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On intuitionistic fuzzy homotopy theory

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Abstract: The homotopy theory is used some areas in mathematics and it has some applications in different areas. The fuzzy homotopy theory was introduced by authors [4] in 2006. After this paper, some topological other structures were studied by several authors [2, 3, 5, 6].

In this paper, firstly, we defined the intuitionistic fuzzy homotopic functions using topological properties. Then, we got some properties of intuitionistic fuzzy homotopic functions and concept of intuitionistic fuzzy homotopy theory.
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1 Introduction

The theory of fuzzy sets (FSs) was first stated by Zadeh in 1965, [7]. Let X be a set, then the function $\mu_A : X \to [0, 1]$ is called a fuzzy set over X and it is shown $\mu_A \in FS(X)$. From now on, we will use A instead of μ_A. For $x \in X$, $\mu_A(x)$ is called the membership degree of x on A. And for $A \in FS(X)$, the complement of A is defined using the equation $coA(x) = 1 - A(x)$.

It is obvious from the definition above that the sum of the membership degree and nonmembership degree is equal to 1. But in real life one can think that two certain objects are in relation R with eachother having a determinate degree. Besides this, this person may not be sure about it. This means that there is a possibility of existence of an uncertainity about the degree of the relationship between these two objects. In the theory of fuzzy sets, it does not have a meaning to incorporate this uncertainity in the degrees of membership.

Atanassov defined intuitionistic fuzzy sets in 1983, [1], in order to give a possible solution for this problem. While the nonmembership degree for each element of the universe is fixed in fuzzy set theory, in intuitionistic fuzzy set theory, nonmembership degree is a more or less independent degree; satisfying the condition that it is smaller than $1 - membership\ degree$. So, if X a universe, then there exist two membersip and nonmembership degrees for each $x \in X$, respectively $\mu_A(x)$ and $\nu_A(x)$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

IFS A is determined with the membership and non-membership of $\mu_A(x) \in FS(X)$, $\nu_A(x) \in FS(X)$ for $x \in X$ respectively. For each $x \in X$, $\pi_A(X) = 1 - \mu_A(x) - \nu_A(x)$ is called hesitation degree of intuitionistic index of x at A.

Although the sum of the degrees of membership and not being a member of an element in FS theory is 1, in IFS theory this sum is less than 1. Besides this, if $A \in IFS(X)$, then μ_A, $\nu_A \in FS(X)$ and $1 - \mu_A \leq \nu_A$ and $1 - \nu_A \leq \mu_A$. The length of the interval $[\mu_A(x), 1 - \nu_A(x)]$ which is given by $\pi_A(X)$, can be considered as hesitation modelling degree between two membership degrees.

An IFS A is said to be contained in an IFS B (notation $A \subseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

The intersection (resp.the union) of two IFSs A and B on X is defined as the IFSs

$$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \,|x \in X\},$$

respectively,

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \,|x \in X\}.$$

Definition 1. *Let X, Y be two sets. If $R \in IFS(X \times Y)$, then R is called intuitionistic fuzzy relation (IFR) from X to Y. If R is a relation from X to Y, then it is clear that there is an IFS of R over $X \times Y$. For $x \in X, y \in Y$, their membership degrees in R at $\mu_R(x, y)$ becomes their non-membership degrees in R at $\nu_R(x, y)$.*

Definition 2. *Let X, Y, Z be sets and $R \in IFR(X, Y)$, $S \in IFR(Y, Z)$.*

$$R \circ S = \{(x, z), \mu_{R \circ S}(x, z), \nu_{R \circ S}(x, z) | x \in X, z \in Z\}.$$

Here, the IFS which is defined by

$$\mu_{R \circ S}(x, z) = \sup_{y \in Y}\{\mu_R(x, y) \wedge \mu_S(y, z)\}$$

$$\nu_{R \circ S}(x, z) = \inf_{y \in Y}\{\nu_R(x, y) \vee \nu_S(y, z)\}$$

is called the composition of R and S.

For any intuitionistic fuzzy set $A \in IFS(X)$ and any $t, s \in [0, 1]$, the $(t, s) - cut$ of A is defined as the following:

$$A_{t,s} = \{x \in X | \mu_A(x) \geq t \wedge \upsilon_A(x) \leq s\}.$$

For $(t, s) - cut$ we have the following properties:

1. $(A \cup B)_{t,s} = \ A_{t,s} \cup B_{t,s}$

2. $(A \cap B)_{t,s} = \ A_{t,s} \cap B_{t,s}$

$(t, s)A$ is a intuitionistic fuzzy set which is defined by, $x \in X, t, s \in [0, 1]$,

$$((t, s)A)(x) = (t, s) \wedge A(x)$$

The Resolution Theorems are given as

$$A = \cup\{(t, s)A_{t,s}|t, s \in [0, 1]\}$$

We know that, for any $x \in X, t, s \in [0, 1]$, x_λ is a intuitionistic fuzzy point and if $A \in F(X)$ is a fuzzy set then

$$A = \cup\{x_{t,s}|x_{t,s} \in A\}$$

thus, we can write,

$$A = \cup\{x_{A(x)}|x \in X\}$$

If $A \in F(X)$ and $B \in F(Y)$, $x_\lambda \in A, y_\eta \in B$, and by denoting

$$\langle x_\lambda, y_\eta\rangle \ = (x, y)_{\lambda\wedge\eta}$$

then

$$A \times B = \cup\{\langle x_{t,s}, y_{m.n}\rangle \, |x_{t.s} \in A, \ y_{m,n} \in B\} = \ \cup\{(x_{A(x)}, y_{B(y)})|x \in X, \ y \in Y\}$$

Definition 3. *Let $A \in IFS(X)$, $B \in IFS(Y)$, $C \in IFS(Z)$, $R \subseteq A \times B$ $Q \subseteq B \times C$ are intuitionistic fuzzy relations from A to B and from B to C resp., and put*

$$Q \circ R = \cup\{(t, s)(Q_{t,s} \circ R_{t,s}|t, s \in [0, 1]\}.$$

This is called composition of Q and R.

Definition 4. *For $A \in IFS(X)$, an intuitionistic fuzzy relation R on A is called an intuitionistic fuzzy equivalence relation on A, if for any $t, s \in [0, 1]$, $R_{t,s}$ is an equivalence relation on $A_{t,s}$.*

Lemma 1. *R is an intuitionistic fuzzy equivalence relation on A if and only if A satisfies the following relation;*

1. $(\forall x \in X), (R(x, x) = 1)$ *(reflexivity)*

2. $(\forall x, y \in X), (R(x, y) = R(y, x))$ *(symmetry)*

3. $R \circ R \subseteq R$ *(transitivity).*

Let R be an equivalence relation on A and $\langle x_{t,s}, y_{m,n}\rangle \in R$, $x, y \in X$, then we say "$x_{t,s}$ equivalent to $y_{m,n}$" or "$x_{t,s}$ and $y_{m,n}$ are equivalent."

For any $x_{t,s} \in A$, using $(x_{t,s})_R = \ \cup\{y_{m,n}| \, \langle x_{t,s}, y_{m,n}\rangle \in R\}$. This is called R-equivalence class of $x_{t,s}$, and simply denoted by $(x_{t,s})$. Now, we put

$$A/B = \{(x_{t,s})|x_{t,s} \in A\}.$$

We denoted, $R^{-1} = \cup\{\lambda R_\lambda^{-1} : t, s \in [0, 1]\}$, it is inverse relation of R.

If, $R = \cup\{\langle x_{t,s}, y_{m,n}\rangle \, |x, y \in X\}$ then $R^{-1} = \cup\{\langle y_{m,n}, x_{t,s}\rangle \, |x, y \in X\}$ and $(R^{-1})^{-1} = R$.

Definition 5. *Let X, Y be two non-empty sets. The mapping $f : X \to IFS(Y)$ is called an intuitionistic fuzzy mapping from X to Y, namely,*

$$x \mapsto f(x) = \{\langle y, f_x^1(y), f_x^2(y)\rangle \,|y \in Y\}$$

where, for a $x \in X, f_x^1, f_x^2 : Y \to [0,1]$ and satisfying 0$\leq f_x^1(y) + f_x^2(y) \leq 1$ for all y $\in Y$.

Given a intuitionistic fuzzy mapping f, for $x \in X, y \in Y$, the value can be denoted by

$$f(x)(y) = \langle f_x^1(y), f_x^2(y)\rangle$$

From the definition of intuitionistic fuzzy mapping, it is easily seen that f_x^1 and f_x^2 are fuzzy sets. So, we can use the following definition.

Definition 6. *For$A \in F(X)$ and $B \in F(Y)$, a fuzzy relation $f \subseteq A \times B$ is called a fuzzy mapping from A to B if f_λ is a mapping from A_λ to B_λ for α any $\lambda \in [0,1]$. When f is a fuzzy mapping from A to B, we denote it by $f : A \to B$. $f : A \to B$ is called a fuzzy injection if f_λ is an injection from A_λ to B_λ for any $\lambda \in [0,1]$, it is called a fuzzy surjection if $(\forall\lambda \in [0,1])(B_{<\lambda>} \subseteq f_\lambda(A_\lambda) \subseteq B_\lambda)$ and it is called a fuzzy bijection if f is injection and surjection.*

Definition 7. *Let X, Y be two non-empty sets. Let $f : X \to IFS(Y)$ is an intuitionistic fuzzy mapping from X to Y. Then f is intuitionititistic fuzzy injection (surjection) then $f_x^1(y)$ and $f_x^2(y)$ are fuzzy injections (surjections) and it is called a fuzzy bijection if f is injection and surjection.*

Definition 8. *An intuitionistic fuzzy topology on X is a family τ of IFSs in X satisfying the following axioms*

1. 0, 1 $\in \tau$
2. $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
3. $\cup G_i \in \tau$ for any $\{G_i | i \in I\}$

In this case the pair (X, τ) is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in τ is known as an intuitionistic fuzzy open set in X.

Definition 9. *Let $A \in FS(X)$, $B \in FS(Y)$, (A, τ_1) and (B, τ_2) two topological spaces and fuzzy topological spaces and $f : A \to B$ a fuzzy function. f is fuzzy continuous if and only if for every $t \in I$, $f_t : A_t \to B_t$ is continuous.*

Definition 10. *Let $f : X \to IFS(Y)$ is called an intuitionistic fuzzy mapping from X to Y. f is intuitionistic fuzzy continuous if and only if f_x^1, f_x^2 are fuzzy continuous.*

Definition 11. *Let (A, τ_1) and (B, τ_2) be fuzzy topological spaces and $f, g : A \to B$ are fuzzy continuous functions. f is fuzzy homotope to g if there exist a fuzzy continuous function $F : A \times I \to B$ such that for every $t \in I$,*

$F_{t,s}(x,0) = f_{t,s}(x),$

$F_{t,s}(x,1) = g_{t,s}(x),$

$F_{t,s}(x,(m,n)) = f_{t,s}^{m,n}(x).$

If f and g are fuzzy homotopic functions we write $f \sim g$.

Proposition 1. *It is clear that every continuous function homotopic to itself.*

Definition 12. *Let (A, τ_1) and (B, τ_2) be fuzzy topological spaces. Let $X_0 \subset X$ and there exists functions $f, g : X \to Y$ such that the condition for every $x_0 \in X_0$, $f(x_0) = g(x_0)$ is satisfied. f is intuitionistic fuzzy homotopic to g relative to x_0 and written by $f \sim grel.X_0$, if there exists a function $F : A \times I \to B$such that the follwing conditions hold:*

1. $F_{t,s}(x,0) = f(x), F_{t,s}(x,1) = g(x)$, for every $x \in X$
2. $F_{t,s}(x_0,(m,n)) = f(x_0) = g(x_0)$, for every $x_0 \in X_0$

Remark 1. *In the above definition, if we choose $X_0 = \varnothing$ then f is homotopic to g.*

Theorem 1. *Intuitionistic fuzzy homotopy relation is intuitionistic fuzzy equivaliance relation.*

Proof. Now that for every continuous function f, $f \sim f$. Assume that $f \sim g$, then there exists a function F such that the conditions of the Definition are satisfied. If we rewrite the function F as $F'(x,(t,s)) = F(x, 1-t)$, then we get $g \sim f$. The distributive condition is clear. □

Definition 13. *Let X and Y be topological spaces and A and B are intuitionistic fuzzy topological spaces on X and Y respectively. Let $f, g \subset A \times B$ intuitionistic fuzzy continuous and $f \sim g$. If $|img| = 1$ then it is called that f is homotopic to arbitrary.*

Definition 14. *X is contractibility or X may deformated to a point if the identity definition on X is homotopic to arbitrary.*

Theorem 2. *Let X be a intuitionistic fuzzy topological space and Y can be deformated to a point then all of the intuitionistic fuzzy continuous function $f : A \to B$ is homotopic to arbitrary.*

Proof. B may deformated intuitionistic fuzzy topological space then for every $t, s \in I$, $B_{t,s}$ may deformated topological subspace. Therefore there exists a function $g : B \to B$ such that for $y_0 \in B_{t,s}$ arbitrary, $g_{t,s}(y) = y_0$ such that $1_{Bt,s} : B_{t,s} \to B_{t,s}$ is homotopic to g$_{t,s}$ i.e. $1_{Bt,s} \sim g_{t,s}$. Therefore, there exists a continuous function$f_{t,s} : B_{t,s} \times I^2 \to B_{t,s}$ such that for every $y \in Y$, $F_{t,s}(y,0) = 1_{Bt,s}(y,0) = 1_{Bt,s}(y), F_{t,s}(y,1) = g_{t,s}(y)$. We assume that, $f : A_{t,s} \to B_{t,s}$ intuitionistic fuzzy continuous function. We define that intuitionistic fuzzy function $G_{t,s} : A_{t,s} \times I^2 \to B_{t,s} as G_{t,s}(x,(m,n)) = F_{t,s}(f_{t,s}(x),(m,n))$. It is clear that $G_{t,s}$ is continuous function for every$t, s \in I$ and $G_{t,s}(x,0) = F_{t,s}(f_{t,s}(x),0) = f_{t,s}(x)$, $G_{t,s}(x,1) = F_{t,s}(f_{t,s}(x),1) = y_0$. Thus f is homotopic to arbitrary for every $t, s \in I$. Therefore f is intuitionistic fuzzy homotopic to arbitrary. □

Theorem 3. *Let $A \in IFS(X), B \in IFS(Y), C \in IFS(Z)$ be intuitionistic fuzzy topological spaces and $f \subset A \times B, g \subset B \times C$ be intuitionistic fuzzy continuous functions. If $g \sim h$ then $g \circ f$ and $h \circ f$ are intuitionistic fuzzy continuous and $g \circ f \sim h \circ f$.*

Proof. If $g \sim h$ than there exists a continuous intuitionistic fuzzy function F such that $F_{t,s}(y,0) = g_{t,s}$ and $F_{t,s}(y,1) = f_{t,s}$ for every $t, s \in I$. Let us define a function G with respect to F such that $G_{t,s}(x,(m,n)) = F_{t,s}(f(x),(m,n))$ for every$t, s \in I$. It is clear that G is intuitionistic fuzzy continuous function. However $G_{t,s}(x,0) = F_{t,s}(f_{t,s}(x),0) = g_{t,s}f_{t,s}$ and $G_{t,s}(x,1) = F_{t,s}(f_{t,s}(x),1) = h_{t,s}f_{t,s}$ for every $t, s \in I$, too. Therefore, $g_{t,s}f_{t,s} \sim h_{t,s}f_{t,s}$, for every $t, s \in I$ thus $gf \sim hf$. □

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