - d)).a\rangle$
$\to_{166}$	$\langle \max(b, \min(a, c)), \min(a, \max(b, d))\rangle$
$\to_{186}$	$\langle \overline{\mathrm{sg}}(d - b) + \mathrm{sg}(d - b).\max(b, c), \mathrm{sg}(d - b).\min(a, d)\rangle$

These implications generate the following negations (Table 2).

Table 2. List of the intuitionistic fuzzy negations, generated by the intuitionistic fuzzy implications that satisfy the mixed *Modus Ponens*

$\neg_1$	$\to_1, \to_4, \to_5, \to_{13}, \to_{61}, \to_{71}, \to_{125}, \to_{127}, \to_{166}, \to_{186}$	$\langle b, a\rangle$
$\neg_2$	$\to_2, \to_3, \to_{11}, \to_{20}$	$\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle$
$\neg_3$	$\to_9, \to_{17}$	$\langle b, ab + a^2\rangle$
$\neg_4$	$\to_{18}$	$\langle b, 1 - b\rangle$
$\neg_6$	$\to_{24}$	$\langle \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a)\rangle$
$\neg_7$	$\to_{28}, \to_{29}$	$\langle \overline{\mathrm{sg}}(1 - b), a\rangle$
$\neg_{13}$	$\to_{77}$	$\langle \mathrm{sg}(1 - a), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{15}$	$\to_{81}$	$\langle \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{18}$	$\to_{100}$	$\langle b.\,\mathrm{sg}(a), a.\,\mathrm{sg}(b)\rangle$
$\neg_{26}$	$\to_{109}, \to_{110}, \to_{112}$	$\langle b, ab + \overline{\mathrm{sg}}(1 - a)\rangle$

The relationships between the intuitionistic fuzzy implications that satisfy the mixed MP are shown on Figure 1.

3 Conclusion

In a next research, we will give the list of the intuitionistic fuzzy conjunctions and disjunctions generated by the above implications and negations and will discuss some of their properties.

In future, some of the so collected implications will be used in a program realization of a new logical programming language. The proven properties of implications in Theorems 1 and 2 provide the opportunity for making standard logical deductions and deductions with a certain

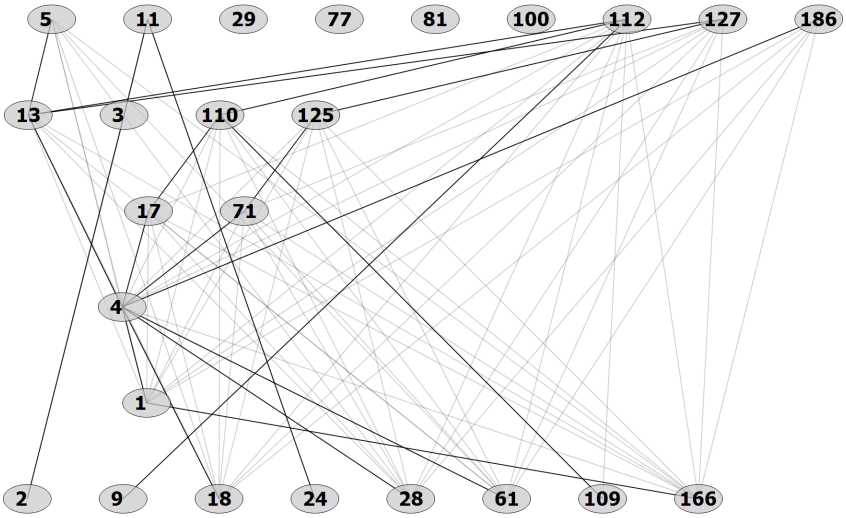

Figure 1. Relationships between the intuitionistic fuzzy implications that satisfy the mixed MP

degree of probability. The second type of deductions can be incorporated as new rules with a specific probability when dealing with ambiguous or incomplete initial data.

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