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On intuitionistic fuzzy norms and distances generated by the intuitionistic fuzzy subtractions $-'_{11}$ and $-''_{11}$

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Abstract: Two of the interesting intuitionistic fuzzy subtractions are $-'_{11}$ and $-''_{11}$. They are used for introduction of 5 new intuitionistic fuzzy norms and 10 new intuitionistic fuzzy distances. For them it is proved that they are intuitionistic fuzzy pairs.

Keywords: Intuitionistic fuzzy distance, Intuitionistic fuzzy norm, Intuitionistic fuzzy subtraction.
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1 On intuitionistic fuzzy subtractions $-'_{11}$ and $-''_{11}$

Different versions of operations "implication", "negation" and "subtraction" were introduced over the Intuitionistic fuzzy sets (IFS, see [3]). Here, following [3, 5], we will give the definitions of two of the "subtraction" operations: $-'_{11}$ and $-''_{11}$.

First, we shall give some definitions.

In the definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ \\ 0 & \text{if } x \le 0 \end{cases},$$

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ \\ 1 & \text{if } x \le 0 \end{cases}$$

Let the IFSs

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

and

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}$$

be given (for the description of their components see [1, 3]). Then

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}.$$

The negation $\neg_{11}$, that generates the definitions of "subtraction" operations $-'_{11}$ and $-''_{11}$ has the form (see, e.g., [3, 4]):

$$\neg_{11}A = \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle | x \in E\},$$

The definitions of operation "subtraction" use the well-known formula from set theory:

$$A - B = A \cap \neg B.$$

In the IFS-case, we also can define the operation "subtraction" by:

$$A -'_i B = A \cap \neg_i B, \tag{1}$$

where $\neg_i$ is one of the IF-negations, but, as we discussed in [2], the Law for Excluded Middle is not always valid in IFS theory. By this reason, we can introduce a new series of "subtraction" operations, that have the form:

$$A -''_i B = \neg_i \neg_i A \cap \neg_i B, \tag{2}$$

where $i = 1, 2, ..., 34$.

In [3, 5], the properties of negation $\neg_{11}$ and both IF-subtractions generated by it are studied.

Below, we will make the subsequent step of our research.

2 Norms and distances, generated by operation $-'_{11}$

Using (1) and (2), we obtain the following forms of "subtraction" operations:

$$A -'_{11} B = \{\langle x, \min(\mu_A(x), \mathrm{sg}(\nu_B(x))), \max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x)))\rangle | x \in E\}$$

and

$$A -''_{11} B = \{\langle x, \min(\overline{\mathrm{sg}}(\nu_A(x)), \mathrm{sg}(\nu_B(x))), \max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(\nu_B(x)))\rangle | x \in E\}.$$

Now, following [3], two norms of element $x \in E$ will be defined about set $A \subseteq E$ by:

$$||x||'_{11} = \langle \min(\mu_A(x), \mathrm{sg}(\nu_A(x))), \max(\nu_A(x), \overline{\mathrm{sg}}(\nu_A(x)))\rangle$$

and

$$||x||''_{11} = \langle \min(\overline{\mathrm{sg}}(\nu_A(x)), \mathrm{sg}(\nu_A(x))), \max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(\nu_A(x)))\rangle.$$

We must check that both pairs are intuitionistic fuzzy pairs, i.e., the sum of their components is smaller or equal to 1.

Really, for a given IFS A and for each $x \in E$ we obtain that if $\nu_A(x) = 0$, then

$$\min(\mu_A(x), \mathrm{sg}(\nu_A(x))) + \max(\nu_A(x), \overline{\mathrm{sg}}(\nu_A(x)))$$

$$= \min(\mu_A(x), 0) + \max(\nu_A(x), 1) = 1$$

and

$$\min(\overline{\mathrm{sg}}(\nu_A(x)), \mathrm{sg}(\nu_A(x))) + \max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(\nu_A(x)))$$
$$= \min(\overline{\mathrm{sg}}(\nu_A(x)), 0) + \max(\mathrm{sg}(\nu_A(x)), 1) = 1;$$

if $\nu_A(x) > 0$, then

$$\min(\mu_A(x), \mathrm{sg}(\nu_B(x))) + \max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x)))$$

$$= \min(\mu_A(x), 1) + \max(\nu_A(x), 0) = \mu_A(x) + \nu_A(x) \leq 1$$

and

$$\min(\overline{\mathrm{sg}}(\nu_A(x)), \mathrm{sg}(\nu_A(x))) + \max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(\nu_A(x)))$$
$$= \min(\overline{\mathrm{sg}}(\nu_A(x)), 1) + \max(\mathrm{sg}(\nu_A(x)), 0)$$
$$= \overline{\mathrm{sg}}(\nu_A(x)) + \mathrm{sg}(\nu_A(x)) = 1.$$

On the other hand, we see that

$$||x||''_{11} = \langle 0, 1\rangle,$$

i.e., this norm is not interesting. By this reason, we will discuss only the first one.

All norms, defined over IFSs up to now, excluding these from [3], are real numbers, and eventually, belong to the interval $[0, 1]$.

Let $e^*, o^*, u^* \in E$, so that

$$\mu_A(e^*) = 1,\ \nu_A(e^*) = 0,$$

$$\mu_A(o^*) = 0,\ \nu_A(o^*) = 1,$$

$$\mu_A(u^*) = 0,\ \nu_A(u^*) = 0.$$

Then,

$$||e^*||'_{11} = ||o^*||'_{11} = ||u^*||'_{11} = \langle 0, 1\rangle.$$

Now, similarly to [3], we will introduce the following five distances between the values of two elements $x, y \in E$ about the IFS A, by analogy with the first norm:

$$d'_{11,str\_opt}(A)(x,y) = \langle \min(\mu_A(x), \text{sg}(\nu_A(y))) + \min(\mu_A(y), \text{sg}(\nu_A(x)))$$

$$- \min(\mu_A(x), \text{sg}(\nu_A(y))). \min(\mu_A(y), \text{sg}(\nu_A(x))),$$

$$\max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))). \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))\rangle,$$

$$d'_{11,opt}(A)(x,y) = \langle \max(\min(\mu_A(x), \text{sg}(\nu_A(y))), \min(\mu_A(y), \text{sg}(\nu_A(x)))),$$

$$\min(\max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))), \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x))))\rangle,$$

$$d'_{11,aver}(A)(x,y) = \langle \frac{\min(\mu_A(x), \text{sg}(\nu_A(y))) + \min(\mu_A(y), \text{sg}(\nu_A(x)))}{2},$$

$$\frac{\max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))}{2}\rangle,$$

$$d'_{11,pes}(A)(x,y) = \langle \min(\mu_A(x), \text{sg}(\nu_A(y)), \mu_A(y), \text{sg}(\nu_A(x))),$$

$$\max(\nu_A(x), \overline{\text{sg}}(\nu_A(y)), \nu_A(y), \overline{\text{sg}}(\nu_A(x)))\rangle,$$

$$d'_{11,str\_pes}(A)(x,y) = \langle \min(\mu_A(x), \text{sg}(\nu_A(y))). \min(\mu_A(y), \text{sg}(\nu_A(x))),$$

$$\max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$

$$- \max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))). \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))\rangle.$$

Theorem 1. The five distances are intuitionistic fuzzy pairs.
Proof: We will check the fifth case. Let

$$X \equiv \min(\mu_A(x), \text{sg}(\nu_A(y))). \min(\mu_A(y), \text{sg}(\nu_A(x)))$$

$$+ \max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$

$$- \max(\nu_A(x), \overline{\text{sg}}(\nu_A(y))). \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x))).$$

If $\nu_A(y) = 0$. Then,

$$X = \min(\mu_A(x), 0). \min(\mu_A(y), \text{sg}(\nu_A(x)))$$

$$+ \max(\nu_A(x), 1) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$

$$- \max(\nu_A(x), 1). \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$

$$= 0 + 1 + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x))) - \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x))) = 1.$$

If $\nu_A(y) > 0$. Then,

$$X = \min(\mu_A(x), 1).\min(\mu_A(y), \text{sg}(\nu_A(x)))$$
$$+\max(\nu_A(x), 0) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$
$$-\max(\nu_A(x), 0).\max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$
$$= \mu_A(x).\min(\mu_A(y), \text{sg}(\nu_A(x)))$$
$$+\nu_A(x) + \max(\nu_A(y), \overline{\text{sg}}(\nu_A(x)))$$
$$-\nu_A(x).\max(\nu_A(y), \overline{\text{sg}}(\nu_A(x))).$$

If $\nu_A(x) = 0$. Then,

$$X = \mu_A(x).\min(\mu_A(y), 0) + \nu_A(x) + \max(\nu_A(y), 1) - \nu_A(x).\max(\nu_A(y), 1)$$
$$= 0 + 0 + 1 - 0 = 1.$$

If $\nu_A(x) > 0$. Then,

$$X = \mu_A(x).\min(\mu_A(y), 1) + \nu_A(x) + \max(\nu_A(y), 0) - \nu_A(x).\max(\nu_A(y), 0)$$
$$= \mu_A(x).\mu_A(y) + \nu_A(x) + \nu_A(y) - \nu_A(x).\nu_A(y)$$
$$\leq (1 - \nu_A(x)).(1 - \nu_A(y)) + \nu_A(x) + \nu_A(y) - \nu_A(x).\nu_A(y) = 1.$$

Therefore, $d'_{11,str\_pes}(A)(x, y)$ is an intuitionistic fuzzy pair. □

Now, we must mention the validity of the following equalities:

$$d'_{11,str\_opt}(A)(e^*, o^*) = \langle 1, 0\rangle,$$
$$d'_{11,str\_opt}(A)(e^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,str\_opt}(A)(o^*, u^*) = \langle 0, 0\rangle,$$
$$d'_{11,opt}(A)(e^*, o^*) = \langle 1, 0\rangle,$$
$$d'_{11,opt}(A)(e^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,opt}(A)(o^*, u^*) = \langle 0, 0\rangle,$$
$$d'_{11,aver}(A)(e^*, o^*) = \langle \frac{1}{2}, \frac{1}{2}\rangle,$$
$$d'_{11,aver}(A)(e^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,aver}(A)(o^*, u^*) = \langle 0, \frac{1}{2}\rangle,$$
$$d'_{11,pes}(A)(e^*, o^*) = \langle 0, 1\rangle,$$
$$d'_{11,pes}(A)(e^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,pes}(A)(o^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,str\_pes}(A)(e^*, o^*) = \langle 0, 1\rangle,$$

$$d'_{11,str\_pes}(A)(e^*, u^*) = \langle 0, 1\rangle,$$
$$d'_{11,str\_pes}(A)(o^*, u^*) = \langle 0, 1\rangle.$$

Finally, similarly to [3], we will introduce the following five distances between the values of element $x \in E$ about two IFSs A and B, by analogy with the first norm:

$$\begin{aligned}
d'_{11,str_opt}(A,B)(x) = \langle &\min(\mu_A(x), \mathrm{sg}(\nu_B(x))) + \min(\mu_B(x), \mathrm{sg}(\nu_A(x))) \\
&- \min(\mu_A(x), \mathrm{sg}(\nu_B(x))).\min(\mu_B(x), \mathrm{sg}(\nu_A(x))), \\
&\max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x))).\max(\nu_B(x), \overline{\mathrm{sg}}(\nu_A(x)))\rangle, \\
d'_{11,opt}(A,B)(x) = \langle &\max(\min(\mu_A(x), \mathrm{sg}(\nu_B(x))), \min(\mu_B(x), \mathrm{sg}(\nu_A(x)))), \\
&\min(\max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x))), \max(\nu_B(x), \overline{\mathrm{sg}}(\nu_A(x))))\rangle, \\
d'_{11,aver}(A,B)(x) = \langle &\frac{\min(\mu_A(x), \mathrm{sg}(\nu_B(x))) + \min(\mu_B(x), \mathrm{sg}(\nu_A(x)))}{2}, \\
&\frac{\max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x))) + \max(\nu_B(x), \overline{\mathrm{sg}}(\nu_A(x)))}{2}\rangle, \\
d'_{11,pes}(A,B)(x) = \langle &\min(\mu_A(x), \mathrm{sg}(\nu_B(x)), \mu_B(x), \mathrm{sg}(\nu_A(x))), \\
&\max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x)), \nu_B(x), \overline{\mathrm{sg}}(\nu_A(x)))\rangle, \\
d'_{11,str_pes}(A,B)(x) = \langle &\min(\mu_A(x), \mathrm{sg}(\nu_B(x))).\min(\mu_B(x), \mathrm{sg}(\nu_A(x))), \\
&\max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x))) + \max(\nu_B(x), \overline{\mathrm{sg}}(\nu_A(x))) \\
&- \max(\nu_A(x), \overline{\mathrm{sg}}(\nu_B(x))).\max(\nu_B(x), \overline{\mathrm{sg}}(\nu_A(x)))\rangle.
\end{aligned}$$

Theorem 2. The five last distances are intuitionistic fuzzy pairs.

3 Conclusion

In a next authors' research, some applications of the new norm and distances in the area of data bases will be discussed.

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