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The limit theorems on the interval valued events

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Abstract: Interval valued event ($IV-$ event) is a pair $A = (\mu_A, \nu_A)$ of fuzzy events such that $\mu_A \leq \nu_A$. The $IV-$ theory is isomorphic to the intuitionistic fuzzy theory. The paper contains a construction of mathematical apparatus and the proofs of some limit theorems in a space of $IV-$ events.
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1 Introduction

The algebraic structure studied in this paper have two aspects: the first one is practical, the second is theoretical one. Fuzzy sets and their generalization – Atanassov's intuitionistic fuzzy sets ($IF-$ sets) – in both give directions new possibilities. The whole $IF-$ theory can be motivated by practical problems and applications [3, 4, 5, 6].

The main contribution of the presented theory is a new point of view on human thinking and creation. We consider algebraic models for multivalued logic: $IF-$events and $IV-$events. But the more important idea is in building the probability theory on $IF-$ events. The theoretical description of uncertainty has two parts in the present time: objective – probability and statistics, and subjective – fuzzy sets. We show that both parts can be considered together.

Let us consider a theory dual to the $IF-$ events theory, theory of $IV-$ events. A prerequisity of $IV-$ theory is in the fact that it considers natural ordering and operations of vectors. On the other hand the $IV-$ theory is isomorphic to the $IF-$ theory [1, 2].

2 The $IV-$events

We shall start with a measurable space $(\Omega, \mathcal{S})$, where Ω is a non-empty set and $\mathcal{S}$ a σ-algebra of subsets of Ω, i.e. $\mathcal{S}$ is closed under complements and countable unions, $\Omega \in \mathcal{S}$. Usually a fuzzy event is a measurable mapping $f : \Omega \to [0,1]$, i.e. $f^{-1}(J) = \{\omega \in \Omega; f(\omega) \in J\} \in \mathcal{S}$ for every interval $J \subseteq [0,1]$.

Definition 1 *Interval valued event ($IV-$ event) is a pair $A = (\mu_A, \nu_A)$ of fuzzy events (i.e. $\mu_A, \nu_A : (\Omega, \mathcal{S}) \to [0,1]$ are fuzzy events) such that $\mu_A \leq \nu_A$. We denote the set of all $IV-$events by symbol $\mathcal{F}$.*

Definition 2 *We define two binary operations $\boxplus, \boxdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ as follows*

$$A \boxplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B) \wedge 1),$$

$$A \boxdot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B - 1) \vee 0),$$

and a partial ordering on the set $\mathcal{F}$

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B,\ \nu_A \leq \nu_B.$$

Remark 1 *Evidently $(0_\Omega, 0_\Omega)$ is the least element of $\mathcal{F}$, $(1_\Omega, 1_\Omega)$ is the greatest element of $\mathcal{F}$.*

Definition 3 *Probability is considered as a mapping*

$$P : \mathcal{F} \to \mathcal{J},\ \mathcal{J} = \{[a,b];\ a, b \in R,\ a \leq b\}$$

satisfying the following conditions

i) $P((0_\Omega, 0_\Omega)) = [0,0], P((1_\Omega, 1_\Omega)) = [1,1]$;

ii) $A \boxdot B = (0_\Omega, 0_\Omega) \Rightarrow P(A \boxplus B) = P(A) \boxplus P(B)$;

iii) $A_n \nearrow A \Rightarrow P(A_n) \nearrow P(A)$,

where $A_n \nearrow A$ means that $\mu_{A_n} \nearrow \mu_A,\ \nu_{A_n} \nearrow \nu_A$.

In the clasical probability space $(\Omega, \mathcal{S}, P)$ a random variable is consider as an $\mathcal{S}$-measurable mapping

$$\xi : \Omega \longrightarrow R,$$

for which holds: if $I \subset R$ is an interval then $\xi^{-1}(I) \in \mathcal{S}$.

Definition 4 *An observable is a mapping*

$$x : \mathcal{B}(R) \longrightarrow \mathcal{F}$$

satisfying the following conditions

i) $x(R) = (1,1),\ x(\emptyset) = (0,0)$;

ii) $A \cap B = \emptyset \Rightarrow x(A) \boxdot x(B) = (0,0),\ x(A \cup B) = x(A) \boxplus x(B)$;

iii) $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$.

Definition 5 *The state is a mapping* $m : \mathcal{F} \longrightarrow [0,1]$ *satisfying the conditions*

i) $m(0_\Omega, 0_\Omega) = 0, m(1_\Omega, 1_\Omega) = 1$;

ii) $A \boxdot B = (0_\Omega, 0_\Omega) \Longrightarrow m(A \boxplus B) = m(A) + m(B)$;

iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$.

Proposition 1 *If* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *is an observable, and* $m : \mathcal{F} \to [0,1]$ *is a state, then the mappping*

$$m_x = m \circ x : \mathcal{B}(R) \to [0,1],$$

defined by the formula

$$m_x(A) = m(x(A))$$

is a probability measure.

Proof:

i) $m_x(R) = m(x(R)) = m(1,1) = 1$;

ii) If $A \cap B = \emptyset$, then $x(A) \boxdot x(B) = (0,0)$;

hence

$$m_x(A \cup B) = m(x(A \cup B)) = m(x(A) \boxplus x(B)) = m(x(A)) + m(x(B)) = m_x(A) + m_x(B);$$

iii) $A_n \nearrow A$ implies $x(A_n) \nearrow x(A)$,

hence

$$m_x(A_n) = m(x(A_n)) \nearrow m(x(A)) = m_x(A).$$

$\square$

Proposition 2 *Let* $x : \mathcal{B}(R) \to \mathcal{F}$ *be an observable,* $m : \mathcal{F} \to [0,1]$ *be a state. We define a function* $F : R \to [0,1]$ *by the formula*

$$F(s) = m(x(-\infty, s)).$$

Then the function F *is non-decreasing, left continuous in any point* $s \in R$,

$$\lim_{s \to \infty} F(s) = 1, \quad \lim_{s \to -\infty} F(s) = 0.$$

Proof:
If $s < t$, then

$$x((-\infty, t)) = x((-\infty, s)) \boxplus x(\langle s, t\rangle) \geq x((-\infty, s))$$

hence

$$F(t) = m((-\infty, t)) \geq m(x((-\infty, s)) = F(s),$$

F is non-decreasing.
If $s_n \nearrow s$ then

$$x((-\infty, s_n)) \nearrow x((-\infty, s)),$$

hence

$$F(s_n) = m(x((-\infty, s_n))) \nearrow m(x((-\infty, s))) = F(s),$$

F is left continuous in any $s \in R$.
Similarly,

$$s_n \nearrow \infty \to x((-\infty, s_n)) \nearrow x((-\infty, \infty)) = (1, 1).$$

Therefore

$$F(s_n) = m(x((-\infty, s_n))) \nearrow s_n((1, 1)) = 1$$

for every $s_n \nearrow \infty$, hence $\lim\limits_{s\to\infty} F(s) = 1$.
Similarly we obtain

$$s_n \searrow -\infty \Longrightarrow -s_n \nearrow \infty,$$

hence

$$m(x((-s_n, s_n))) \nearrow m(x(R)) = 1.$$

$$1 = \lim_{n\to\infty} F(-s_n) = \lim_{n\to\infty} (x((-s_n, s_n))) + \lim_{n\to\infty} F(s_n) = 1 + \lim_{n\to\infty} F(s_n),$$

hence

$$\lim_{n\to\infty} F(s_n) = 0$$

for any $s_n \searrow -\infty$. □

3 The laws of large numbers

If we want to define the sum $\xi + \eta$ of two observables, one of possibilities is the following way. Put

$$T = (\xi, \eta) : \Omega \to R^2,$$

$$g : R^2, g(s, t) = s + t,$$

$$\xi + \eta = g \circ T : \Omega \to \Omega.$$

Namely, it is convenient for the constructing of preimages

$$(\xi + \eta)^{-1}(A) = T^{-1}(g^{-1}(A)).$$

In our $IV-$ case, we have two observables

$$x, y : \mathcal{B}(R) \to \mathcal{F},$$

hence $x + y$ could be defined as a morphism

$$(x + y)(A) = h(g^{-1}(A)),$$

where $h : \mathcal{B}(R^2) \to \mathcal{F}$ is a morphism connecting with x, y. In the classical case it was realized by the formula

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D).$$

In our $IV-$ case, instead of intersection, we shall use the product of $IV-$ sets defined by the formula

$$A \boxdot B = (\mu_A, \nu_A) \boxdot (\mu_B, \nu_B) = (\mu_A \cdot \mu_B, \nu_A \cdot \nu_B).$$

Definition 6 *Let $x_1, x_2, \cdots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be observables. By the joint observable of $x_1, x_2, \cdots, x_n$ we consider a mapping $h : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions*

i) $h(R^n) = (1, 0)$;

ii) $A \cap B = \emptyset \to h(A \cup B) = h(A) \boxplus h(B)$;

iii) $A_n \nearrow A \to h(A_n) \nearrow h(A)$;

iv) $h(C_1 \times C_2 \times \ldots \times C_n) = x_1(C_1) \cdot x_2(C_2) \cdot \ldots \cdot x_n(C_n)$, *for any* $C_1, C_2, \ldots, C_n \in \mathcal{B}(R)$.

Theorem 1 *For any observables $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ there exist their joint observable $h : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$.*

Proof:

We shall prove it for $n = 2$. Consider two observables $x, y : \mathcal{B}(R) \to \mathcal{F}$. Since $x(A) \in \mathcal{F}$, we shall write

$$x(A) = (x^\flat(A), x^*(A))$$

and similarly

$$y(B) = (y^\flat(B), y^*(B)).$$

From the definition of product $x(C) \cdot y(D)$ the following equalities hold:

$$x(C).y(D) = (x^\flat(C), x^*(C)).(y^\flat(D), y^*(D)) = (x^\flat(C).y^\flat(D), x^*(C).y^*(D)).$$

We shall construct similarly

$$(h^\flat(K), h^*(K).$$

Let us fix $\omega \in \Omega$ and let us put

$$\mu_A = x^\flat(A)(\omega),$$

$$\nu_B = x^\flat(B)(\omega),$$

$$h^\flat(K) = \mu \times \nu(K).$$

$\mu \times \nu$ is the product of probability measures μ, ν.

Then,

$$h^\flat(C \times D)(\omega) = \mu \times \nu(C \times D) = \mu(C) \cdot \nu(D) = x^\flat(C) \cdot y^\flat(D)(\omega),$$

hence

$$h^\flat(C \times D) = x^\flat(C) \cdot y^\flat(D).$$

Analogously,

$$h^*(C \times D) = x^*(C) \cdot y^*(D).$$

If we define

$$h(A) = (h^\flat(A), h^*(A)), A \in \mathcal{B}(R^2),$$

then there holds

$$h(C \times D) = (x^\flat(C), y^\flat(D), x^*(C) \cdot y^*(D)) = x(C) \cdot y(D).$$

□

Then, the previous theorem can be applied for obtaining the sum

$$x_1 + x_2 + \ldots + x_n = h \circ g^{-1}$$

with

$$g(u_1, \ldots u_n) = u_1 + \ldots + u_n$$

or for the arithmetic means

$$\frac{1}{n}(x_1 + x_2 + \ldots + x_n) = h \circ g^{-1},$$

with

$$g(u_1, \ldots, u_n) = \frac{1}{n}(u_1 + \ldots + u_n).$$

4 The weak law of large numbers

We shall consider an event A whose probability is p. We make n independent tests. Let k is a number of the tests in which an event A occured. The laws of large numbers state, that the relative frequency $\frac{k_n}{n}$ of event A convergence to the probability p. It is known, that k_n is the random variable with binomial distribution with the parameters n, p. It can be expressed as the map

$$k_n = \sum_{i=1}^{n} \chi_{A_i},$$

where $A_1, A_2, \ldots, A_n$ are independent events. We hence talk about convergence

$$\frac{1}{n}\sum_{i=1}^{n} \chi_{A_i} \to p.$$

Generally, we can consider instead of a sequence of characteristic functions

$$\chi_{A_1}, \chi_{A_2}, \dots$$

the sequence of independent random variables

$$\xi_1, \xi_2, \dots$$

then the aritmetic mean

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i$$

converges to a normal distribution.

Definition 7 *Let $y_1, y_2, \dots$ be a sequence of observables $y_n : \mathcal{B}(R) \to \mathcal{F}$, for $n = 1, 2, \dots$ and a mapping $m : \mathcal{F} \to [0;1]$ be a state.*

1. *A sequence converges in distribution to a function $F : R \to [0,1]$, if for all $t \in R$ there holds*
$$\lim_{n\to\infty} m(y_n((-\infty;t))) = F(t)$$
2. *A sequence converges by a measure to $(0_\Omega, 0_\Omega)$, if for all $\epsilon > 0$ there holds*
$$\lim_{n\to\infty} (y_n((-\epsilon, \epsilon))) = 1;$$
3. *A sequence converges to $(0_\Omega, 0_\Omega)$ almost everywhere, if*
$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m(\bigvee_{n=k}^{k+i}\left(-\frac{1}{p}, \frac{1}{p}\right) = 1.$$

Definition 8 *Let $x_1, x_2, \dots$ be observables $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be a joint observable of observables $x_1, x_2, \dots, x_n$. We define the functions $y_n = g_n(x_1, \dots, x_n)$, where the functions $g_n : R^n \to R$, are given by formula $y_n = h_n \circ g_n^{-1}$.*

Theorem 2 *Let $x_1, x_2, \dots$ be a sequence of observables, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be a joint observable of observables $x_1, x_2, \dots, x_n$ and $y_n = g_n(x_1, \dots, x_n)$, for $n = 1, 2, \dots,\ g_n : R^n \to R$. Then there exist the probability space (Ω, S, P) and a sequence of random variables $(\xi_n)_{n=1}^{\infty},\ x_n : \Omega \to R$, such that*

if

$$\eta_n = g_n(\xi_1, \dots, \xi_n),$$

then

1. *A sequence $y_1, y_2, \dots$ converges in a distribution to function F if and only if a sequence $\eta_1, \eta_2, \dots$ converges in a distribution to function F.*
2. *A sequence $y_1, y_2, \dots$ converges to $(0_\Omega, 0_\Omega)$ by a measure m if and only if $\eta_1, \eta_2, \dots$ converges to 0 by a measure P.*

3. *If* $\eta_1, \eta_2, \dots$ *coverges* $P-$*almost everywhere to* 0*, then* $y_1, y_2, \dots$ *coverges* $m-$*almost everywhere to* $(0_\Omega, 0_\Omega)$.

Proof: By Kolmogorov theorem there exists just one probability measure $P : \sigma(C) \to [0,1]$, where C is the set of all cylinders, such that

$$P \circ \pi_n^{-1} = m \circ h_n,$$

for $n = 1, 2, \dots$, where $\pi_n : R^N \to R$ is a projection.

Let $\xi_n : R^N \to R$

$$\xi(((u_i)_{i=1}^{\infty})) = u_n$$

for $n = 1, 2, \dots$. Then

$$P(\eta_n^{-1}(A)) = P((g_n(\xi_1, \dots, \xi_n))^{-1}(A)) = P(\pi_n^{-1}(g_n^{-1}(A))) = m(y_n(A)).$$

Hence

$$m(y_n((-\infty, t))) = P(\eta^{-1}((-\infty, t))).$$

Analogously there hold

$$m(y_n((-\epsilon, \epsilon))) = P(\eta^{-1}((-\epsilon, \epsilon))).$$

From the above equalities follows the validity of the first and second equivalence.

Now we shall show the validity of the third implication.

$$1 = P\left(\bigcap_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right)$$

$$P\left(h_{k+i}\left(\bigcap_{n=k}^{k+i}\left\{(t_1, \dots, t_{k+i}) : g_n(t_1, \dots, t_n) \in \left(-\frac{1}{t}, \frac{1}{t}\right)\right\}\right)\right) \leq$$

$$\leq m\left(\bigwedge_{n=k}^{k+i} h_{k+i}\left(\left\{(t_1, \dots, t_{k+i}) : (t_1, \dots, t_n) \in g_n^{-1}\left(\left(-\frac{1}{t}, \frac{1}{t}\right)\right)\right\}\right)\right) =$$

$$= m\left(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}\left(\left(-\frac{1}{t}, \frac{1}{t}\right)\right)\right) = m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{t}, \frac{1}{t}\right)\right)\right).$$

Hence

$$1 = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) \leq$$

$$\leq \lim_{t\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{t}, \frac{1}{t}\right)\right)\right) \leq 1.$$

□

Theorem 3 *(The weak law of large numbers) Let be* m *a state,* $x_1, x_2, \ldots$ *be a sequence of independent integrable observables with the same probability distribution and* $m_{x_1} = m_{x_2} = \ldots$. *Let* $a = E(x_1) = E(x_2) = \ldots$*, then there exists a sequence of observables* (y_n)*, where*

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n} - a,\ (n = 1, 2, \ldots),$$

converges in measure m *to* $(0_\Omega, 0_\Omega)$.

Proof: Let $h_n : \mathcal{B}(R) \to \mathcal{F}$ be the joint observable of observables $x_1, x_2, \ldots, x_n,\ g_n : R^n \to R$ be a function given by formula $g_n(w_1, w_2, \ldots, w_n) = \frac{w_1 + w_2 + \ldots + w_n}{n} - a$, and $y_n : \mathcal{B}(R) \to \mathcal{F};\ y_n = g_n(x_1, x_2, \ldots x_n) = h_n \circ g_n^{-1}$, for $n = 1, 2, \ldots$.

Let us consider the probability space $(\Omega, \mathcal{S}, P)$ and a sequence $(\xi_n)_{n=1}^{\infty}$ the random variables $\xi_n : R^n \to R$.

For every $n \in N$ we define a random variable $\xi_n : R^N \to R,\ \xi_n((u_i)_{i=1}^{\infty}) = u_n$ and the mapping $\eta_n : R^n \to R$ by the formula

$$\eta_n = g_n(\xi_1, \xi_2, \ldots, \xi_n) = g_n \circ \pi_n = \frac{1}{n}\sum_{i=1}^{n} v_i - a.$$

We get the equalities

$$P \circ \xi_n^{-1} = P\xi_n = m_{x_n} = m \circ \xi_n,$$
$$P \circ T_n^{-1} = m_{x_1} \times m_{x_2} \times \ldots \times m_{x_n} = m \circ h_n.$$

Then, an average ξ_n is

$$E(\xi_n) = \int_{\Omega} x_n dP = \int_{\infty}^{-\infty} t dP_{\xi_n}(t) = \int_{\infty}^{-\infty} t dm_{x_n}(t) = E(\xi_n) = a.$$

If the observables $x_1, x_2, \ldots, x_n$ are independent, then the random variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent, too.

For $n = 2$

$$T_2 = (\xi_1, \xi_2) \in A \times B$$

hence

$$P(\xi_1^{-1}(A) \cap \xi_2^{-1}(B) = P \circ T_2^{-1}(C) = m \circ h_2(C) = m \circ h_2(A \times B) =$$
$$= m_{x_1}(A) \times m_{x_2}(B) = P(\xi_1)(A) \cdot P(\xi_2)(B).$$

Therefore, for every $\epsilon > 0$

$$\lim_{n\to\infty} P = \left(\left\{\omega \in \Omega; \frac{\xi_1 + \ldots + \xi_n}{n} - a < \epsilon\right\}\right) = 1$$

hold the equalities

$$\lim_{n\to\infty} P((\eta_n^{-1})((-\epsilon, \epsilon))) = \lim_{n\to\infty} (\{\omega \in \Omega; |\eta_n(\omega) - 0 < \epsilon|\}) =$$
$$\lim_{n\to\infty} \left(\left\{\omega \in \Omega; \left|\frac{\xi_1 + \ldots + \xi_n}{n} - a < \epsilon\right|\right\}\right) = 1.$$

□

5 The strong law of large numbers

Theorem 4 *Let $x_1, x_2, \ldots$ be a sequence of independent observables that have an integrable square, then a sequence of observables*

$$y_n = \frac{x_1 - E(x_1) + x_2 - E(x_2) + \ldots + x_n - E(x_n)}{n}, \ n = 1, 2, \ldots$$

converges to $(0_\Omega, 0_\Omega)$ almost everywhere.

Proof: Let $h_n(n = 1, 2, \ldots) : \mathcal{B}(R) \to \mathcal{F}$ be the joint observables of the observables $x_1, x_2, \ldots,$ and the functions $g_n : R^n \to R$ be given by the formula

$$g_n(\omega_1, \omega_2, \ldots, \omega_n) = \frac{1}{n} \left(\omega_1 - E(x_1) + \omega_2 - E(x_2) + \ldots + \omega_n - E(x_n)\right)$$

and let $y_n : \mathcal{B}(R) \to \mathcal{F}$ be mappings such that

$$y_n = h_n \circ g_n^{-1}, \ (n = 1, 2, \ldots)$$

Let us consider a probability space $(\Omega, \mathcal{S}, P)$ and a sequence of random variables $\xi_n : R^n \to R, \ (n = 1, 2, \ldots)$. We put

$$\eta_n = g_n(\xi_1, \ldots, \xi_n) = \frac{1}{n} \left(\xi_1 - E(x_1) + \xi_2 - E(x_2) + \ldots + \xi_n - E(x_n)\right).$$

Then, for the mean value there holds

$$E(\xi_n) = \int_\Omega \xi_n dP = \int_{-\infty}^{\infty} t dP_{\xi_n}(t) = \int_{-\infty}^{\infty} t dm_{x_n}(t) = E(x_n).$$

If the observables $x_1, x_2, \ldots$ are independent, then the random variables $\xi_1, \xi_2, \ldots$ are independent, too.

A dispersion

$$\sigma^2(\xi_n) = \int_{-\infty}^{\infty} (t - E(\xi_n))^2 dm_{\xi_n}(t) = \int_{-\infty}^{\infty} (t - E(x_n))^2 dm_{x_n}(t) = \sigma^2(x_n).$$

Hence, the sequence $\eta_n = \frac{1}{n} \sum_{i=1}^{n} (\xi_i - E(\xi_1))$ converges $P-$ almost everywhere to 0 and following $y_1, y_2, \ldots$ it converges $m-$almost everywhere to $(0_\Omega, 0_\Omega)$. □

6 Conclusion

We have proved some versions of the laws of large number for sequences of the independent observables in the space of the interval valued events. The central limit theorem on $IV-$ events was proved in [6]. Research about $IV-$ events can continue for a conditional probability.

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