

Operators on circular interval-valued intuitionistic fuzzy sets

P. Gajalaxmi¹ , D. Jayanthi²  and M. Priyadharshini³ 

¹ Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, Tamil Nadu, India
e-mail: pgajalaxmi1004@gmail.com

² Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, Tamil Nadu, India
e-mail: jeyanthi_mat@avinuty.ac.in

³ Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, Tamil Nadu, India
e-mail: priyadharshini1501@gmail.com

Received: 22 October 2024

Revised: 26 May 2025

Accepted: 21 November 2025

Online First: 15 March 2026

Abstract: A circular interval-valued intuitionistic fuzzy set extends the concept of a circular intuitionistic fuzzy set by incorporating membership and non-membership degrees, which are represented as closed intervals with a given radius r , both confined within the range $[0, 1]$. The geometrical representation of a circular interval-valued intuitionistic fuzzy set is given. A few fundamental operators and their properties have been discussed. To develop a circular interval-valued intuitionistic fuzzy set from an interval-valued intuitionistic fuzzy set, two methods have been presented, and an instance is provided to illustrate.

Keywords: Interval-valued intuitionistic fuzzy sets, Circular interval-valued intuitionistic fuzzy sets, Circular intuitionistic fuzzy set, Operators.

2020 Mathematics Subject Classification: 03E72, 28E10.



1 Introduction

To cope with skepticism, in 1965, Zadeh [29] introduced fuzzy set (FS), which is a function to the closed interval between zero and one, which are their membership degrees. Even though fuzzy is helpful in various real-life applications, it has a drawback that experts found it difficult to quantify their opinion into an interval; hence, as its extension, interval-valued fuzzy sets (IVFSs) [19, 25] were introduced whose membership degrees are themselves a closed interval. To tackle more vagueness and uncertainty, Atanassov [6] introduced intuitionistic fuzzy set (IFS) in 1986, which has both membership and non-membership degrees but the sum of both should be less than or equal to one. However, due to the world's rapid advancement, the IFSs lack the ability to accurately characterize uncertain information in situations where the membership and non-membership degrees cannot be expressed as exact real numbers. Consequently, Atanassov and Gargov [9] in 1989 introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is capable of handling these kinds of circumstances. The membership and non-membership degrees in IVIFS are characterized as a closed interval, and the supremum of those two intervals added together must be less than or equal to one. As the next extension of IFS, in 2020, Atanassov [7] introduced circular IFS (C-IFS), in which the circular area around the IFS was taken into account. As further extension, Atanassov [8] introduced elliptic - IFS (E-IFS) in which each element will be represented as an ellipse around an IVIFS. The evolution of FS is provided in Figure 1.

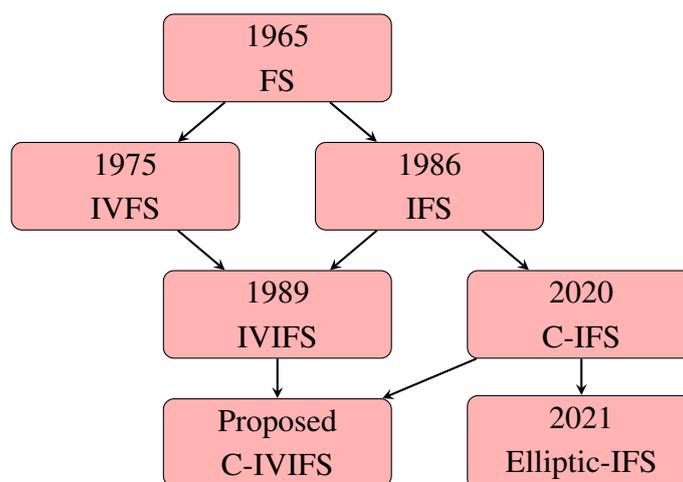


Figure 1. Evolution of FS

Recently, many researchers [1, 3, 5, 10, 11, 13, 14, 16, 17, 20, 23, 26–28] have been rigorously working on the applications of C-IFSs. Otay and Kahraman [24] applied C-IFS to a multi-expert supplier problem using the VIKOR methodology. Garg *et al.* [18] used the Evaluation based on Distance from Average Solution (EDAS) method to apply C-IFS to multi-criteria decision-making (MCDM) situations. Decision-making (DM) problems in C-IFS were discussed by many researchers. Many researchers [2, 12, 15, 21] applied C-IFS to DM using the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method. The TOMada de Decisão Iterativa Multicritério (TODIM) method in C-IFS was applied by Ashraf *et al.* [4]. The VIKOR method in C-IFS was applied by several researchers, including

Kahraman [22] and Otay [24].

The motivation of this research is that, after the introduction of C-IFS by Atanassov [7], many researchers working rapidly on the applications and extension of this set, but they have only membership and non-membership degrees. Hence, we extend C-IFS into Circular IVIFS (C-IVIFS) as their degrees of membership and non-membership are closed intervals, and each element in C-IVIFS is represented as a circle around IVIFS, so C-IVIFS is more capable of handling the uncertainties.

The article is structured as follows: Section 2 is the preliminaries, in which basic definitions are given. In Section 3, the definition of C-IVIFS is defined, and the geometrical representation of C-IVIFS is provided along with the representation of C-IFS. Further, the basic operators of C-IVIFS such as complement, proper subset, subset, equality, intersection, union, algebraic sum, and product are defined, and the properties of those defined operators are analyzed and proved. In Section 4, the methodologies for the conversion of IVIFS into C-IVIFS are provided, and to illustrate the conversion, an example is also given. Section 5 is the conclusion part, in which the overview of the study is discussed.

2 Preliminaries

Some definitions from literature that are useful for this article are given in this section.

Definition 2.1. [9] Consider X to be a non-empty set. The set

$$A = \{ \langle x, [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)] \rangle | x \in X \}$$

is an IVIFS over X where $[\mu_A^-(x), \mu_A^+(x)] = \mu_A(x)$ and $[\nu_A^-(x), \nu_A^+(x)] = \nu_A(x)$ are respectively the membership and non-membership closed intervals over X such that

$$0 \leq \sup(\mu_A(x)) + \sup(\nu_A(x)) \leq 1 \Rightarrow 0 \leq \mu_A^+(x) + \nu_A^+(x) \leq 1.$$

Definition 2.2. [7] Consider X to be a non-empty set. A circular intuitionistic fuzzy set is

$$A = \{ \langle x, \mu_A(x), \nu_A(x), r \rangle | x \in X \},$$

where the functions $\mu_A : X \rightarrow [0, 1]$, $\nu_A : X \rightarrow [0, 1]$ are respectively the degree of membership and non-membership, $0 \leq \mu_A + \nu_A \leq 1$ and $r \in [0, 1]$ is the radius of the circle around each element $x \in X$.

Definition 2.3. [7, 12] Conversion of IFS into C-IFS can be done in two ways:

- Arithmetic mean: For a set of IFSs, $\{ \langle \mu_{i,1}, \nu_{i,1} \rangle, \langle \mu_{i,2}, \nu_{i,2} \rangle, \dots \}$, C-IFS is

$$\langle \mu(x_i), \nu(x_i), r \rangle = \left\langle \frac{\sum_{j=1}^n \mu_{i,j}}{n}, \frac{\sum_{j=1}^n \nu_{i,j}}{n} \right\rangle$$

where

$$r = \max_{1 \leq j \leq n} \left| \sqrt{(\mu(x_i) - \mu_{i,j})^2 + (\nu(x_i) - \nu_{i,j})^2} \right|$$

- Geometric mean: For a set of IFSs, $\{\langle \mu_{i,1}, \nu_{i,1} \rangle, \langle \mu_{i,2}, \nu_{i,2} \rangle, \dots \}$, C-IFS is

$$\langle \mu(x_i), \nu(x_i), r \rangle = \left\langle \sqrt[n]{\prod_{j=1}^n \mu_{i,j}}, \sqrt[n]{\prod_{j=1}^n \nu_{i,j}} \right\rangle$$

where

$$r = \max_{1 \leq j \leq n} \left| \sqrt{(\mu(x_i) - \mu_{i,j})^2 + (\nu(x_i) - \nu_{i,j})^2} \right|$$

3 Circular interval-valued intuitionistic fuzzy sets

In this section, C-IVIFS is defined as a special case of E-IFS and the geometrical representation of C-IVIFS is provided. Some basic operators are defined and the properties associated with these operators are proved.

Definition 3.1. Consider X to be a non-empty set. The set

$$A = \{ \langle x, [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle \mid x \in X \}$$

is called circular interval-valued intuitionistic fuzzy set where $[\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)]$ are respectively the membership and non-membership intervals with $0 \leq \mu_A^+(x) + \nu_A^+(x) \leq 1$ and $r \in [0, 1]$ is the radius of the circle around each element $x \in X$:

$$\text{Center: } (\mu_A(x), \nu_A(x)) = \left(\frac{\mu_A^-(x) + \mu_A^+(x)}{2}, \frac{\nu_A^-(x) + \nu_A^+(x)}{2} \right)$$

$$\begin{aligned} \text{Radius: } r &= \sqrt{(\mu_A^-(x) - \mu_A(x))^2 + (\nu_A^-(x) - \nu_A(x))^2} \\ &= \sqrt{(\mu_A^+(x) - \mu_A(x))^2 + (\nu_A^+(x) - \nu_A(x))^2} \end{aligned}$$

In short, C-IVIFS $A = \{ \langle x, [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle \mid x \in X \}$ can be written as $A = \langle [\mu_A^-, \mu_A^+], [\nu_A^-, \nu_A^+], r \rangle$.

The geometrical representation of C-IVIFS along with the representation of C-IFS is provided in Figure 2.

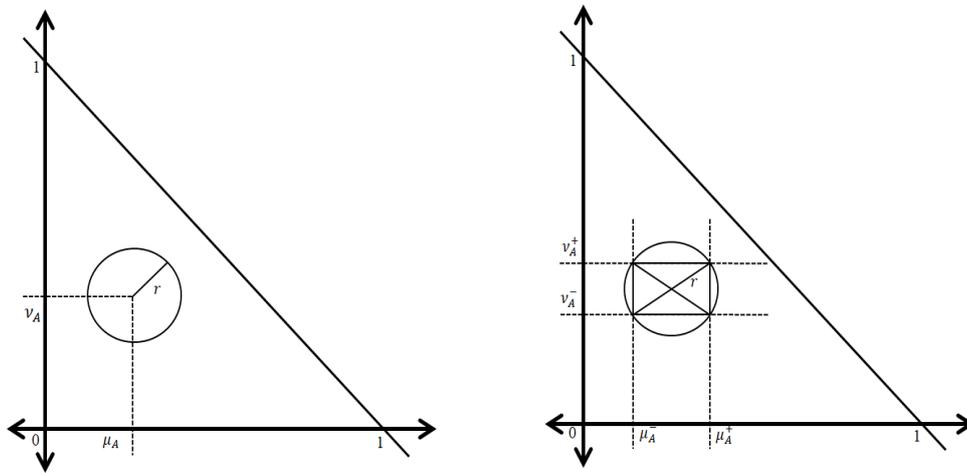


Figure 2. Geometrical representation of C-IFS and C-IVIFS

Definition 3.2. Let $A = \langle [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle$ be a C-IVIFS in X , then the complement of A is represented as A^c or A' where, $A^c = \langle [\nu_A^-(x), \nu_A^+(x)], [\mu_A^-(x), \mu_A^+(x)], r \rangle$.

Definition 3.3. Let $A = \langle [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle$ and $B = \langle [\mu_B^-(x), \mu_B^+(x)], [\nu_B^-(x), \nu_B^+(x)], s \rangle$ be two C-IVIFSs in X , then the following operations hold:

- $A \subset B \Leftrightarrow r < s \ \& \ (\mu_A^-(x) < \mu_B^-(x), \mu_A^+(x) < \mu_B^+(x), \nu_A^-(x) \geq \nu_B^-(x), \nu_A^+(x) \geq \nu_B^+(x))$
(or)
 $\Leftrightarrow r < s \ \& \ (\mu_A^-(x) \leq \mu_B^-(x), \mu_A^+(x) \leq \mu_B^+(x), \nu_A^-(x) > \nu_B^-(x), \nu_A^+(x) > \nu_B^+(x))$
(or)
 $\Leftrightarrow r < s \ \& \ (\mu_A^-(x) < \mu_B^-(x), \mu_A^+(x) < \mu_B^+(x), \nu_A^-(x) > \nu_B^-(x), \nu_A^+(x) > \nu_B^+(x))$
- $A \supset B \Leftrightarrow B \subset A$
- $A \subseteq B \Leftrightarrow r \leq s \ \& \ (\mu_A^-(x) \leq \mu_B^-(x), \mu_A^+(x) \leq \mu_B^+(x), \nu_A^-(x) \geq \nu_B^-(x), \nu_A^+(x) \geq \nu_B^+(x))$
- $A \supset B \Leftrightarrow B \supset A$
- $A = B \Leftrightarrow A \supseteq B \ \& \ B \subseteq A$
- $A \cap_{\min} B = \langle [\min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))], [\max(\nu_A^-(x), \nu_B^-(x)), \max(\nu_A^+(x), \nu_B^+(x))], \min(r, s) \rangle$
- $A \cap_{\max} B = \langle [\min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))], [\max(\nu_A^-(x), \nu_B^-(x)), \max(\nu_A^+(x), \nu_B^+(x))], \max(r, s) \rangle$
- $A \cup_{\min} B = \langle [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))], [\min(\nu_A^-(x), \nu_B^-(x)), \min(\nu_A^+(x), \nu_B^+(x))], \min(r, s) \rangle$
- $A \cup_{\max} B = \langle [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))], [\min(\nu_A^-(x), \nu_B^-(x)), \min(\nu_A^+(x), \nu_B^+(x))], \max(r, s) \rangle$
- $A \oplus_{\min} B = \langle [\mu_A^-(x) + \mu_B^-(x) - \mu_A^-(x)\mu_B^-(x), \mu_A^+(x) + \mu_B^+(x) - \mu_A^+(x)\mu_B^+(x)], [\nu_A^-(x)\nu_B^-(x), \nu_A^+(x)\nu_B^+(x)], \max(r, s) \rangle$
- $A \oplus_{\max} B = \langle [\mu_A^-(x) + \mu_B^-(x) - \mu_A^-(x)\mu_B^-(x), \mu_A^+(x) + \mu_B^+(x) - \mu_A^+(x)\mu_B^+(x)], [\nu_A^-(x)\nu_B^-(x), \nu_A^+(x)\nu_B^+(x)], \max(r, s) \rangle$
- $A \otimes_{\min} B = \langle [\mu_A^-(x)\mu_B^-(x), \mu_A^+(x)\mu_B^+(x)], [\nu_A^-(x) + \nu_B^-(x) - \nu_A^-(x)\nu_B^-(x), \nu_A^+(x) + \nu_B^+(x) - \nu_A^+(x)\nu_B^+(x)], \min(r, s) \rangle$
- $A \otimes_{\max} B = \langle [\mu_A^-(x)\mu_B^-(x), \mu_A^+(x)\mu_B^+(x)], [\nu_A^-(x) + \nu_B^-(x) - \nu_A^-(x)\nu_B^-(x), \nu_A^+(x) + \nu_B^+(x) - \nu_A^+(x)\nu_B^+(x)], \max(r, s) \rangle$.

Theorem 3.1. For the three C-IVIFSs $A = \langle [\mu_A^-, \mu_A^+], [\nu_A^-, \nu_A^+], r \rangle$, $B = \langle [\mu_B^-, \mu_B^+], [\nu_B^-, \nu_B^+], s \rangle$ and $C = \langle [\mu_C^-, \mu_C^+], [\nu_C^-, \nu_C^+], t \rangle$ in X , the following holds:

1. $A \cap_{\min} B = B \cap_{\min} A$
2. $A \cap_{\max} B = B \cap_{\max} A$
3. $A \cup_{\min} B = B \cup_{\min} A$
4. $A \cup_{\max} B = B \cup_{\max} A$
5. $A \oplus_{\min} B = B \oplus_{\min} A$
6. $A \oplus_{\max} B = B \oplus_{\max} A$

7. $A \otimes_{\min} B = B \otimes_{\min} A$
8. $A \otimes_{\max} B = B \otimes_{\max} A$
9. $(A \cap_{\min} B) \cap_{\min} C = A \cap_{\min} (B \cap_{\min} C)$
10. $(A \cap_{\max} B) \cap_{\max} C = A \cap_{\max} (B \cap_{\max} C)$
11. $(A \cup_{\min} B) \cup_{\min} C = A \cup_{\min} (B \cup_{\min} C)$
12. $(A \cup_{\max} B) \cup_{\max} C = A \cup_{\max} (B \cup_{\max} C)$
13. $(A \oplus_{\min} B) \oplus_{\min} C = A \oplus_{\min} (B \oplus_{\min} C)$
14. $(A \oplus_{\max} B) \oplus_{\max} C = A \oplus_{\max} (B \oplus_{\max} C)$
15. $(A \otimes_{\min} B) \otimes_{\min} C = A \otimes_{\min} (B \otimes_{\min} C)$
16. $(A \otimes_{\max} B) \otimes_{\max} C = A \otimes_{\max} (B \otimes_{\max} C)$
17. $(A \cap_{\min} B) \cup_{\min} C = (A \cup_{\min} C) \cap_{\min} (B \cup_{\min} C)$
18. $(A \cap_{\min} B) \cup_{\max} C = (A \cup_{\max} C) \cap_{\min} (B \cup_{\max} C)$
19. $(A \cap_{\max} B) \cup_{\min} C = (A \cup_{\min} C) \cap_{\max} (B \cup_{\min} C)$
20. $(A \cap_{\max} B) \cup_{\max} C = (A \cup_{\max} C) \cap_{\max} (B \cup_{\max} C)$
21. $(A \cap_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cap_{\min} (B \oplus_{\min} C)$
22. $(A \cap_{\min} B) \oplus_{\max} C = (A \oplus_{\max} C) \cap_{\min} (B \oplus_{\max} C)$
23. $(A \cap_{\max} B) \oplus_{\min} C = (A \oplus_{\min} C) \cap_{\max} (B \oplus_{\min} C)$
24. $(A \cap_{\max} B) \oplus_{\max} C = (A \oplus_{\max} C) \cap_{\max} (B \oplus_{\max} C)$
25. $(A \cap_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cap_{\min} (B \otimes_{\min} C)$
26. $(A \cap_{\min} B) \otimes_{\max} C = (A \otimes_{\max} C) \cap_{\min} (B \otimes_{\max} C)$
27. $(A \cap_{\max} B) \otimes_{\min} C = (A \otimes_{\min} C) \cap_{\max} (B \otimes_{\min} C)$
28. $(A \cap_{\max} B) \otimes_{\max} C = (A \otimes_{\max} C) \cap_{\max} (B \otimes_{\max} C)$
29. $(A \cup_{\min} B) \cap_{\min} C = (A \cap_{\min} C) \cup_{\min} (B \cap_{\min} C)$
30. $(A \cup_{\min} B) \cap_{\max} C = (A \cap_{\max} C) \cup_{\min} (B \cap_{\max} C)$
31. $(A \cup_{\max} B) \cap_{\min} C = (A \cap_{\min} C) \cup_{\max} (B \cap_{\min} C)$
32. $(A \cup_{\max} B) \cap_{\max} C = (A \cap_{\max} C) \cup_{\max} (B \cap_{\max} C)$
33. $(A \cup_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cup_{\min} (B \oplus_{\min} C)$
34. $(A \cup_{\min} B) \oplus_{\max} C = (A \oplus_{\max} C) \cup_{\min} (B \oplus_{\max} C)$
35. $(A \cup_{\max} B) \oplus_{\min} C = (A \oplus_{\min} C) \cup_{\max} (B \oplus_{\min} C)$
36. $(A \cup_{\max} B) \oplus_{\max} C = (A \oplus_{\max} C) \cup_{\max} (B \oplus_{\max} C)$
37. $(A \cup_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cup_{\min} (B \otimes_{\min} C)$
38. $(A \cup_{\min} B) \otimes_{\max} C = (A \otimes_{\max} C) \cup_{\min} (B \otimes_{\max} C)$

39. $(A \cup_{\max} B) \otimes_{\min} C = (A \otimes_{\min} C) \cup_{\max} (B \otimes_{\min} C)$
40. $(A \cup_{\max} B) \otimes_{\max} C = (A \otimes_{\max} C) \cup_{\max} (B \otimes_{\max} C)$
41. $A \cap_{\min} A = A$
42. $A \cap_{\max} A = A$
43. $A \cup_{\min} A = A$
44. $A \cup_{\max} A = A$
45. $(A' \cap_{\min} B')' = A \cup_{\min} B$
46. $(A' \cap_{\max} B')' = A \cup_{\max} B$
47. $(A' \cup_{\min} B')' = A \cap_{\min} B$
48. $(A' \cup_{\max} B')' = A \cap_{\max} B$
49. $(A' \oplus_{\min} B')' = A \oplus_{\min} B$
50. $(A' \oplus_{\max} B')' = A \oplus_{\max} B$
51. $(A' \otimes_{\min} B')' = A \oplus_{\min} B$
52. $(A' \otimes_{\max} B')' = A \oplus_{\max} B$
53. $(A')' = A$
54. $(A \cup_{\min} B)' = A' \cap_{\min} B'$
55. $(A \cup_{\max} B)' = A' \cap_{\max} B'$
56. $(A \cap_{\min} B)' = A' \cup_{\min} B'$
57. $(A \cap_{\max} B)' = A' \cup_{\max} B'$

Proof. Here we prove some of the statements, and the rest are proved by analogy.

1. LHS

$$A \cap_{\min} B = \langle [\min(\mu_A^-, \mu_B^-), \min(\mu_A^+, \mu_B^+)], [\max(\nu_A^-, \nu_B^-), \max(\nu_A^+, \nu_B^+)], \min(r, s) \rangle;$$

RHS

$$B \cap_{\min} A = \langle [\min(\mu_B^-, \mu_A^-), \min(\mu_B^+, \mu_A^+)], [\max(\nu_B^-, \nu_A^-), \max(\nu_B^+, \nu_A^+)], \min(s, r) \rangle.$$

Hence, $A \cap_{\min} B = B \cap_{\min} A$.

Similarly (2)–(4) can also be proved.

5. LHS

$$A \oplus_{\min} B = \langle [\mu_A^- + \mu_B^- - \mu_A^- \mu_B^-, \mu_A^+ + \mu_B^+ - \mu_A^+ \mu_B^+], [\nu_A^- \nu_B^-, \nu_A^+ \nu_B^+], \min(r, s) \rangle;$$

RHS

$$B \oplus_{\min} A = \langle [\mu_B^- + \mu_A^- - \mu_B^- \mu_A^-, \mu_B^+ + \mu_A^+ - \mu_B^+ \mu_A^+], [\nu_B^- \nu_A^-, \nu_B^+ \nu_A^+], \min(s, r) \rangle.$$

Hence, $A \oplus_{\min} B = B \oplus_{\min} A$.

Similarly (6)–(8) can also be proved.

9. LHS

$$\begin{aligned}
(A \cap_{\min} B) \cap_{\min} C &= \langle [\min(\mu_A^-, \mu_B^-), \min(\mu_A^+, \mu_B^+)], [\max(\nu_A^-, \nu_B^-), \max(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\cap_{\min} C \\
&= \langle [\min(\min(\mu_A^-, \mu_B^-), \mu_C^-), \min(\min(\mu_A^+, \mu_B^+), \mu_C^+)], \\
&\quad [\max(\max(\nu_A^-, \nu_B^-), \nu_C^-), \max(\max(\nu_A^+, \nu_B^+), \nu_C^+)], \min(\min(r, s), t) \rangle \\
&= \langle [\min(\mu_A^-, \min(\mu_B^-, \mu_C^-)), \min(\mu_A^+, \min(\mu_B^+, \mu_C^+))], \\
&\quad [\max(\nu_A^-, \max(\nu_B^-, \nu_C^-)), \max(\nu_A^+, \max(\nu_B^+, \nu_C^+))], \min(r, \min(s, t)) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
A \cap_{\min} (B \cap_{\min} C) &= A \cap_{\min} \langle [\min(\mu_B^-, \mu_C^-), \min(\mu_B^+, \mu_C^+)], [\max(\nu_B^-, \nu_C^-), \max(\nu_B^+, \nu_C^+)], \\
&\quad \min(s, t) \rangle \\
&= \langle [\min(\mu_A^-, \min(\mu_B^-, \mu_C^-)), \min(\mu_A^+, \min(\mu_B^+, \mu_C^+))], \\
&\quad [\max(\nu_A^-, \max(\nu_B^-, \nu_C^-)), \max(\nu_A^+, \max(\nu_B^+, \nu_C^+))], \min(r, \min(s, t)) \rangle.
\end{aligned}$$

Hence, $(A \cap_{\min} B) \cap_{\min} C = A \cap_{\min} (B \cap_{\min} C)$.

Similarly (10)–(12) can also be proved.

13. LHS

$$\begin{aligned}
(A \otimes_{\min} B) \oplus_{\min} C &= \langle [\mu_A^- + \mu_B^- - \mu_A^- \mu_B^-, \mu_A^+ + \mu_B^+ - \mu_A^+ \mu_B^+], [\nu_A^- \nu_B^-, \nu_A^+ \nu_B^+], \min(r, s) \rangle \\
&\oplus_{\min} C \\
&= \langle [\mu_A^- + \mu_B^- - \mu_A^- \mu_B^- + \mu_C^- - (\mu_A^- + \mu_B^- - \mu_A^- \mu_B^-) \mu_C^-, \\
&\quad \mu_A^+ + \mu_B^+ - \mu_A^+ \mu_B^+ + \mu_C^+ - (\mu_A^+ + \mu_B^+ - \mu_A^+ \mu_B^+) \mu_C^+], \\
&\quad [\nu_A^- \nu_B^- \nu_C^-, \nu_A^+ \nu_B^+ \nu_C^+], \min(\min(r, s), t) \rangle \\
&= \langle [\mu_A^- + \mu_B^- - \mu_A^- + \mu_B^- + \mu_C^- - (\mu_A^- + \mu_B^- - \mu_A^- + \mu_B^-) \mu_C^-, \\
&\quad \mu_A^+ + \mu_B^+ - \mu_A^+ + \mu_B^+ + \mu_C^+ - (\mu_A^+ + \mu_B^+ - \mu_A^+ + \mu_B^+) \mu_C^+], \\
&\quad [\nu_A^- \nu_B^- \nu_C^-, \nu_A^+ \nu_B^+ \nu_C^+], \min(r, \min(s, t)) \rangle \\
&= \langle [\mu_A^- + \mu_B^- + \mu_C^- - \mu_B^- \mu_C^- - \mu_A^- \mu_B^- - \mu_A^- \mu_C^- + \mu_A^- \mu_B^- \mu_C^-, \\
&\quad \mu_A^+ + \mu_B^+ + \mu_C^+ - \mu_B^+ \mu_C^+ - \mu_A^+ \mu_B^+ - \mu_A^+ \mu_C^+ + \mu_A^+ \mu_B^+ \mu_C^+], \\
&\quad [\nu_A^- \nu_B^- \nu_C^-, \nu_A^+ \nu_B^+ \nu_C^+], \min(r, \min(s, t)) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
A \oplus_{\min} (B \oplus_{\min} C) &= A \oplus_{\min} \langle [\mu_B^- + \mu_C^- - \mu_B^- \mu_C^-, \mu_B^+ + \mu_C^+ - \mu_B^+ \mu_C^+], [\nu_B^- \nu_C^-, \nu_B^+ \nu_C^+], \\
&\quad \min(s, t) \rangle \\
&= \langle [\mu_A^- + \mu_B^- + \mu_C^- - \mu_B^- \mu_C^- - \mu_A^- \mu_B^- - \mu_A^- \mu_C^- + \mu_A^- \mu_B^- \mu_C^-, \\
&\quad \mu_A^+ + \mu_B^+ + \mu_C^+ - \mu_B^+ \mu_C^+ - \mu_A^+ \mu_B^+ - \mu_A^+ \mu_C^+ + \mu_A^+ \mu_B^+ \mu_C^+], \\
&\quad [\nu_A^- \nu_B^- \nu_C^-, \nu_A^+ \nu_B^+ \nu_C^+], \min(r, \min(s, t)) \rangle.
\end{aligned}$$

Hence, $(A \oplus_{\min} B) \oplus_{\min} C = A \oplus_{\min} (B \oplus_{\min} C)$.

Similarly (14)–(16) can also be proved.

17. LHS

$$\begin{aligned}
(A \cap_{\min} B) \cup_{\min} C &= \langle [\min(\mu_A^-, \mu_B^-), \min(\mu_A^+, \mu_B^+)], [\max(\nu_A^-, \nu_B^-), \max(\nu_A^+, \nu_B^+)], \\
&\quad \min(r, s) \rangle \cup_{\min} C \\
&= \langle [\max(\min(\mu_A^-, \mu_B^-), \mu_C^-), \max(\min(\mu_A^+, \mu_B^+), \mu_C^+)], \\
&\quad [\min(\max(\nu_A^-, \nu_B^-), \nu_C^-), \min(\max(\nu_A^+, \nu_B^+), \nu_C^+)], \min(\min(r, s), t) \rangle \\
&= \langle [\min(\max(\mu_A^-, \mu_C^-), \max(\mu_B^-, \mu_C^-)), \min(\max(\mu_A^+, \mu_C^+), \max(\mu_B^+, \mu_C^+))], \\
&\quad [\max(\min(\nu_A^-, \nu_C^-), \min(\nu_B^-, \nu_C^-)), \max(\min(\nu_A^+, \nu_C^+), \min(\nu_B^+, \nu_C^+))], \\
&\quad \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\max(\mu_A^-, \mu_C^-), \max(\mu_A^+, \mu_C^+)], [\min(\nu_A^-, \nu_C^-), \min(\nu_A^+, \nu_C^+)], \min(r, t) \rangle \\
&\quad \cap_{\min} \langle [\max(\mu_B^-, \mu_C^-), \max(\mu_B^+, \mu_C^+)], [\min(\nu_B^-, \nu_C^-), \min(\nu_B^+, \nu_C^+)], \\
&\quad \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \cup_{\min} C) \cap_{\min} (B \cup_{\min} C) &= \langle [\max(\mu_A^-, \mu_C^-), \max(\mu_A^+, \mu_C^+)], [\min(\nu_A^-, \nu_C^-), \min(\nu_A^+, \nu_C^+)], \\
&\quad \min(r, t) \rangle \cap_{\min} \langle [\max(\mu_B^-, \mu_C^-), \max(\mu_B^+, \mu_C^+)], \\
&\quad [\min(\nu_B^-, \nu_C^-), \min(\nu_B^+, \nu_C^+)], \min(s, t) \rangle
\end{aligned}$$

Hence, $(A \cap_{\min} B) \cup_{\min} C = (A \cup_{\min} C) \cap_{\min} (B \cup_{\min} C)$.

Similarly (18)–(20) can also be proved.

21. LHS

$$\begin{aligned}
(A \cap_{\min} B) \oplus_{\min} C &= \langle [\min(\mu_A^-, \mu_B^-), \min(\mu_A^+, \mu_B^+)], [\max(\nu_A^-, \nu_B^-), \max(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\quad \oplus_{\min} C \\
&= \langle [\min(\mu_A^-, \mu_B^-) + \mu_C^- - \min(\mu_A^-, \mu_B^-)\mu_C^-, \\
&\quad \min(\mu_A^+, \mu_B^+) + \mu_C^+ - \min(\mu_A^+, \mu_B^+)\mu_C^+], \\
&\quad [\max(\nu_A^-, \nu_B^-)\nu_C^-, \max(\nu_A^+, \nu_B^+)\nu_C^+], \min(\min(r, s), t) \rangle \\
&= \langle [\min(\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_B^- + \mu_C^- - \mu_B^-\mu_C^-), \\
&\quad \min(\mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+)], \\
&\quad [\max(\nu_A^-\nu_C^-, \nu_B^-\nu_C^-), \max(\nu_A^+\nu_C^+, \nu_B^+\nu_C^+)], \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+], [\nu_A^-\nu_C^-, \nu_A^+\nu_C^+], \min(r, t) \rangle \\
&\quad \cap_{\min} \langle [\mu_B^- + \mu_C^- - \mu_B^-\mu_C^-, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+], [\nu_B^-\nu_C^-, \nu_B^+\nu_C^+], \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \oplus_{\min} C) \cap_{\min} (B \oplus_{\min} C) &= \langle [\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+], [\nu_A^-\nu_C^-, \nu_A^+\nu_C^+], \\
&\quad \min(r, t) \rangle \cap_{\min} \langle [\mu_B^- + \mu_C^- - \mu_B^-\mu_C^-, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+], \\
&\quad [\nu_B^-\nu_C^-, \nu_B^+\nu_C^+], \min(s, t) \rangle.
\end{aligned}$$

Hence, $(A \cap_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cap_{\min} (B \oplus_{\min} C)$.

Similarly (22)–(24) can also be proved.

25. LHS

$$\begin{aligned}
(A \cap_{\min} B) \otimes_{\min} C &= \langle [\min(\mu_A^-, \mu_B^-), \min(\mu_A^+, \mu_B^+)], [\max(\nu_A^-, \nu_B^-), \max(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\quad \otimes_{\min} C \\
&= \langle [\min(\mu_A^-, \mu_B^-)\mu_C^-, \min(\mu_A^+, \mu_B^+)\mu_C^+], \\
&\quad [\max(\nu_A^-, \nu_B^-) + \nu_C^- - \max(\nu_A^-, \nu_B^-)\nu_C^-, \\
&\quad \max(\nu_A^+, \nu_B^+) + \nu_C^+ - \max(\nu_A^+, \nu_B^+)\nu_C^+], \min(\min(r, s), t) \rangle \\
&= \langle [\min(\mu_A^-\mu_C^-, \mu_B^-\mu_C^-), \min(\mu_A^+\mu_C^+, \mu_B^+\mu_C^+)], \\
&\quad [\max(\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_B^- + \nu_C^- - \nu_B^-\nu_C^-), \\
&\quad \max(\nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+)], \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\mu_A^-\mu_C^-, \mu_A^+\mu_C^+], [\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+], \min(r, t) \rangle \\
&\quad \cap_{\min} \langle [\mu_B^-\mu_C^-, \mu_B^+\mu_C^+], [\nu_B^- + \nu_C^- - \nu_B^-\nu_C^-, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+], \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \otimes_{\min} C) \cap_{\min} (B \otimes_{\min} C) &= \langle [\mu_A^-\mu_C^-, \mu_A^+\mu_C^+], [\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+], \min(r, t) \rangle \\
&\quad \cap_{\min} \\
&\quad \langle [\mu_B^-\mu_C^-, \mu_B^+\mu_C^+], [\nu_B^- + \nu_C^- - \nu_B^-\nu_C^-, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+], \min(s, t) \rangle.
\end{aligned}$$

Hence, $(A \cap_{\min} B) \otimes_{\min} C = (A \otimes_{\min} C) \cap_{\min} (B \otimes_{\min} C)$.

Similarly (26)–(28) can also be proved.

29. LHS

$$\begin{aligned}
(A \cup_{\min} B) \cap_{\min} C &= \langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\quad \cap_{\min} C \\
&= \langle [\min(\max(\mu_A^-, \mu_B^-), \mu_C^-), \min(\max(\mu_A^+, \mu_B^+), \mu_C^+)], \\
&\quad [\max(\min(\nu_A^-, \nu_B^-), \nu_C^-), \max(\min(\nu_A^+, \nu_B^+), \nu_C^+)], \min(\min(r, s), t) \rangle \\
&= \langle [\max(\min(\mu_A^-, \mu_C^-), \min(\mu_B^-, \mu_C^-)), \max(\min(\mu_A^+, \mu_C^+), \min(\mu_B^+, \mu_C^+))], \\
&\quad [\min(\max(\nu_A^-, \nu_C^-), \max(\nu_B^-, \nu_C^-)), \min(\max(\nu_A^+, \nu_C^+), \max(\nu_B^+, \nu_C^+))], \\
&\quad \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\min(\mu_A^-, \mu_C^-), \min(\mu_A^+, \mu_C^+)], [\max(\nu_A^-, \nu_C^-), \max(\nu_A^+, \nu_C^+)], \min(r, t) \rangle \\
&\quad \cup_{\min} \\
&\quad \langle [\min(\mu_B^-, \mu_C^-), \min(\mu_B^+, \mu_C^+)], [\max(\nu_B^-, \nu_C^-), \max(\nu_B^+, \nu_C^+)], \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \cap_{\min} C) \cup_{\min} (B \cap_{\min} C) &= \langle [\min(\mu_A^-, \mu_C^-), \min(\mu_A^+, \mu_C^+)], [\max(\nu_A^-, \nu_C^-), \max(\nu_A^+, \nu_C^+)], \\
&\quad \min(r, t) \rangle \cup_{\min} \langle [\min(\mu_B^-, \mu_C^-), \min(\mu_B^+, \mu_C^+)], \\
&\quad [\max(\nu_B^-, \nu_C^-), \max(\nu_B^+, \nu_C^+)], \min(s, t) \rangle.
\end{aligned}$$

Hence, $(A \cup_{\min} B) \cap_{\min} C = (A \cap_{\min} C) \cup_{\min} (B \cap_{\min} C)$.

Similarly (30)–(32) can also be proved.

33. LHS

$$\begin{aligned}
(A \cup_{\min} B) \oplus_{\min} C &= \langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\quad \oplus_{\min} C \\
&= \langle [\max(\mu_A^-, \mu_B^-) + \mu_C^- - \max(\mu_A^-, \mu_B^-)\mu_C^-, \\
&\quad \max(\mu_A^+, \mu_B^+) + \mu_C^+ - \max(\mu_A^+, \mu_B^+)\mu_C^+], \\
&\quad [\min(\nu_A^-, \nu_B^-)\nu_C^-, \min(\nu_A^+, \nu_B^+)\nu_C^+], \min(\min(r, s), t) \rangle \\
&= \langle [\max(\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_B^- + \mu_C^- - \mu_B^-\mu_C^-), \\
&\quad \max(\mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+)], \\
&\quad [\min(\nu_A^-\nu_C^-, \nu_B^-\nu_C^-), \min(\nu_A^+\nu_C^+, \nu_B^+\nu_C^+)], \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+], [\nu_A^-\nu_C^-, \nu_A^+\nu_C^+], \min(r, t) \rangle \\
&\quad \cup_{\min} \langle [\mu_B^- + \mu_C^- - \mu_B^-\mu_C^-, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+], [\nu_B^-\nu_C^-, \nu_B^+\nu_C^+], \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \oplus_{\min} C) \cup_{\min} (B \oplus_{\min} C) &= \langle [\mu_A^- + \mu_C^- - \mu_A^-\mu_C^-, \mu_A^+ + \mu_C^+ - \mu_A^+\mu_C^+], [\nu_A^-\nu_C^-, \nu_A^+\nu_C^+], \\
&\quad \min(r, t) \rangle \cup_{\min} \langle [\mu_B^- + \mu_C^- - \mu_B^-\mu_C^-, \mu_B^+ + \mu_C^+ - \mu_B^+\mu_C^+], \\
&\quad [\nu_B^-\nu_C^-, \nu_B^+\nu_C^+], \min(s, t) \rangle.
\end{aligned}$$

Hence, $(A \cup_{\min} B) \oplus_{\min} C = (A \oplus_{\min} C) \cup_{\min} (B \oplus_{\min} C)$.

Similarly (34)–(36) can also be proved.

37. LHS

$$\begin{aligned}
(A \cup_{\min} B) \otimes_{\min} C &= \langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle \\
&\quad \otimes_{\min} C \\
&= \langle [\max(\mu_A^-, \mu_B^-)\mu_C^-, \max(\mu_A^+, \mu_B^+)\mu_C^+], \\
&\quad [\min(\nu_A^-, \nu_B^-) + \nu_C^- - \min(\nu_A^-, \nu_B^-)\nu_C^-, \\
&\quad \min(\nu_A^+, \nu_B^+) + \nu_C^+ - \min(\nu_A^+, \nu_B^+)\nu_C^+], \min(\min(r, s), t) \rangle \\
&= \langle [\max(\mu_A^-\mu_C^-, \mu_B^-\mu_C^-), \max(\mu_A^+\mu_C^+, \mu_B^+\mu_C^+)], \\
&\quad [\min(\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_B^- + \nu_C^- - \nu_B^-\nu_C^-), \\
&\quad \min(\nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+)], \min(\min(r, t), \min(s, t)) \rangle \\
&= \langle [\mu_A^-\mu_C^-, \mu_A^+\mu_C^+], [\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+], \min(r, t) \rangle \cup_{\min} \\
&\quad \langle [\mu_B^-\mu_C^-, \mu_B^+\mu_C^+], [\nu_B^- + \nu_C^- - \nu_B^-\nu_C^-, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+], \min(s, t) \rangle;
\end{aligned}$$

RHS

$$\begin{aligned}
(A \otimes_{\min} C) \cup_{\min} (B \otimes_{\min} C) &= \langle [\mu_A^-\mu_C^-, \mu_A^+\mu_C^+], [\nu_A^- + \nu_C^- - \nu_A^-\nu_C^-, \nu_A^+ + \nu_C^+ - \nu_A^+\nu_C^+], \min(r, t) \rangle \\
&\quad \cup_{\min} \\
&\quad \langle [\mu_B^-\mu_C^-, \mu_B^+\mu_C^+], [\nu_B^- + \nu_C^- - \nu_B^-\nu_C^-, \nu_B^+ + \nu_C^+ - \nu_B^+\nu_C^+], \min(s, t) \rangle.
\end{aligned}$$

Hence, $(A \cup_{\min} B) \otimes_{\min} C = (A \otimes_{\min} C) \cup_{\min} (B \otimes_{\min} C)$.

Similarly (38)–(40) can also be proved.

It is to be noted that for (41)–(44), the proofs are obvious.

45. LHS

$$\begin{aligned} (A' \cap_{\min} B')' &= (\langle [\nu_A^-, \nu_A^+], [\mu_A^-, \mu_A^+], r \rangle \cap_{\min} \langle [\nu_B^-, \nu_B^+], [\mu_B^-, \mu_B^+], s \rangle)' \\ &= (\langle [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], \min(r, s) \rangle)' \\ &= \langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle; \end{aligned}$$

RHS

$$A \cup_{\min} B = \langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle.$$

Hence, $(A' \cap_{\min} B')' = A \cup_{\min} B$.

Similarly (46)–(48) can also be proved.

49. LHS

$$\begin{aligned} (A' \oplus_{\min} B')' &= (\langle [\nu_A^-, \nu_A^+], [\mu_A^-, \mu_A^+], r \rangle \oplus_{\min} \langle [\nu_B^-, \nu_B^+], [\mu_B^-, \mu_B^+], s \rangle)' \\ &= \langle [\nu_A^- + \nu_B^- - \nu_A^- \nu_B^-, \nu_A^+ + \nu_B^+ - \nu_A^+ \nu_B^+], [\mu_A^- \mu_B^-, \mu_A^+ \mu_B^+], \min(r, s) \rangle' \\ &= \langle [\mu_A^- \mu_B^-, \mu_A^+ \mu_B^+], [\nu_A^- + \nu_B^- - \nu_A^- \nu_B^-, \nu_A^+ + \nu_B^+ - \nu_A^+ \nu_B^+], \min(r, s) \rangle; \end{aligned}$$

RHS

$$A \otimes_{\min} B = \langle [\mu_A^- \mu_B^-, \mu_A^+ \mu_B^+], [\nu_A^- + \nu_B^- - \nu_A^- \nu_B^-, \nu_A^+ + \nu_B^+ - \nu_A^+ \nu_B^+], \min(r, s) \rangle$$

Hence, $(A' \oplus_{\min} B')' = A \otimes_{\min} B$.

Similarly (50)–(52) can also be proved.

53. $(A')' = (\langle [\nu_A^-, \nu_A^+], [\mu_A^-, \mu_A^+], r \rangle)' = \langle [\mu_A^-, \mu_A^+], [\nu_A^-, \nu_A^+], r \rangle = A$.

Hence, $(A')' = A$.

54. LHS

$$\begin{aligned} (A \cup_{\min} B)' &= (\langle [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], \min(r, s) \rangle)' \\ &= \langle [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], \min(r, s) \rangle; \end{aligned}$$

RHS

$$\begin{aligned} A' \cap_{\min} B' &= \langle [\nu_A^-, \nu_A^+], [\mu_A^-, \mu_A^+], r \rangle \cap_{\min} \langle [\nu_B^-, \nu_B^+], [\mu_B^-, \mu_B^+], s \rangle \\ &= \langle [\min(\nu_A^-, \nu_B^-), \min(\nu_A^+, \nu_B^+)], [\max(\mu_A^-, \mu_B^-), \max(\mu_A^+, \mu_B^+)], \min(r, s) \rangle. \end{aligned}$$

Hence, $(A \cup_{\min} B)' = A' \cap_{\min} B'$.

Similarly (55)–(57) can also be proved. □

4 Construction of C-IVIFS from IVIFS

For the construction of C-IFS from IFS, Atanassov [7] employed an arithmetic mean-based approach, whereas Bozuyigit and Unver [12] used a geometric mean-based method. Here, in this section, two distinct techniques for the construction of C-IVIFS from IVIFS are addressed, which are arithmetic mean method and geometric mean method.

4.1 Conversion:

Suppose there is a collection of IVIFSs

$$A = \{ \langle [\mu_1^-(x_1), \mu_1^+(x_1)], [\nu_1^-(x_1), \nu_1^+(x_1)] \rangle, \langle [\mu_2^-(x_2), \mu_2^+(x_2)], [\nu_2^-(x_2), \nu_2^+(x_2)] \rangle, \dots, \langle [\mu_n^-(x_n), \mu_n^+(x_n)], [\nu_n^-(x_n), \nu_n^+(x_n)] \rangle \}$$

Arithmetic Mean (AM):

$$\text{AM of all membership intervals, } [\mu_A^-(x), \mu_A^+(x)] = \left[\frac{\sum_{j=1}^n \mu_j^-(x_j)}{n}, \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} \right]$$

$$\text{AM of all non-membership intervals, } [\nu_A^-(x), \nu_A^+(x)] = \left[\frac{\sum_{j=1}^n \nu_j^-(x_j)}{n}, \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} \right]$$

$$\text{Intersection point of diagonals, } (\mu_A(x), \nu_A(x)) = \left(\frac{\mu_A^-(x) + \mu_A^+(x)}{2}, \frac{\nu_A^-(x) + \nu_A^+(x)}{2} \right)$$

$$\begin{aligned} \therefore \text{Radius, } r &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^-(x_j) - \mu_A(x))^2 + (\nu_A^-(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^+(x_j) - \mu_A(x))^2 + (\nu_A^+(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^-(x_j) - \mu_A(x))^2 + (\nu_A^+(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^+(x_j) - \mu_A(x))^2 + (\nu_A^-(x_j) - \nu_A(x))^2} \end{aligned}$$

Hence, $A_{\text{arith}} = \langle [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle$ is a C-IVIFS.

Geometric Mean (GM):

$$\text{GM of all membership intervals, } [\mu_A^-(x), \mu_A^+(x)] = \left[\sqrt[n]{\prod_{j=1}^n \mu_j^-(x_j)}, \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} \right]$$

$$\text{GM of all non-membership intervals, } [\nu_A^-(x), \nu_A^+(x)] = \left[\sqrt[n]{\prod_{j=1}^n \nu_j^-(x_j)}, \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)} \right]$$

$$\text{Intersection point of diagonals, } (\mu_A(x), \nu_A(x)) = \left(\frac{\mu_A^-(x) + \mu_A^+(x)}{2}, \frac{\nu_A^-(x) + \nu_A^+(x)}{2} \right)$$

$$\begin{aligned} \therefore \text{Radius, } r &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^-(x_j) - \mu_A(x))^2 + (\nu_A^-(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^+(x_j) - \mu_A(x))^2 + (\nu_A^+(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^-(x_j) - \mu_A(x))^2 + (\nu_A^+(x_j) - \nu_A(x))^2} \\ &= \max_{1 \leq j \leq n} \sqrt{(\mu_A^+(x_j) - \mu_A(x))^2 + (\nu_A^-(x_j) - \nu_A(x))^2} \end{aligned}$$

Hence, $A_{\text{geo}} = \langle [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)], r \rangle$ is a C-IVIFS.

Validation of the above definitions:

Let

$$A = \{ \langle [\mu_1^-(x_1), \mu_1^+(x_1)], [\nu_1^-(x_1), \nu_1^+(x_1)] \rangle, \langle [\mu_2^-(x_2), \mu_2^+(x_2)], [\nu_2^-(x_2), \nu_2^+(x_2)] \rangle, \dots, \langle [\mu_n^-(x_n), \mu_n^+(x_n)], [\nu_n^-(x_n), \nu_n^+(x_n)] \rangle \}$$

be a collection of IVIFSs. Then

$$A_{\text{arith}} = \left\langle \left[\frac{\sum_{j=1}^n \mu_j^-(x_j)}{n}, \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} \right], \left[\frac{\sum_{j=1}^n \nu_j^-(x_j)}{n}, \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} \right], r \right\rangle,$$

$$A_{\text{geo}} = \left\langle \left[\sqrt[n]{\prod_{j=1}^n \mu_j^-(x_j)}, \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} \right], \left[\sqrt[n]{\prod_{j=1}^n \nu_j^-(x_j)}, \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)} \right], r \right\rangle$$

are C-IVIFSs.

Proof. We have to prove:

$$(i) \quad 0 \leq \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} + \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} \leq 1;$$

$$(ii) \quad 0 \leq \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} + \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)} \leq 1.$$

$$(i) \quad \text{Obviously, } 0 \leq \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} + \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n}. \text{ Now,}$$

$$\begin{aligned} \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} + \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} &= \frac{\sum_{j=1}^n (\mu_j^+(x_j) + \nu_j^+(x_j))}{n} \\ &\leq \frac{\sum_{j=1}^n 1}{n} && [\because \mu_j^+(x_j) + \nu_j^+(x_j) \leq 1] \\ &= 1. \end{aligned}$$

$$\text{Hence, } 0 \leq \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} + \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} \leq 1.$$

$$(ii) \quad \text{Obviously, } 0 \leq \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} + \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)}. \text{ Now,}$$

$$\begin{aligned} \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} + \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)} &\leq \frac{\sum_{j=1}^n \mu_j^+(x_j)}{n} + \frac{\sum_{j=1}^n \nu_j^+(x_j)}{n} \\ &= \frac{\sum_{j=1}^n (\mu_j^+(x_j) + \nu_j^+(x_j))}{n} \\ &\leq \frac{\sum_{j=1}^n 1}{n} && [\because \mu_j^+(x_j) + \nu_j^+(x_j) \leq 1] \\ &= 1. \end{aligned}$$

$$\text{Hence, } 0 \leq \sqrt[n]{\prod_{j=1}^n \mu_j^+(x_j)} + \sqrt[n]{\prod_{j=1}^n \nu_j^+(x_j)} \leq 1.$$

Hence, A_{arith} and A_{geo} are C-IVIFSs. □

Example 4.1. *Let*
$$A = \{ \langle x_1, [0.44, 0.80], [0.16, 0.20] \rangle, \langle x_2, [0.4, 0.65], [0.19, 0.22] \rangle, \langle x_3, [0.46, 0.64], [0.18, 0.24] \rangle, \langle x_4, [0.5, 0.6], [0.2, 0.3] \rangle \},$$
$$B = \{ \langle x_1, [0.38, 0.63], [0.24, 0.35] \rangle, \langle x_2, [0.51, 0.65], [0.14, 0.30] \rangle, \langle x_3, [0.32, 0.50], [0.24, 0.40] \rangle \}$$

be IVIFSs in X .

Then the corresponding C -IVIFSs, as determined by using both AM and GM, are $A = \langle x, [0.45, 0.67], [0.18, 0.24], 0.47 \rangle$ and $B = \langle x, [0.40, 0.59], [0.20, 0.35], 0.33 \rangle$

5 Conclusion

Circular interval-valued intuitionistic fuzzy set is introduced, which is circular intuitionistic fuzzy set's extension. Some basic operators of circular interval-valued intuitionistic fuzzy set and the properties based on those proposed operators are introduced and proved. Furthermore, two methods have been deliberated to construct circular interval-valued intuitionistic fuzzy set from interval-valued intuitionistic fuzzy set with an instance. The future scope of this paper is that the circular interval-valued intuitionistic fuzzy set can be used in multi-criteria decision-making, group decision-making, and pattern recognition problems.

References

- [1] Alimohammadlou, M., Alinejad, S., Khoshsepehr, Z., Safari, M., Jafari, Y., Tajodin, A., & Mohammadi, S. S. (2023). Circular intuitionistic fuzzy AHP: An application in manufacturing sector. In: *Analytic hierarchy process with fuzzy sets extensions: Applications and discussions* (pp. 369–394). Springer International Publishing.
- [2] Alkan, N., & Kahraman, C. (2022). Circular intuitionistic fuzzy TOPSIS method: Pandemic hospital location selection. *Journal of Intelligent & Fuzzy Systems*, 42(1), 295–316.
- [3] Alsattar, H. A., Mourad, N., Zaidan, A. A., Deveci, M., Qahtan, S., Jayaraman, V., & Khalid, Z. (2023). Developing IoT sustainable real-time monitoring devices for food supply chain systems based on climate change using circular intuitionistic fuzzy set. *IEEE Internet of Things Journal*, 11(16), 26680–26689.
- [4] Ashraf, S., Chohan, M. S., Muhammad, S., & Khan, F. (2023). Circular intuitionistic fuzzy TODIM approach for material selection for cryogenic storage tank for liquid nitrogen transportation. *IEEE Access*, 11, 98458–98468.
- [5] Ashraf, S., Sohail, M., Chohan, M. S., Paokanta, S., & Park, C. (2024). Higher-order circular intuitionistic fuzzy time series forecasting methodology: Application of stock change index. *Demonstratio Mathematica*, 57(1), Article 20230115.
- [6] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

- [7] Atanassov, K. T. (2020). Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5), 5981–5986.
- [8] Atanassov, K. (2021). Elliptic intuitionistic fuzzy sets. *Comptes rendus de l'Académie bulgare des Sciences*, 74(6), 812–819.
- [9] Atanassov, K., & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31(3), 343–349.
- [10] Atanassov, K., & Marinov, E. (2021). Four distances for circular intuitionistic fuzzy sets. *Mathematics*, 9(10), Article ID 1121.
- [11] Boltürk, E., & Kahraman, C. (2022). Interval-valued and circular intuitionistic fuzzy present worth analyses. *Informatica*, 33(4), 693–711.
- [12] Bozyiğit, M. C., & Ünver, M. (2024). Parametric circular intuitionistic fuzzy information measures and multi-criteria decision making with extended TOPSIS. *Granular Computing*, 9(2), Article ID 43.
- [13] Çakır, E., & Demircioğlu, E. (2024). Multi-criteria evaluation of battery electric vehicles via circular intuitionistic fuzzy PROMETHEE. In: *IEEE 18th International Symposium on Applied Computational Intelligence and Informatics (SACI)* (pp. 151–156).
- [14] Çakır, E., & Taş, M. A. (2023). Circular intuitionistic fuzzy decision making and its application. *Expert Systems with Applications*, 225, Article ID 120076.
- [15] Çaloğlu Büyükselçuk, E. (2024). Evaluation of industrial IoT service providers with TOPSIS based on circular intuitionistic fuzzy sets. *Computers, Materials & Continua*, 80(1), 715–746.
- [16] Çaloğlu Büyükselçuk, E., & Sarı, Y. C. (2023). The best whey protein powder selection via VIKOR based on circular intuitionistic fuzzy sets. *Symmetry*, 15(7), Article ID 1313.
- [17] Chen, T. Y. (2023). Evolved distance measures for circular intuitionistic fuzzy sets and their exploitation in the technique for order preference by similarity to ideal solutions. *Artificial Intelligence Review*, 56(7), 7347–7401.
- [18] Garg, H., Ünver, M., Olgun, M., & Türkarlan, E. (2023). An extended EDAS method with circular intuitionistic fuzzy value features and its application to multi-criteria decision-making process. *Artificial Intelligence Review*, 56(3), 3173–3204.
- [19] Gorzalczany, M. B. (1987). A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems*, 21(1), 1–17.
- [20] Imanov, G., & Aliyev, A. (2021). Circular intuitionistic fuzzy sets in evaluation of human capital. *Revista Científica del Instituto Iberoamericano de Desarrollo Empresarial*, 1, 1–13.

- [21] Kahraman, C., & Alkan, N. (2021). Circular intuitionistic fuzzy TOPSIS method with vague membership functions: Supplier selection application context. *Notes on Intuitionistic Fuzzy Sets*, 27(1), 24–52.
- [22] Kahraman, C., & Otay, I. (2022). Extension of VIKOR method using circular intuitionistic fuzzy sets. In: *Intelligent and Fuzzy Techniques for Emerging Conditions and Digital Transformation: Proceedings of the INFUS 2021 Conference* (Vol. 2, pp. 48–57). Springer International Publishing.
- [23] Khan, M. J., Kumam, W., & Alreshidi, N. A. (2022). Divergence measures for circular intuitionistic fuzzy sets and their applications. *Engineering Applications of Artificial Intelligence*, 116, Article ID 105455.
- [24] Otay, I., & Kahraman, C. (2022). A novel circular intuitionistic fuzzy AHP & VIKOR methodology: An application to a multi-expert supplier evaluation problem. *Pamukkale Üniversitesi Mühendislik Bilimleri Dergisi*, 28(1), 194–207.
- [25] Turksen, I. B. (1996). Interval-valued strict preference with Zadeh triples. *Fuzzy Sets and Systems*, 78, 183–195.
- [26] Wang, J. C., & Chen, T. Y. (2024). A compromise decision-support technique with an augmented scoring function within circular intuitionistic fuzzy settings. *Engineering Applications of Artificial Intelligence*, 128, Article ID 107359.
- [27] Xu, C., & Wen, Y. (2023). New measure of circular intuitionistic fuzzy sets and its application in decision making. *AIMS Mathematics*, 8(10), 24053–24074.
- [28] Yusoff, B., Pratama, D., Kilicman, A., & Abdullah, L. (2023). Circular intuitionistic fuzzy ELECTRE III model for group decision analysis. *Informatica*, 34(4), 881–908.
- [29] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.