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Trapezoidal approximations of intuitionistic fuzzy numbers expressed by value, ambiguity, width and weighted expected value

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Abstract Some parameters for intuitionistic fuzzy numbers are introduced: value, ambiguity, width and weighted expected value. Trapezoidal approximation operators given in previous papers are expressed by these parameters.

Key words: Fuzzy number, Intuitionistic fuzzy number, Trapezoidal fuzzy number, Value, Ambiguity, Width, Expected value

1 Intuitionistic fuzzy numbers

The section contains some basic on intuitionistic fuzzy numbers: definition, expected interval and expected value, metric. We mention that in the paper [12] some aspects are also pointed out.

Definition 1.1 *([1]-[3]) Let $X \neq \emptyset$ be a given set. An intuitionistic fuzzy set in X is an object A given by*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in X\},$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

for every $x \in X$.

Definition 1.2 *An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in \mathbb{R}\}$ such that μ_A and $1 - \nu_A$, where*

$$(1 - \nu_A)(x) = 1 - \nu_A(x), \forall x \in X,$$

are fuzzy numbers is called an intuitionistic fuzzy number.

We denote by $A = \langle \mu_A, \nu_A \rangle$ an intuitionistic fuzzy number and by $IF(\mathbb{R})$ the set of all intuitionistic fuzzy numbers. It is obvious that any fuzzy number u can be represented as an intuitionistic fuzzy number by $\langle u, 1-u \rangle$. An intuitionistic fuzzy number $A = \langle \mu_A, \nu_A \rangle$ such that μ_A and $1 - \nu_A$ are trapezoidal fuzzy numbers is called a trapezoidal intuitionistic fuzzy number. We denote by $IF^T(\mathbb{R})$ the set of all trapezoidal intuitionistic fuzzy numbers.

With respect to the α-cuts of the fuzzy number $1 - \nu_A$ the following equalities are immediate:

$$(1 - \nu_A)_L(\alpha) = \nu_{A_L}(1 - \alpha)$$

and

$$(1 - \nu_A)_U(\alpha) = \nu_{A_U}(1 - \alpha),$$

for every $\alpha \in [0, 1]$.

A metric on the set of fuzzy numbers, which is an extension of the Euclidean distance, is defined by (see, e. g., [5])

$$d^2(u, v) = \int_0^1 (u_L(\alpha) - v_L(\alpha))^2 \, d\alpha + \int_0^1 (u_U(\alpha) - v_U(\alpha))^2 \, d\alpha,$$

where u and v are arbitrary fuzzy numbers with α-cuts $u_\alpha = [u_L(\alpha), u_U(\alpha)]$ and $v_\alpha = [v_L(\alpha), v_U(\alpha)], \alpha \in [0, 1]$.

In the intuitionistic fuzzy case the metric d becomes (see [4])

$$\begin{aligned} \widetilde{d}^2(A, B) = &\frac{1}{2}\int_0^1 (\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha))^2 \, d\alpha + \frac{1}{2}\int_0^1 (\mu_{A_U}(\alpha) - \mu_{B_U}(\alpha))^2 \, d\alpha \\ &+ \frac{1}{2}\int_0^1 (\nu_{A_L}(\alpha) - \nu_{B_L}(\alpha))^2 \, d\alpha + \frac{1}{2}\int_0^1 (\nu_{A_U}(\alpha) - \nu_{B_U}(\alpha))^2 \, d\alpha, \end{aligned}$$

where $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R})$ and $B = \langle \mu_B, \nu_B \rangle \in IF(\mathbb{R})$,

$$[u_L(\alpha), u_U(\alpha)]$$

denotes the α-cut, $\alpha \in [0, 1]$, of the fuzzy number u and $[\omega_L(\alpha), \omega_U(\alpha)]$ denotes the α-cut, $\alpha \in [0, 1]$, of the fuzzy set ω such that $1 - \omega$ is a fuzzy number.

The trapezoidal approximation of an intuitionistic fuzzy number is the nearest trapezoidal fuzzy number to an intuitionistic fuzzy number, with respect to metric $\widetilde{d}$, with or without some additional conditions.

Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,

$$\begin{aligned} \mu_A &= ([\mu_{A_L}(\alpha), \mu_{A_U}(\alpha)])_{\alpha \in [0,1]}, \\ \nu_A &= ([\nu_{A_L}(\alpha), \nu_{A_U}(\alpha)])_{\alpha \in [0,1]}. \end{aligned}$$

We denote

$$m_L = \int_0^1 \mu_{A_L}(\alpha)\, d\alpha, m_U = \int_0^1 \mu_{A_U}(\alpha)\, d\alpha,$$
$$n_L = \int_0^1 \nu_{A_L}(\alpha)\, d\alpha, n_U = \int_0^1 \nu_{A_U}(\alpha)\, d\alpha,$$
$$M_L = \int_0^1 \alpha\mu_{A_L}(\alpha)\, d\alpha, M_U = \int_0^1 \alpha\mu_{A_U}(\alpha)\, d\alpha,$$
$$N_L = \int_0^1 \alpha\nu_{A_L}(\alpha)\, d\alpha, N_U = \int_0^1 \alpha\nu_{A_U}(\alpha)\, d\alpha,$$
$$m = \int_0^1 \mu_{A_L}^2(\alpha)\, d\alpha, M = \int_0^1 \mu_{A_U}^2(\alpha)\, d\alpha,$$
$$n = \int_0^1 \nu_{A_L}^2(\alpha)\, d\alpha, N = \int_0^1 \nu_{A_U}^2(\alpha)\, d\alpha.$$

2 Value, ambiguity, width and weighted expected value of an intuitionistic fuzzy number

An important notion connected with the concept of fuzzy number is the expected interval $EI(u)$ of a fuzzy number u introduced in [10] and [13]. It is given by

$$EI(u) = [E_*(u), E^*(u)] = \left[\int_0^1 u_L(\alpha)\, d\alpha, \int_0^1 u_U(\alpha)\, d\alpha\right].$$

The expected value $EV(u)$ of a fuzzy number u is defined by

$$EV(u) = \frac{E_*(u) + E^*(u)}{2}.$$

Its generalization, called weighted expected value, is defined (see [11]) as

$$EV_q(u) = (1-q)\int_0^1 u_L(\alpha)\, d\alpha + q\int_0^1 u_U(\alpha)\, d\alpha,$$

where $q \in [0,1]$. The operator can be extended in a natural way to intuitionistic fuzzy numbers by

$$\widetilde{EV}_q(\langle\mu_A, \nu_A\rangle) = \frac{1}{2}\left(EV_q(\mu_A) + EV_q(1-\nu_A)\right),$$

for every $A = \langle\mu_A, \nu_A\rangle \in IF(\mathbb{R})$.

To simplify the task of representing and handling fuzzy numbers the value and the ambiguity of a fuzzy number $u, u_\alpha = [u_L(\alpha), u_U(\alpha)], \alpha \in [0,1]$, defined by

$$Val(u) = \int_0^1 \alpha\left(u_L(\alpha) + u_U(\alpha)\right) d\alpha$$

and

$$Amb(u) = \int_0^1 \alpha\left(u_U(\alpha) - u_L(\alpha)\right) d\alpha,$$

were introduced in [9]. The width of a fuzzy number (see [8]) defined by

$$w(u)=\int_0^1 (u_U(\alpha)-u_L(\alpha))\,d\alpha$$

is a parameter which characterizes the nonspecificity of a fuzzy number.

These parameters can be extended to intuitionistic fuzzy numbers by

$$\begin{aligned}\widetilde{Val}(\langle\mu_A,\nu_A\rangle) &= \frac{1}{2}(Val(\mu_A)+Val(1-\nu_A)) \\ &= \frac{1}{2}\int_0^1 \alpha(\mu_{A_L}(\alpha)+\mu_{A_U}(\alpha))\,d\alpha \\ &+ \frac{1}{2}\int_0^1 (1-\alpha)(\nu_{A_L}(\alpha)+\nu_{A_U}(\alpha))\,d\alpha,\end{aligned}$$

$$\begin{aligned}\widetilde{Amb}(\langle\mu_A,\nu_A\rangle) &= \frac{1}{2}(Amb(\mu_A)+Amb(1-\nu_A)) \\ &= \frac{1}{2}\int_0^1 \alpha(\mu_{A_U}(\alpha)-\mu_{A_L}(\alpha))\,d\alpha \\ &+ \frac{1}{2}\int_0^1 (1-\alpha)(\nu_{A_U}(\alpha)-\nu_{A_L}(\alpha))\,d\alpha\end{aligned}$$

and

$$\begin{aligned}\widetilde{w}(\langle\mu_A,\nu_A\rangle) &= \frac{1}{2}(w(\mu_A)+w(1-\nu_A)) \\ &= \frac{1}{2}\int_0^1 (\mu_{A_U}(\alpha)-\mu_{A_L}(\alpha))\,d\alpha \\ &+ \frac{1}{2}\int_0^1 (\nu_{A_U}(\alpha)-\nu_{A_L}(\alpha))\,d\alpha,\end{aligned}$$

for every $A=\langle\mu_A,\nu_A\rangle\in IF(\mathbb{R})$.

With the above notations we get

$$\widetilde{EV}_q(A)=(1-q)\frac{m_L+n_L}{2}+q\frac{m_U+n_U}{2}, \tag{1}$$

$$\widetilde{Val}(A)=\frac{1}{2}(M_L+M_U)+\frac{1}{2}(n_L-N_L+n_U-N_U), \tag{2}$$

$$\widetilde{Amb}(A)=\frac{1}{2}(M_U-M_L)+\frac{1}{2}(n_U-N_U-n_L+N_L), \tag{3}$$

$$\widetilde{w}(A)=\frac{1}{2}(m_U-m_L)+\frac{1}{2}(n_U-n_L). \tag{4}$$

3 Approximation without condition

The main result of the paper [7] is the following trapezoidal approximation.

Theorem 3.1 *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,*

$$\mu_A = \left(\left[\mu_{A_L}(\alpha), \mu_{A_U}(\alpha) \right] \right)_{\alpha \in [0,1]},$$
$$\nu_A = \left(\left[\nu_{A_L}(\alpha), \nu_{A_U}(\alpha) \right] \right)_{\alpha \in [0,1]}$$

and

$$T_{\widetilde{d}}(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to intuitionistic fuzzy number A.

(i) If

$$m_L - m_U - 3M_L + 3M_U - 2n_L + 2n_U + 3N_L - 3N_U \geq 0, \tag{5}$$

then

$$t_1 = 2m_L - 3M_L - n_L + 3N_L, \tag{6}$$
$$t_2 = -m_L + 3M_L + 2n_L - 3N_L, \tag{7}$$
$$t_3 = -m_U + 3M_U + 2n_U - 3N_U, \tag{8}$$
$$t_4 = 2m_U - 3M_U - n_U + 3N_U; \tag{9}$$

(ii) If

$$m_L - m_U - 3M_L + 3M_U - 2n_L + 2n_U + 3N_L - 3N_U < 0, \tag{10}$$
$$3m_L + m_U - 5M_L - 3M_U - 2n_L - 2n_U + 5N_L + 3N_U \leq 0, \tag{11}$$
$$-m_L - 3m_U + 3M_L + 5M_U + 2n_L + 2n_U - 3N_L - 5N_U \leq 0 \tag{12}$$

then

$$t_1 = \frac{7}{4}m_L + \frac{1}{4}m_U - \frac{9}{4}M_L - \frac{3}{4}M_U - \frac{1}{2}n_L - \frac{1}{2}n_U + \frac{9}{4}N_L + \frac{3}{4}N_U, \tag{13}$$
$$t_2 = t_3 = -\frac{1}{2}m_L - \frac{1}{2}m_U + \frac{3}{2}M_L + \frac{3}{2}M_U + n_L + n_U - \frac{3}{2}N_L - \frac{3}{2}N_U, \tag{14}$$
$$t_4 = \frac{1}{4}m_L + \frac{7}{4}m_U - \frac{3}{4}M_L - \frac{9}{4}M_U - \frac{1}{2}n_L - \frac{1}{2}n_U + \frac{3}{4}N_L + \frac{9}{4}N_U; \tag{15}$$

(iii) If

$$3m_L + m_U - 5M_L - 3M_U - 2n_L - 2n_U + 5N_L + 3N_U > 0 \tag{16}$$
$$m_L + 2m_U - 6M_U + n_L - 4n_U + 6N_U > 0 \tag{17}$$

then

$$t_1 = t_2 = t_3 = \frac{2}{5}m_L - \frac{1}{5}m_U + \frac{3}{5}M_U + \frac{2}{5}n_L + \frac{2}{5}n_U - \frac{3}{5}N_U, \tag{18}$$
$$t_4 = -\frac{1}{5}m_L + \frac{8}{5}m_U - \frac{9}{5}M_U - \frac{1}{5}n_L - \frac{1}{5}n_U + \frac{9}{5}N_U. \tag{19}$$

(iv) If

$$-m_L - 3m_U + 3M_L + 5M_U + 2n_L + 2n_U - 3N_L - 5N_U > 0 \tag{20}$$
$$-2m_L - m_U + 6M_L + 4n_L - n_U - 6N_L > 0 \tag{21}$$

then

$$t_1 = \frac{8}{5}m_L - \frac{1}{5}m_U - \frac{9}{5}M_L - \frac{1}{5}n_L - \frac{1}{5}n_U + \frac{9}{5}N_L, \tag{22}$$

$$t_2 = t_3 = t_4 = -\frac{1}{5}m_L + \frac{2}{5}m_U + \frac{3}{5}M_L + \frac{2}{5}n_L + \frac{2}{5}n_U - \frac{3}{5}N_L. \tag{23}$$

We present an equivalent form of this theorem in terms of weighted expected value, value, ambiguity and width.

Theorem 3.2 *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,*

$$\mu_A = \left(\left[\mu_{A_L}(\alpha), \mu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]},$$
$$\nu_A = \left(\left[\nu_{A_L}(\alpha), \nu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]}$$

and

$$T_{\widetilde{d}}(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to intuitionistic fuzzy number A.

(i) If

$$\widetilde{Amb}(A) \geq \frac{1}{3}\widetilde{w}(A) \tag{24}$$

then $T_{\widetilde{d}}(A)$ is given by (6)-(9);

(ii) If

$$\widetilde{Amb}(A) < \frac{1}{3}\widetilde{w}(A), \tag{25}$$

$$\widetilde{Val}(A) \leq \widetilde{EV}_{\frac{3}{4}}(A) - \frac{1}{4}\widetilde{Amb}(A) \tag{26}$$

and

$$\widetilde{Val}(A) \geq \widetilde{EV}_{\frac{1}{4}}(A) + \frac{1}{4}\widetilde{Amb}(A) \tag{27}$$

then $T_{\widetilde{d}}(A)$ is given by (13)-(15);

(iii) If

$$\widetilde{Val}(A) < \widetilde{EV}_{\frac{1}{4}}(A) + \frac{1}{4}\widetilde{Amb}(A) \tag{28}$$

and

$$\widetilde{Val}(A) < \widetilde{EV}_{\frac{2}{3}}(A) - \widetilde{Amb}(A) \tag{29}$$

then $T_{\widetilde{d}}(A)$ is given by (18)-(19);

(iv) If

$$\widetilde{Val}(A) > \widetilde{EV}_{\frac{3}{4}}(A) - \frac{1}{4}\widetilde{Amb}(A) \tag{30}$$

and

$$\widetilde{Val}(A) > \widetilde{EV}_{\frac{1}{3}}(A) + \widetilde{Amb}(A) \tag{31}$$

then $T_{\widetilde{d}}(A)$ is given by (22)-(23).

Proof. Taking into account the equalities (1)-(4) we easy obtain the equivalences between (5) and (24), (12) and (26), (11) and (27), (17) and (29), (21) and (31). ■

4 Approximation preserving the expected interval

The main result of the paper [6] is the following trapezoidal approximation.

Theorem 4.1 *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,*

$$\mu_A = \left(\left[\mu_{A_L}(\alpha), \mu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]},$$
$$\nu_A = \left(\left[\nu_{A_L}(\alpha), \nu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]}$$

and

$$T^e_{\widetilde{d}}(A) = \left(t^e_1(A), t^e_2(A), t^e_3(A), t^e_4(A)\right) = \left(t^e_1, t^e_2, t^e_3, t^e_4\right)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to intuitionistic fuzzy number A, preserving the expected interval.

(*i*) *If*

$$m_L - m_U - 3M_L + 3M_U - 2n_L + 2n_U + 3N_L - 3N_U \geq 0, \tag{32}$$

then

$$t^e_1 = 2m_L - 3M_L - n_L + 3N_L, \tag{33}$$
$$t^e_2 = -m_L + 3M_L + 2n_L - 3N_L, \tag{34}$$
$$t^e_3 = -m_U + 3M_U + 2n_U - 3N_U, \tag{35}$$
$$t^e_4 = 2m_U - 3M_U - n_U + 3N_U. \tag{36}$$

(*ii*) *If*

$$m_L - m_U - 3M_L + 3M_U - 2n_L + 2n_U + 3N_L - 3N_U < 0, \tag{37}$$
$$2m_L + m_U - 3M_L - 3M_U - n_L - 2n_U + 3N_L + 3N_U \leq 0, \tag{38}$$
$$-m_L - 2m_U + 3M_L + 3M_U + 2n_L + n_U - 3N_L - 3N_U \leq 0 \tag{39}$$

then

$$t^e_1 = \frac{3}{2}m_L + \frac{1}{2}m_U - \frac{3}{2}M_L - \frac{3}{2}M_U - n_U + \frac{3}{2}N_L + \frac{3}{2}N_U, \tag{40}$$
$$t^e_2 = t^e_3 = -\frac{1}{2}m_L - \frac{1}{2}m_U + \frac{3}{2}M_L + \frac{3}{2}M_U + n_L + n_U - \frac{3}{2}N_L - \frac{3}{2}N_U, \tag{41}$$
$$t^e_4 = \frac{1}{2}m_L + \frac{3}{2}m_U - \frac{3}{2}M_L - \frac{3}{2}M_U - n_L + \frac{3}{2}N_L + \frac{3}{2}N_U; \tag{42}$$

(*iii*) *If*

$$2m_L + m_U - 3M_L - 3M_U - n_L - 2n_U + 3N_L + 3N_U > 0 \tag{43}$$

then

$$t^e_1 = t^e_2 = t^e_3 = \frac{1}{2}m_L + \frac{1}{2}n_L, \tag{44}$$
$$t^e_4 = -\frac{1}{2}m_L + m_U - \frac{1}{2}n_L + n_U; \tag{45}$$

(*iv*) *If*

$$-m_L - 2m_U + 3M_L + 3M_U + 2n_L + n_U - 3N_L - 3N_U > 0 \tag{46}$$

then

$$t_1^e = m_L - \frac{1}{2}m_U + n_L - \frac{1}{2}n_U, \tag{47}$$

$$t_2^e = t_3^e = t_4^e = \frac{1}{2}m_U + \frac{1}{2}n_U. \tag{48}$$

We present an equivalent form in terms of weighted expected value, value, ambiguity and width.

Theorem 4.2 *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,*

$$\mu_A = \left(\left[\mu_{A_L}(\alpha), \mu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]},$$
$$\nu_A = \left(\left[\nu_{A_L}(\alpha), \nu_{A_U}(\alpha)\right]\right)_{\alpha\in[0,1]}$$

and

$$T_{\widetilde{d}}^e(A) = \left(t_1^e(A), t_2^e(A), t_3^e(A), t_4^e(A)\right) = \left(t_1^e, t_2^e, t_3^e, t_4^e\right)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to intuitionistic fuzzy number A, preserving the expected interval.

(i) If

$$\widetilde{Amb}(A) \geq \frac{1}{3}\widetilde{w}(A) \tag{49}$$

then $T_{\widetilde{d}}^e(A)$ is given by (33)-(36);

(ii) If

$$\widetilde{Amb}(A) < \frac{1}{3}\widetilde{w}(A), \tag{50}$$

$$\widetilde{Val}(A) \leq \widetilde{EV}_{\frac{2}{3}}(A) \tag{51}$$

and

$$\widetilde{Val}(A) \geq \widetilde{EV}_{\frac{1}{3}}(A) \tag{52}$$

then $T_{\widetilde{d}}^e(A)$ is given by (40)-(42);

(iii) If

$$\widetilde{Val}(A) < \widetilde{EV}_{\frac{1}{3}}(A) \tag{53}$$

then $T_{\widetilde{d}}^e(A)$ is given by (44)-(45);

(iv) If

$$\widetilde{Val}(A) > \widetilde{EV}_{\frac{2}{3}}(A) \tag{54}$$

then $T_{\widetilde{d}}^e(A)$ is given by (47)-(48).

Proof. Taking into account the equalities (1)-(4) we easy obtain the equivalences between (32) and (49), (39) and (51), (38) and (52). ■

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