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On the probability on IF-sets and MV-algebras

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Abstract. Starting with a family F of IF-events we present a simple construction of an MV-algebra M such that F can be embedded to M. Moreover, there is a one-to-one correspondence between probabilities on F and M.

1 IF-sets

We start with a measurable space $(\Omega, \mathcal{S})$ and the tribe $\mathcal{T}$ of all $\mathcal{S}$-measurable functions $f : \Omega \to [0, 1]$ Put $\mathcal{F} = \{(f, g); f, g \in \mathcal{T}, f + g \leq 1\}$.

We shall consider the set $\mathcal{F}$ together with the operations $\oplus, \odot$ defined as follows. They are based on the Lukasiewicz connectives:

$$(f, g) \oplus (h, k) = (f \oplus h, g \odot k),$$

$$(f, g) \odot (h, k) = (f \odot h, g \oplus k).$$

(Recall that $a \oplus b = min(a + b, 1)$, and $a \odot b = max(a + b - 1, 0)$.)

Definition 1.1 *Define* $\mathcal{M} = \{(f, g); f, g \in \mathcal{T}\}$ *together with the Lukasiewicz operations*

$$(f, g) \oplus (h, k) = (f \oplus h, g \odot k),$$

$$(f, g) \odot (h, k) = (f \odot h, g \oplus k).$$

the negation

$$\neg(f, g) = (1 - f, 1 - g)$$

the zero element $\mathbf{0}$ = (0,1), and the unit element $\mathbf{1}$ = (1,0).

Theorem 1.2 *The system* $(\mathcal{M}, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1})$ *is an* MV*-algebra.*

Proof. We use the Mundici theorem ([2,11]): any MV-algebra is an interval $[0, u]$ in an available l-group G. Consider the set $\mathcal{G} = \{(f, g); f, g : \Omega \to R, f, g$ are measurable $\}$. The ordering $\leq$ is induced by the IF-ordering, hence $(f, g) \leq (h, k) \iff f \leq h, g \geq k$. Evidently $(\mathcal{G}, \leq)$ is a lattice, $(f, g) \vee (h, k) = (f \vee h, g \wedge k), (f, g) \wedge (h, k) = (f \wedge h, g \vee k)$. Now we shall define the group operation $+$ by the following formula:

$(f, g) + (h, k) = (f + h, g + k - 1)$.

It is not difficult to see that $+$ is commutative and associative, and $(0, 1)$ is the neutral element. The inverse element to (f, g) is the couple $(-f, 2 - g)$, since

$(f, g) + (-f, 2 - g) = (f - f, g + 2 - g - 1) = (0, 1)$,

therefore

$(f, g) - (h, k) = (f, g) + (-h, 2 - k) = (f - h, g - k + 1)$.

If we put $u = (1, 0)$, then $\mathcal{M} = \{(f, g) \in \mathcal{G}; (0, 1) \leq (f, g) \leq (1, 0)\} = \{(f, g) \in \mathcal{G}; 0 \leq f \leq 1, 0 \leq g \leq 1\}$ with the MV-algebra operations, i.e.

$(f, g) \oplus (h, k) = ((f, g) + (h, k)) \wedge (1, 0) = (f + h, g + k - 1) \wedge (1, 0) = ((f + h) \wedge 1, (g + g - 1) \vee 0) = (f \oplus h, g \odot k)$,

and similarly

$(f, g) \odot (h, k) = (f \odot h, g \oplus k)$.

2 Probability

Inspired by [13] and starting with a probability space $(\Omega, \mathcal{S}, P)$ and the family $\mathcal{T}$ of all $\mathcal{S}$-measurable functions to [0,1]., P. Grzegorzewski and E. Mrowka defined in [4] the probability $\mathcal{P}((f, g))$ of any $(f, g) \in \mathcal{F}$ as the interval

$$\mathcal{P}((f, g)) = [\int_\Omega f dP, 1 - \int_\Omega g dP].$$

By the construction they obtained a mapping $\mathcal{P}$ from the family of all IF-events (= Atanassov events) to the family of all compact subintervals of the unit interval [0,1].

In [6] a descriptive characterization of $\mathcal{P}$ has been obtained. It is based on the Lukasiewicz connectives: Generalizing the approach the following definition has been introduced ([7]).

Definition 2.1 *An IF-probability on* $\mathcal{F}$ *is a mapping* $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ *satisfying the following conditions:*

(i) $\mathcal{P}((0, 1)) = [0, 0], \mathcal{P}((1, 0)) = [1, 1]$;

(ii) $\mathcal{P}((f, g)) + \mathcal{P}((h, k)) = \mathcal{P}((f, g) \oplus (h, k)) + \mathcal{P}((f, g) \odot (h, k))$ *for all* $(f, g), (h, k) \in \mathcal{F}$;

(iii) $(f_n, g_n) \nearrow (f, g) \Longrightarrow \mathcal{P}((f_n, g_n)) \nearrow \mathcal{P}((f, g))$.

(Recall that $(f_n, g_n) \nearrow (f, g)$ *means that* $(f_n, g_n) \nearrow f, g_n \searrow g$. *On the other hand* $[a_n, b_n] \nearrow [a, b]$ *means* $a_n \nearrow a, b_n \nearrow b$.*)*

In this Section we find a one - to - one correspondence between the set of probabilities defined below and probabilities defined on the family $\mathcal{M}$.

Similarly as the probability a mapping $p : \mathcal{F} \to [0, 1]$ can be defined owing some analogous properties.

Definition 2.2 *An IF-state is a mapping $p : \mathcal{F} \to [0,1]$ satisfying the following conditions:*

(i) $p((0,1)) = 0, p((1,0)) = 1;$

(ii) $p((f,g)) + p((h,k)) = p((f \oplus h, g \odot k)) + p((f \odot h, g \oplus k));$

(iii) $(f_n, g_n) \nearrow (f,g) \Longrightarrow p((f_n, g_n)) \nearrow p((f,g)).$

Theorem 2.3 *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$. For any $(f,g) \in \mathcal{F}$ denote*

$$\mathcal{P}((f,g)) = [\mathcal{P}^\flat((f,g)), \mathcal{P}^\sharp((f,g))]$$

Then $\mathcal{P}$ is an IF-probability if and only of $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ are IF states.

Proof. It follows directly from the definitions.

The preceding proposition shows that a characterization of states leads to a characterization of probabilities, too. Therefore we shall now characterize states on $\mathcal{F}$ by states on

$$\mathcal{M} = \{(f,g); f,g \in \mathcal{T}\}.$$

Probabilities and states on $\mathcal{M}$ are defined as mappings $\mathcal{P} : \mathcal{M} \to \mathcal{J}$, or $p : \mathcal{M} \to [0,1]$ respectively by the properties analogous to the properties stated in Definitions 1.1 and 2.1

Theorem 2.4 *To any state $p : \mathcal{F} \to [0,1]$ there exists exactly one state $\bar{p} : \mathcal{M} \to [0,1]$ such that $\bar{p}|\mathcal{F} = p$.*

Proof. For any $(f,g) \in \mathcal{M}$ define

$$\bar{p}(f,g) = p(f,0) - p(0,1-g).$$

Let $(f,g) \in \mathcal{F}$. Since $(0,1-g) \oplus (f,g) = (f,0), (0,1-g) \odot (f,g) = (0,1)$, we have $p((0,1-g)) + p((f,g)) = p((f,0))$, hence $\bar{p}|\mathcal{F} = p$.

If $(f,g), (h,k) \in \mathcal{M}$, then

$$\bar{p}((f,g)) + \bar{p}((h,k)) =$$
$$= p((f,0)) + p((h,0)) - p((0,1-g)) - p((0,1-k)) =$$
$$= p((f \oplus h, 0)) + p((f \odot h, 0)) -$$
$$-p((0, 1-g \odot k)) - p((0, 1-g \oplus k)) =$$
$$= \bar{p}((f \oplus h, g \odot k)) + \bar{p}((f \odot h, g \oplus k)) =$$
$$= \bar{p}((f,g) \oplus (h,k)) + \bar{p}((f,g) \odot (h,k)),$$

hence $\bar{p}$ is additive. We shall prove that $\bar{p}$ is continuous. Let $(f_n, g_n) \nearrow (f,g)$, hence $f_n \nearrow f, g_n \searrow g$, and $1 - g_n \nearrow 1 - g$. Therefore

$$\lim_{n\to\infty} \bar{p}((f_n, g_n)) =$$
$$= \lim_{n\to\infty} (p((f_n, 0)) - p((0, 1-g_n))) =$$
$$= \lim_{n\to\infty} p((f_n, 0)) - \lim_{n\to\infty} p((0, 1-g_n)) =$$
$$= p((f,0)) - p((0,1-g)) = \bar{p}((f,g)).$$

Theorem 2.5 *There is one - to one correspondence between states on $\mathcal{F}$ and states on $\mathcal{M}$.*

Proof. Let $\Pi(\mathcal{F})$ be the set of all states on $\mathcal{F}, \Pi(\mathcal{M})$ the set of all states on $\mathcal{M}$. Define $\psi : \Pi(\mathcal{F}) \to \Pi(\mathcal{M}))$ by the formula $\psi(p) = \bar{p}$. Then ψ is injective, since $\bar{p}|\mathcal{F} = p$. Of course, it is also surjective, because for any $\mu \in \Pi(\mathcal{M})$ it is $\psi(\mu|\mathcal{F}) = \mu$, and $\mu|\mathcal{F} \in \Pi(\mathcal{F})$.

Theorem 2.6 *To any probability $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ there exists exactly one probability $\bar{\mathcal{P}} : \mathcal{M} \to \mathcal{J}$ such that $\bar{\mathcal{P}}|\mathcal{F} = \mathcal{P}$.*

Theorem 2.7 *There is one - to - one correspondence between probabilities on $\mathcal{F}$ and probabilities on $\mathcal{M}$.*

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