Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy bimodal topological structures

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Abstract: Two new intuitionistic fuzzy operators from a modal type are defined. Some properties of these operators are studied. Based on them, four new intuitionistic fuzzy modal topological structures are introduced and after this, four intuitionistic fuzzy bimodal topological structures are constructed. All they are examples of modal and bimodal topological structures.
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1 Introduction

The concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) was introduced in [2]. During the last year, it was developed and modified in a series of author's papers. The basic definitions of the IFMTS follow the Polish mathematician Kazimierz Kuratowski's definitions of a topological structures with closure ($\mathcal{C}, cl$-) and interior ($\mathcal{I}, in$-) operators, respectively, given in [6].

Following [3], we mention that the IFMTS is of a closure or of an interior type in respect of the form of its topological operator (of closure ($\mathcal{C}$) or interior ($\mathcal{I}$) type), respectively. On the other hand, the same notation is used about the form of the modal operator, because at least conditionally we can accept that the modal operator (see, e.g., [5, 7]) "possibility" ($\Diamond$) is related to operation "union" ($\cup$, as well as the topological operator closure), while modal operator "necessity" ($\Box$) is related to operation "intersection" ($\cap$, as well as the topological interior).

Let E by a fixed universe,

$$E^* = \{\langle x, 1, 0\rangle | x \in E\},$$

$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$\mathcal{O}, \mathcal{Q} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ be operators of a closure and of an interior types related to operations $\Delta, \nabla : \mathcal{P}(E^*) \times \mathcal{P}(E^*) \to \mathcal{P}(E^*)$, respectively, $\circ, \bullet : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ be modal operators. Let for every two IFSs $A, B \in \mathcal{P}(E^*)$ and for the (standard) intuitionistic fuzzy negation of the IFS

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

(see [1]) that has the form:

$$\neg A = \{\langle x, \nu, \mu\rangle | x \in E\}.$$

Let the following equalities hold:

$$\begin{array}{llll} A\Delta B & = \neg(\neg A \nabla \neg B), & A\nabla B & = \neg(\neg A \Delta \neg B), \\ \mathcal{O}(A) & = \neg(\mathcal{Q}(\neg A)), & \mathcal{Q}(A) & = \neg(\mathcal{O}(\neg A)), \\ \circ A & = \neg \bullet \neg A, & \bullet A & = \neg \circ \neg A. \end{array}$$

Now, we can define the following four structures: cl-cl-IFMTS, in-in-IFMTS, cl-in-IFMTS, in-cl-IFMTS, that for every two IFSs $A, B \in \mathcal{P}(E^*)$ the following nine conditions hold, respectively:

	cl-cl-IFMTS		in-in-IFMTS
CC1	$\mathcal{O}(A\Delta B) = \mathcal{O}(A)\Delta\mathcal{O}(B)$	II1	$\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$
CC2	$A \subseteq \mathcal{O}(A)$	II2	$\mathcal{Q}(A) \subseteq A$
CC3	$\mathcal{O}(O^*) = O^*$	II3	$\mathcal{Q}(E^*) = E^*$
CC4	$\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$	II4	$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$
CC5	$\circ(A\nabla B) = \circ A \nabla \circ B$	II5	$\bullet(A\nabla B) = \bullet A \nabla \bullet B$
CC6	$\circ A \subseteq A$	II6	$\bullet A \subseteq A$
CC7	$\circ E^* = E^*$	II7	$\bullet O^* = O^*$
CC8	$\circ \circ A = \circ A$	II8	$\bullet \bullet A = \bullet A$
CC9	$\circ\, \mathcal{O}(A) = \mathcal{O}(\circ A)$	II9	$\bullet\, \mathcal{Q}(A) = \mathcal{Q}(\bullet A)$

	cl-in-IFMTS		*in-cl*-IFMTS
CI1	$\mathcal{O}(A\Delta B) = \mathcal{O}(A)\Delta\mathcal{O}(B)$	IC1	$\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$
CI2	$A \subseteq \mathcal{O}(A)$	IC2	$\mathcal{Q}(A) \subseteq A$
CI3	$\mathcal{O}(O^*) = O^*$	IC3	$\mathcal{Q}(E^*) = E^*$
CI4	$\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$	IC4	$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$
CI5	$\bullet(A\nabla B) = \bullet A\nabla \bullet B$	IC5	$\circ(A\Delta B) = \circ A\Delta \circ B$
CI6	$\bullet A \subseteq A$	IC6	$A \subseteq \circ A$
CI7	$\bullet O^* = O^*$	IC7	$\circ E^* = E^*$
CI8	$\bullet \bullet A = \bullet A$	IC8	$\circ \circ A = \circ A$
CI9	$\bullet\, \mathcal{O}(A) = \mathcal{O}(\bullet A)$	IC9	$\circ\, \mathcal{Q}(A) = \mathcal{Q}(\circ A)$

In Section 2, we introduce two new operators from a modal type that are analogues of the standard modal operators, mentioned above. Some of their basic properties are discussed. In Section 3, we show that the new modal operators participate in the definition of four IFMTSs that satisfy the above described axioms. In section 4, on the basis of the definitions of the IFMTSs and theorems, proved in [2] and from Section 3, two new types of IFMTSs are introduced. In each one of them, two different modal operators are used and by this reason these structures are called IF BiModal TSs (IF2MTSs). Some perspectives of the development of the IFMTSs are discussed in the Conclusion.

All definitions related to IFSs are given in [1]. Here, at the respective places, we give only these definitions that are necessary for the present research.

2 Two new operators of a modal type

On Figure 1, one of the geometrical interpretations of the IFSs (see, [1]) is shown.

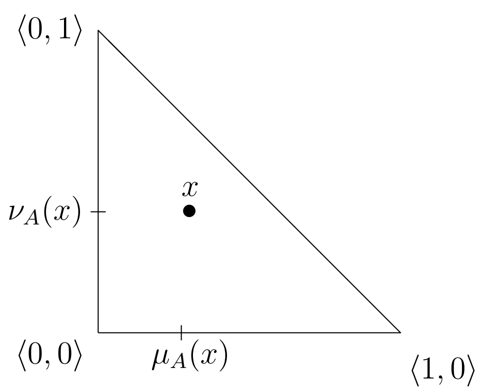

Figure 1. A geometrical interpretation of an element $x \in E$

The first two (simplest) modal operators (see, e.g., [1]) are:

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

The geometrical interpretation of an element $x \in E$ in a result of operators $\Box$ and $\Diamond$ is shown on Fig. 2 and these two results are marked by $\Box x$ and $\Diamond x$.

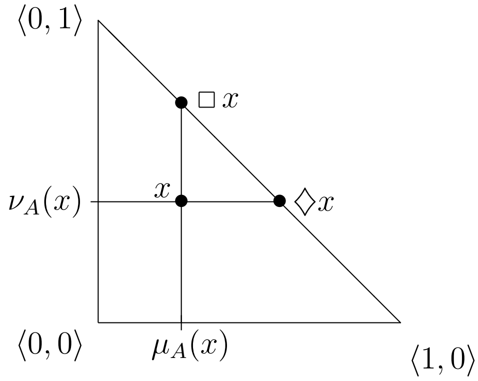

Figure 2. The geometrical interpretation of an elements $\Box x \in E$ and $\Diamond x \in E$

Let us see the two other points on Fig. 2, especially marked on Fig. 3, and let us denote them by:

$$\diamondplus A = \{\langle x, \mu_A(x), 0\rangle | x \in E\};$$
$$\boxplus A = \{\langle x, 0, \nu_A(x)\rangle | x \in E\}.$$

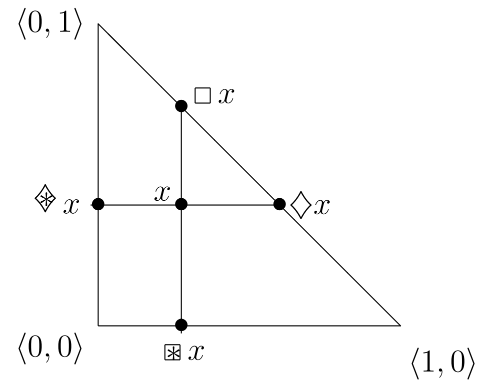

Figure 3. The geometrical interpretation of an elements $\boxplus x \in E$ and $\diamondplus x \in E$

Obviously, for each IFS A, $\boxplus A$ and $\diamondplus A$ are IFSs.

Let us, following [1], define

$$A \subseteq B \quad \text{iff} \quad (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x));$$
$$A \supseteq B \quad \text{iff} \quad B \subseteq A;$$
$$A = B \quad \text{iff} \quad (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x));$$
$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\};$$
$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.$$

For the new operators we can check directly the validity of the following assertions.

Theorem 1. For each IFS A:

(a) $\boxast A \subseteq A \subseteq \diamondast A$,

(b) $\neg \boxast \neg A = \diamondast A$,

(c) $\neg \diamondast \neg A = \boxast A$,

(d) $\boxast \boxast A = \boxast A$,

(e) $\diamondast \diamondast A = \diamondast A$,

(f) $\boxast \diamondast A = O^* = \diamondast \boxast A$.

Theorem 2. For each IFS A:

$$\boxast A \subseteq A \subseteq \Diamond A,$$
$$\Box A \subseteq A \subseteq \diamondast A.$$

Theorem 3. For each IFS A:

$$\Box \boxast A \subseteq \boxast \Box A,$$
$$\Box \diamondast A \subseteq \diamondast \Box A,$$
$$\Diamond \boxast A \subseteq \boxast \Diamond A,$$
$$\diamondast \Diamond A \subseteq \Diamond \diamondast A.$$

Theorem 4. For each IFS A:

$$O^* = \Box \boxast A \subseteq \boxast \Box A \subseteq \left\{ \begin{array}{c} \boxast \Diamond A \\ \Box \diamondast A \end{array} \right\} \subseteq U^* \subseteq \left\{ \begin{array}{c} \diamondast \Box A \\ \Diamond \boxast A \end{array} \right\} \subseteq \diamondast \Diamond A \subseteq \Diamond \diamondast A = E^*.$$

3 Four new intuitionistic fuzzy modal topological structures

Following, e.g., [3], we give some definitions.

If for the set X

$$\mathcal{P}(X) = \{Y | Y \subseteq X\},$$

then for each IFS A over the universe E:

$$\mathcal{P}(O^*) = \{O^*\},$$
$$\mathcal{P}(E^*) = \{A | A \subseteq E^*\}.$$

Let $\mathcal{O}$ and $\mathcal{Q}$ be topological operators such that for each IFS $A \in \mathcal{P}(E^*)$:

$$\mathcal{O}(A) = \neg \mathcal{Q}(\neg A),$$
$$\mathcal{Q}(A) = \neg \mathcal{O}(\neg A).$$

Let $\Delta, \nabla : \mathcal{P}(E^*) \times \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ be operations such that for every two IFSs $A, B \in \mathcal{P}(E^*)$:

$$A \nabla B = \neg(\neg A \Delta \neg B),$$
$$A \Delta B = \neg(\neg A \nabla \neg B).$$

Let $\circ$ and $\bullet$ be modal operators such that for each IFS $A \in \mathcal{P}(E^*)$:

$$\circ A = \neg \bullet \neg A,$$
$$\bullet A = \neg \circ \neg A.$$

In [2], the concept of an IFMTS was introduced and it was extended in [4] to cl-cl-IFMTS, in-in-IFMTS, cl-in-IFMTS and in-cl-IFMTS in respect of the type of the topological operator (from "closure" or from "interior" type) and of the modal operator, that at least conditionally can be from one of these two types.

Now, we repeat the research from [2,4], but about the new modal operators, using the respective conditions (axioms) from the Introduction.

3.1 *cl-cl*-IFMTS

Theorem 5. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \circledast \rangle$ is a cl-cl-IFMTS.
Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$. The check of conditions CC1– CC4 coincide with those in [2] and we omit them.

CC5.

$$\begin{aligned} \circledast (A \cap B) &= \circledast (\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\ &= \circledast \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\ &= \{\langle x, \min(\mu_A(x), \mu_B(x)), 0\rangle | x \in E\} \\ &= \{\langle x, \mu_A(x), 0\rangle | x \in E\} \cap \{\langle x, \mu_B(x), 0\rangle | x \in E\} \\ &= \circledast A \cap \circledast B; \end{aligned}$$

CC6.

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\ &\subseteq \{\langle x, \mu_A(x), 0\rangle | x \in E\} \\ &= \circledast A; \end{aligned}$$

CC7.

$$\begin{aligned} \circledast E^* &= \circledast \{\langle x, 1, 0\rangle | x \in E\} \\ &= \{\langle x, 1, 0\rangle | x \in E\} \\ &= E^*; \end{aligned}$$

CC8.

$$\begin{aligned} \circledast \circledast A &= \circledast \{\langle x, \mu_A(x), 0\rangle | x \in E\} \\ &= \{\langle x, \mu_A(x), 0\rangle | x \in E\} \\ &= \circledast A; \end{aligned}$$

CC9.

$$\begin{aligned}
\diamondplus\, \mathcal{C}(A) &= \diamondplus\, \{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y\in E} \mu_A(y), 0\rangle | x \in E\}) \\
&= \mathcal{C}(\{\langle x, \mu_A(x), 0\rangle | x \in E\}) \\
&= \mathcal{C}(\diamondplus\, A).
\end{aligned}$$

This completes the proof. □

3.2 in-in-IFMTS

Theorem 6. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxtimes \rangle$ is an in-in-IFMTS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$. The check of conditions II1–II4 coincide with those in [2] and we omit them.

II5.

$$\begin{aligned}
\boxtimes (A \cup B) &= \boxtimes (\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cup \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \boxtimes \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 0, \min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 0, \nu_A(x)\rangle | x \in E\} \cup \{\langle x, 0, \nu_B(x)\rangle | x \in E\} \\
&= \boxtimes A \cup \boxtimes B;
\end{aligned}$$

II6.

$$\begin{aligned}
\boxtimes A &= \{\langle x, 0, \nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= A;
\end{aligned}$$

II7.

$$\begin{aligned}
\boxtimes O^* &= \boxtimes \{\langle x, 0, 1\rangle | x \in E\} \\
&= \{\langle x, 0, 1\rangle | x \in E\} \\
&= O^*;
\end{aligned}$$

II8.

$$\begin{aligned}
\boxtimes \boxtimes A &= \boxtimes \{\langle x, 0, \nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, 0, \nu_A(x)\rangle | x \in E\} \\
&= \boxtimes A;
\end{aligned}$$

II9.

$$\begin{aligned}
\boxtimes \mathcal{I}(A) &= \boxtimes \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, 0, \sup_{y\in E} \nu_A(y)\rangle | x \in E\}) \\
&= \mathcal{I}(\{\langle x, 0, \nu_A(x)\rangle | x \in E\}) \\
&= \mathcal{I}(\boxtimes A).
\end{aligned}$$

This completes the proof. □

3.3 cl-in-IFMTS

Theorem 7. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxast \rangle$ is a cl-in-IFMTS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$. The check of conditions CC1–CC4 coincide with those in [2] and we omit them. The checks of the conditions CI6– CI8 coincide with the proofs of conditions II6–II8 in Theorem 6. Hence, it is enough only to show the validity of the conditions CI5 and CI9.

For the validity of condition CI5 we obtain:

$$\begin{aligned}
\boxast(A \cap B) &= \boxast(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \boxast\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 0, \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 0, \nu_A(x)\rangle | x \in E\} \cap \{\langle x, 0, \nu_B(x)\rangle | x \in E\} \\
&= \boxast A \cap \boxast B;
\end{aligned}$$

For the validity of condition CI9, we obtain

$$\begin{aligned}
\boxast\mathcal{C}(A)) &= \boxast\{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, 0, \inf_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \mathcal{C}(\{\langle x, 0, \nu_A(x)\rangle | x \in E\}) \\
&= \mathcal{C}(\boxast A).
\end{aligned}$$

This completes the proof. □

3.4 in-cl-IFMTS

Theorem 8. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamondast \rangle$ is an in-cl-IFMTS.

Proof. Let the IFS $A \in \mathcal{P}(E^*)$ be given. The checks of the conditions IC1–IC4 coincide with the proofs of conditions II1–II4 in Theorem 6. The checks of the conditions IC6– IC8 coincide with the proofs of conditions CC6–CC8 from Theorem 5. Hence, it remains that we check only the validity of the conditions IC5 and IC9.

For the validity of condition IC5 we obtain:

$$\begin{aligned}
\diamondast(A \cup B) &= \diamondast\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \max(\mu_A(x), \mu_B(x)), 0\rangle | x \in E\} \\
&= \{\langle x, \mu_A(x), 0\rangle | x \in E\} \cup \{\langle x, \mu_B(x), 0\rangle | x \in E\} \\
&= \diamondast A \cup \diamondast B;
\end{aligned}$$

For the validity of condition IC9, we obtain

$$\begin{aligned}
\diamondplus\, \mathcal{I}(A) &= \diamondplus\, \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \inf_{y \in E} \mu_A(y), 0\rangle | x \in E\}) \\
&= \mathcal{I}(\{\langle x, \mu_A(x), 0\rangle | x \in E\}) \\
&= \mathcal{I}(\diamondplus A).
\end{aligned}$$

This completes the proof. □

4 Four intuitionistic fuzzy bimodal topological structures

Let $\blacksquare$ and $\blacklozenge$ be modal operators such that for each IFS $A \in \mathcal{P}(E^*)$:

$$\blacksquare A = \neg \blacklozenge \neg A,$$

$$\blacklozenge A = \neg \blacksquare \neg A.$$

Now, on the basis of the results from [2] and Section 3, here we, for a first time, introduce the idea for "Intuitionistic fuzzy bimodal topological structures" (IF2MTS), giving the definitions of the following four structures: cl-(cl, in)-, cl-(in, cl)-, in-(cl, in)-, in-(in, cl)-structures.

	cl-(cl, in)-IFMTS		in-(in, cl)-IFMTS
CCI1	$\mathcal{O}(A \Delta B) = \mathcal{O}(A) \Delta \mathcal{O}(B)$	IIC1	$\mathcal{Q}(A \nabla B) = \mathcal{Q}(A) \nabla \mathcal{Q}(B)$
CCI2	$A \subseteq \mathcal{O}(A)$	IIC2	$\mathcal{Q}(A) \subseteq A$
CCI3	$\mathcal{O}(O^*) = O^*$	IIC3	$\mathcal{Q}(E^*) = E^*$
CCI4	$\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$	IIC4	$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$
CCI5	$\circ(A \nabla B) = \circ A \nabla \circ B$	IIC5	$\bullet(A \nabla B) = \bullet A \nabla \bullet B$
CCI6	$\circ A \subseteq A$	IIC6	$\bullet A \subseteq A$
CCI7	$\circ E^* = E^*$	IIC7	$\bullet O^* = O^*$
CCI8	$\circ \circ A = \circ A$	IIC8	$\bullet \bullet A = \bullet A$
CCI9	$\circ\, \mathcal{O}(A) = \mathcal{O}(\circ A)$	IIC9	$\bullet\, \mathcal{Q}(A) = \mathcal{Q}(\bullet A)$
CCI10	$\blacksquare(A \nabla B) = \blacksquare A \nabla \blacksquare B$	IIC10	$\blacklozenge(A \nabla B) = \blacklozenge A \nabla \blacklozenge B$
CCI11	$\blacksquare A \subseteq A$	IIC11	$\blacklozenge A \subseteq A$
CCI12	$\blacksquare E^* = E^*$	IIC12	$\blacklozenge O^* = O^*$
CCI13	$\blacksquare \blacksquare A = \blacksquare A$	IIC13	$\blacklozenge \blacklozenge A = \blacklozenge A$
CCI14	$\blacksquare\, \mathcal{O}(A) = \mathcal{O}(\blacksquare A)$	IIC14	$\blacklozenge\, \mathcal{Q}(A) = \mathcal{Q}(\blacklozenge A)$
CCI15	$\blacksquare A \subseteq \circ A$	IIC15	$\blacklozenge A \subseteq \bullet A$

	cl-(in, cl)-IFMTS		in-(cl, in)-IFMTS
CIC1	$\mathcal{O}(A\Delta B) = \mathcal{O}(A)\Delta\mathcal{O}(B)$	ICI1	$\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$
CIC2	$A \subseteq \mathcal{O}(A)$	ICI2	$\mathcal{Q}(A) \subseteq A$
CIC3	$\mathcal{O}(O^*) = O^*$	ICI3	$\mathcal{Q}(E^*) = E^*$
CIC4	$\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$	ICI4	$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$
CIC5	$\bullet(A\nabla B) = \bullet A\nabla \bullet B$	ICI5	$\circ(A\Delta B) = \circ A\Delta \circ B$
CIC6	$\bullet A \subseteq A$	ICI6	$A \subseteq \circ A$
CIC7	$\bullet O^* = O^*$	ICI7	$\circ E^* = E^*$
CIC8	$\bullet \bullet A = \bullet A$	ICI8	$\circ \circ A = \circ A$
CIC9	$\bullet\, \mathcal{O}(A) = \mathcal{O}(\bullet A)$	ICI9	$\circ\, \mathcal{Q}(A) = \mathcal{Q}(\circ A)$
CIC10	$\blacklozenge(A\nabla B) = \blacklozenge A\nabla \blacklozenge B$	ICI10	$\blacksquare(A\Delta B) = \blacksquare A\Delta\blacksquare B$
CIC11	$\blacklozenge A \subseteq A$	ICI11	$A \subseteq \blacksquare A$
CIC12	$\blacklozenge O^* = O^*$	ICI12	$\blacksquare E^* = E^*$
CIC13	$\blacklozenge \blacklozenge A = \blacklozenge A$	ICI13	$\blacksquare\blacksquare A = \blacksquare A$
CIC14	$\blacklozenge\ \mathcal{O}(A) = \mathcal{O}(\blacklozenge A)$	ICI14	$\blacksquare\ \mathcal{Q}(A) = \mathcal{Q}(\blacksquare A)$
CIC15	$\bullet\, A \subseteq \blacklozenge A$	ICI15	$\blacksquare A \subseteq \circ A$

Now, we can formulate the following four assertions and their proofs are based on the proofs of Theorems 2, 5–8.

Theorem 9. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Diamond, \boxast \rangle$ is an cl-(cl, in)-IFMTS.

Theorem 10. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Box, \Diamond\!\!\!\ast \rangle$ is an cl-(in, cl)-IFMTS.

Theorem 11. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Diamond, \boxast \rangle$ is an in-(cl, in)-IFMTS.

Theorem 12. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Box, \Diamond\!\!\!\ast \rangle$ is an in-(in, cl)-IFMTS.

5 Conclusion

Initially, we must mention two important things.

First, we mention that the introduced in [2] IFMTSs are the first examples of modal topological structures. The axioms (conditons) CC1–CC9, II1–II9, CI1–CI9, IC1–IC9 are universal for the topological structures in which modal operators are added.

Second, here, for a first time we introduce idea not only for IF2MTS, but in general, for bimodal topological structures. They have the axioms (conditons) CC1–CC15, II1–II15, CI1–CI15, IC1–IC15.

Finally, in future we will search for topological structures having more than two modal operators, i.e., for multimodal topological structures.

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