A multiattribute decision making approach using intuitionistic fuzzy sets

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Abstract

The concept of intuitionistic fuzzy sets is the generalization of the concept of fuzzy sets. The theory of intuitionistic fuzzy sets is well suited to dealing with vagueness. Recently, intuitionistic fuzzy sets have been used to build soft decision making models that can accommodate imprecise information, and two solution concepts about the intuitionistic fuzzy core and the consensus winner for group decision making have also been developed by other researchers using intuitionistic fuzzy sets. However, it seems that there is little investigation on multicriteria and/or group decision making using intuitionistic fuzzy sets with multiple criteria being explicitly taken into account. In this paper, multiattribute decision making using intuitionistic fuzzy sets is investigated, in which multiple criteria are explicitly considered, several linear programming models are constructed to generate optimal weights for attributes, and the corresponding decision making methods have also been proposed. Feasibility and effectiveness of the proposed method are illustrated using a numerical example.

Keywords: fuzzy set, intuitionistic fuzzy set, multiattribute decision making, linear programming model.

1 Introduction

The theory of fuzzy sets proposed by Zadeh [10] in 1965 has attracted wide spread attentions in various fields, especially where conventional mathematical techniques are of limited effectiveness, including biological and social sciences, linguistic, psychology, economics, and more generally soft sciences. In such fields, variables are difficult to quantify and dependencies among variables are so ill-defined that precise characterization in terms of algebraic, difference or differential equations becomes almost impossible. Even in fields where dependencies between variables are well-defined, it might be necessary or advantageous to employ fuzzy rather than crisp algorithms to arrive at a solution [5].

Out of several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [1] in 1986 have been found to be well suited to dealing with vagueness. The concept of an intuitionistic fuzzy set can be viewed as an alternative approach to define a fuzzy set in cases where available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set. In general, the theory of intuitionistic fuzzy sets is the generalization of fuzzy sets. Therefore, it is expected that intuitionistic fuzzy sets could be used to simulate human decision making processes and any activities requiring human expertise and knowledge, which are inevitably imprecise or not totally reliable.

De, et al [3] studied the application of intuitionistic fuzzy sets in medical diagnosis. Szmidt and Kacprzyk [5,6,7,8] considered the use of intuitionistic fuzzy sets for building soft decision making models with imprecise information, and proposed two solution concepts about the intuitionistic fuzzy core and the consensus winner for group decision making via intuitionistic fuzzy sets. However, it seems that so far there has been little research on multicriteria or multiattribute in discrete decision situations and/or group decision making using intuitionistic fuzzy sets, which is indeed one of the most important areas in decision analysis as most real world decision problems involve multiple criteria and a group of decision makers. In this paper, multiattribute decision making using intuitionistic fuzzy sets is investigated, in which attributes are explicitly considered, several corresponding linear programming models are constructed to generate optimal weights of attributes, and the corresponding decision making methods are also proposed.

2 Models and methods for multiattribute decision making using intuitionistic fuzzy sets

2.1 Definition of intuitionistic fuzzy sets

Let X be a universal set. An intuitionistic fuzzy set A in X is an object having the following form [2]

$$A = \{< x, \mu_A(x), \upsilon_A(x) > | x \in X\}$$

where the functions $\mu_A : X \mapsto [0,1]$ and

$\upsilon_A : X \mapsto [0,1]$ define the degree of membership and degree of non-membership of the element $x \in X$ to the set $A \subseteq X$, respectively, and for every $x \in X$,

$$0 \le \mu_A(x) + \upsilon_A(x) \le 1$$

We call $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$ the intuitionistic index of the element x in A. It is the degree of indeterminacy membership of the element x to A. It is obvious that $0 \le \pi_A(x) \le 1$ for every $x \in X$.

Let $A = \{< x, \mu_A(x), \upsilon_A(x) > | x \in X\}$ and $B = \{< x, \mu_B(x), \upsilon_B(x) > | x \in X\}$ be two intuitionistic fuzzy sets in the finite set X. Here, a new distance measure between A and B is defined as follows

$$D(A,B) = \sum_{x \in X} \frac{|\mu_A(x) - \mu_B(x)| + |\upsilon_A(x) - \upsilon_B(x)|}{2} \quad (1)$$

which can be proven to be a metric (omitted) and differs from those introduced in [2] and [9]. This distance measure will be employed in the following.

2.2 Presentation of multiattribute decision making problems under intuitionistic fuzzy environment

Suppose there exists an alternative set $X = \{x_1, x_2, \cdots, x_n\}$ which consists of n noninferior decision making alternatives from which a most preferred alternative is to be selected. Each alternative is assessed on m attributes. Denote the set of all attributes $A = \{a_1, a_2, \cdots, a_m\}$. Assume that μ_{ij} and υ_{ij} are the degree of membership and the degree of non-membership of the alternative $x_j \in X$ on the attribute $a_i \in A$ to the fuzzy concept "excellence", respectively, where $0 \le \mu_{ij} \le 1$, $0 \le \upsilon_{ij} \le 1$ and $0 \le \mu_{ij} + \upsilon_{ij} \le 1$. In other words, the evaluation of $x_j \in X$ on $a_i \in A$ is an intuitionistic fuzzy set. Denote $X_{ij} = \{< x_j, \mu_{ij}, \upsilon_{ij} >\}$. The intuitionistic indices $\pi_{ij} = 1 - \mu_{ij} - \upsilon_{ij}$ are such that the larger π_{ij} the higher a hesitation margin of the decision maker as to the "excellence" of $x_j \in X$ on $a_i \in A$ whose intensity is given by μ_{ij}. π_{ij} allows us to calculate the best final result (and the worst one) we can expect in a process leading to a final optimal decision. During the process the decision maker can change his evaluations in the following way. He can increase his evaluation by adding the value of the intuitionistic index. So in fact his evaluation lies in the closed interval $[\mu_{ij}^l, \mu_{ij}^u] = [\mu_{ij}, \mu_{ij} + \pi_{ij}]$, where $\mu_{ij}^l = \mu_{ij}$ and $\mu_{ij}^u = \mu_{ij} + \pi_{ij}$. Obviously, $0 \le \mu_{ij}^l \le \mu_{ij}^u \le 1$.

Similarly, assume that ρ_i and τ_i are the degree of membership and the degree of non-membership of $a_i \in A$ to the fuzzy concept "importance", respectively, where $0 \le \rho_i \le 1$, $0 \le \tau_i \le 1$ and $0 \le \rho_i + \tau_i \le 1$. The intuitionistic indices $\eta_i = 1 - \rho_i - \tau_i$ are such that the larger η_i the higher a hesitation margin of decision maker as to the "importance" of $a_i \in A$ whose intensity is given by ρ_i. Intuitionistic indices allow us to calculate the biggest weight (and the smallest one) we can expect in a process leading to a final decision. He can increase his evaluating weights by adding the value of the intuitionistic index. So his weight lies in the closed interval $[\omega_i^l, \omega_i^u] = [\rho_i, \rho_i + \eta_i]$, where $\omega_i^l = \rho_i$ and $\omega_i^u = \rho_i + \eta_i$. Obviously, $0 \le \omega_i^l \le \omega_i^u \le 1$.

2.3 Optimization model of multiattribute decision making under intuitionistic fuzzy environment

For each alternative $x_j \in X$, its optimal comprehensive value can be computed via the following programming

$$\max\{z_j = \sum_{i=1}^{m} \beta_{ij}\omega_i\}$$

$$\begin{cases} \mu_{ij}^l \le \beta_{ij} \le \mu_{ij}^u \ (i = 1,2,\cdots,m; j = 1,2,\cdots,n) \\ \sum_{i=1}^{m} \omega_i = 1, \omega_i^l \le \omega_i \le \omega_i^u \ (i = 1,2,\cdots,m) \end{cases} \quad (2)$$

To solve Eq.(2), we can solve the following two linear programmings

$$\min\{z_j^l = \sum_{i=1}^{m} \mu_{ij}^l \omega_i\}$$

$$\begin{cases} \omega_i^l \le \omega_i \le \omega_i^u \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \quad (3)$$

and

$$\max\{z_j^u = \sum_{i=1}^{m} \mu_{ij}^u \omega_i\}$$

$$\begin{cases} \omega_i^l \le \omega_i \le \omega_i^u \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \quad (4)$$

Optimal solutions $\overline{\boldsymbol{\omega}}^j = (\overline{\omega}_1^j, \overline{\omega}_2^j, \cdots, \overline{\omega}_m^j)^{\mathrm{T}}$ and $\overline{\overline{\boldsymbol{\omega}}}^j = (\overline{\overline{\omega}}_1^j, \overline{\overline{\omega}}_2^j, \cdots, \overline{\overline{\omega}}_m^j)^{\mathrm{T}}$ ($j = 1,2,\cdots,n$) can be obtained solving Eqs.(3) and (4) by Simplex method, respectively. In total, $2n$ linear programmings need to be solved since there are n alternatives in the set X.

After generating the corresponding optimal weight vectors, the optimal comprehensive value of the alternative

$x_j \in X$ can be computed as an intuitionistic fuzzy set given by

$$\overline{A}_j = \{< x_j, \sum_{i=1}^{m} \mu_{ij}\overline{\omega}_i^j, \sum_{i=1}^{m} \upsilon_{ij}\overline{\overline{\omega}}_i^j >\} \tag{5}$$

However, optimal solutions of Eqs.(3) and (4) are different in general, i.e., the weight vectors $\overline{\boldsymbol{\omega}}^j \neq \overline{\overline{\boldsymbol{\omega}}}^j$ for all $x_j \in X$, or $\overline{\omega}_i^j \neq \overline{\overline{\omega}}_i^j$ for all $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$. Therefore, the comprehensive values of all n alternatives $x_j \in X$ can not be compared.

Since X is a noninferior set, there exists no evident preference on some alternatives. Hence, for each $x_j \in X$, its objective function z_j^l in Eq.(3) should be assigned a equal weight $1/n$. Eq.(3) is then aggregated into the following linear programming

$$\min\{z_0^l = \sum_{j=1}^{n}\sum_{i=1}^{m} \mu_{ij}^l \omega_i / n\}$$

$$\begin{cases} \omega_i^l \leq \omega_i \leq \omega_i^u \ \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \tag{6}$$

In a similar way, Eq.(4) is aggregated into the following linear programming

$$\max\{z_0^u = \sum_{j=}^{n}\sum_{i=1}^{m} \mu_{ij}^u \omega_i / n\}$$

$$\begin{cases} \omega_i^l \leq \omega_i \leq \omega_i^u \ \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \tag{7}$$

Optimal solutions $\overline{\boldsymbol{\omega}}^0 = (\overline{\omega}_1^0, \overline{\omega}_2^0, \cdots, \overline{\omega}_m^0)^{\mathrm{T}}$ and $\overline{\overline{\boldsymbol{\omega}}}^0 = (\overline{\overline{\omega}}_1^0, \overline{\overline{\omega}}_2^0, \cdots, \overline{\overline{\omega}}_m^0)^{\mathrm{T}}$ can be obtained solving Eqs.(8) and (9) by Simplex method, respectively.

After generating the corresponding optimal weight vectors, the optimal comprehensive value of $x_j \in X$ can be computed as an intuitionistic fuzzy set given by

$$\overline{\overline{A}}_j = \{< x_j, \sum_{i=1}^{m} \mu_{ij}\overline{\omega}_i^0, \sum_{i=1}^{m} \upsilon_{ij}\overline{\overline{\omega}}_i^0 >\} \tag{8}$$

In generating the above intuitionistic fuzzy set only two linear programmings (i.e. Eqs.(6) and (7)) need to be solved. However, the optimal solutions of Eqs.(6) and (7) are normally different, so $\overline{\boldsymbol{\omega}}^0 \neq \overline{\overline{\boldsymbol{\omega}}}^0$ in general, or $\overline{\omega}_i^0 \neq \overline{\overline{\omega}}_i^0$ for all $i = 1,2,\cdots,m$. Therefore, it is possible that $\overline{\overline{z}}_j^l > \overline{\overline{z}}_j^u$. If this is the case, it follows that

$$\overline{\overline{\pi}}_j = 1 - \overline{\overline{z}}_j^l - (1 - \overline{\overline{z}}_j^u) = \overline{\overline{z}}_j^u - \overline{\overline{z}}_j^l < 0$$

However, this is not permitted by Definition 1.

Note that Eq.(6) is equivalent to the following linear programming

$$\max\{\overline{z}_0^l = -\sum_{j=1}^{n}\sum_{i=1}^{m} \mu_{ij}^l \omega_i / n\}$$

$$\begin{cases} \omega_i^l \leq \omega_i \leq \omega_i^u \ \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \tag{9}$$

Since Eqs.(7) and (9) have the same constraints, they can be combined to formulate the following programming

$$\max\{z = \sum_{j=}^{n}\sum_{i=1}^{m} (\mu_{ij}^u - \mu_{ij}^l)\omega_i / n\}$$

$$\begin{cases} \omega_i^l \leq \omega_i \leq \omega_i^u \ \ (i = 1,2,\cdots,m) \\ \sum_{i=1}^{m} \omega_i = 1 \end{cases} \tag{10}$$

The optimal solution $\boldsymbol{\omega}^0 = (\omega_1^0, \omega_2^0, \cdots, \omega_m^0)^{\mathrm{T}}$ can be obtained solving Eq.(10) by Simplex method. Then, the optimal comprehensive value of $x_j \in X$ can be computed as an intuitionistic fuzzy set given by

$$A_j^0 = \{< x_j, \sum_{i=1}^{m} \mu_{ij}\omega_i^0, \sum_{i=1}^{m} \upsilon_{ij}\omega_i^0 >\} \tag{11}$$

Then for each alternative $x_j \in X$, we can prove the property $[\overline{z}_j^l, \overline{z}_j^u] \supset [z_j^{0l}, z_j^{0u}]$ (omitted).

2.4 Multiattribute decision making method under an intuitionistic fuzzy environment

Using Eq.(10), we can obtain n optimal comprehensive values A_j^0 of all $x_j \in X$. Now, we are interested in how a final best compromise alternative or the final ranking order of the alternative set X can be generated.

In a similar way to the TOPSIS method proposed by Hwang and Yoon [4], we define the following index for each alternative $x_j \in X$

$$\xi_j = D(A_j^0, B)/[D(A_j^0, B) + D(A_j^0, G)] \tag{12}$$

where $G = \{< g, 1, 0 >\}$ and $B = \{< b, 0, 1 >\}$ are intuitionistic fuzzy sets corresponding to the evaluation of the ideal alternative g and the negative ideal alternative b, respectively. In this paper, $D(A_j^0, C)$ is chosen to be a distance measure between the intuitionistic fuzzy sets A_j^0 and C given by Eq.(1), where $C = B$ or G.

Obviously, $0 \leq \xi_j \leq 1$ for each $x_j \in X$. Furthermore, $\xi_j = 0$ if $A_j^0 = B$; $\xi_j = 1$ if $A_j^0 = G$. It is easy to see that the higher ξ_j the better the alternative x_j.

Using Eq.(1), Eq.(12) can be simply written as follows

$$\xi_j = (z_j^{0l} + z_j^{0u})/2 \tag{13}$$

Thus, the best alternative x_{j^*} can be generated so that $\xi_{j^*} = \max\{\xi_j \mid x_j \in X\}$ and the alternatives are ranked according to the increasing order of ξ_j for all $x_j \in X$.

3 An numerical example

Consider an air-condition system selection problem. Suppose there exist three air-condition systems x_1, x_2 and x_3. Denote the alternative set by $X = \{x_1, x_2, x_3\}$. Suppose three attributes a_1 (economical), a_2 (function) and a_3 (being operative) are taken into consideration in the selection problem. Denote the set of all attributes by $A = \{a_1, a_2, a_3\}$. Using statistical methods, the degrees μ_{ij} of membership and the degrees υ_{ij} of non-membership for $x_j \in X$ on $a_i \in A$ to the fuzzy concept "excellence" can be obtained, respectively

$$((\mu_{ij}, \upsilon_{ij})) = \begin{pmatrix} (0.75, 0.10) & (0.80, 0.15) & (0.40, 0.45) \\ (0.60, 0.25) & (0.68, 0.20) & (0.75, 0.05) \\ (0.80, 0.20) & (0.45, 0.50) & (0.60, 0.30) \end{pmatrix}$$

In a similar way, the degrees ρ_i of membership and the degrees τ_i of non-membership for $a_i \in A$ to the fuzzy concept "importance" can be obtained, respectively

$$((\rho_i, \tau_i)) = \begin{pmatrix} (0.25, 0.25) & (0.35, 0.40) & (0.30, 0.65) \end{pmatrix}$$

The following programming can be obtained via Eq.(10)

$$\max\{z = 0.35\omega_1 + 0.47\omega_2 + 0.50\omega_3\}$$

$$\begin{cases} 0.25 \le \omega_1 \le 0.75 \\ 0.35 \le \omega_2 \le 0.60 \\ 0.30 \le \omega_3 \le 0.35 \\ \omega_1 + \omega_2 + \omega_3 = 1 \end{cases} \tag{14}$$

Solving Eq.(14), its optimal solution can be obtained as follows

$$\boldsymbol{\omega}^0 = (0.25, 0.40, 0.35)^{\mathrm{T}}$$

Using Eq.(11), we can obtain optimal comprehensive values $A_1^0 = \{< x_1, 0.7075, 0.1950 >\}$, $A_2^0 = \{< x_2, 0.6295, 0.2925 >\}$ and $A_3^0 = \{< x_3, 0.610, 0.2375 >\}$ for x_1, x_2 and x_3, respectively. Hence, the following indices can be generated using Eq.(13)

$$\xi_1 = 0.7563, \ \xi_2 = 0.6685, \ \xi_3 = 0.6863$$

Then, the best alternative is x_1. The optimal ranking order of the alternatives is given by $x_1 \succ x_3 \succ x_2$.

Conclusions

In the above analysis, we have proposed several linear programming models and methods for multiattribute decision making under "intuitionistic fuzziness". In such decision situations, attributes are explicitly considered and are not compound, which differ from the ways used by Szmidt, et al [6,7,8]. Moreover, the evaluations of each alternative on each attribute to the fuzzy concept " excellence " are given using intuitionistic fuzzy sets, and the weights of each attribute are also given using intuitionistic fuzzy sets. This allows us to use flexible ways to simulate real decision situations, thereby building more realistic scenarios describing possible future events. In conclusion, multiattribute decision making models using intuitionistic fuzzy sets can represent a wide spectrum of possibilities, which enables the explicit consideration of the best and the worst results one can expect.

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