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Norms over *Q*-intuitionistic fuzzy subgroups of a group

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Abstract: In this work, by using norms (t-norms and t-conorms), Q-intuitionistic fuzzy subgroups of a group are defined and investigated some of their properties and structured characteristics.
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1 Introduction

Undoubtedly the notion of fuzzy set theory initiated by Zadeh [50] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair $\{0, 1\}$ any more but on the continuous interval [0, 1] was born. As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by Atanassov [3,4]. The concept of fuzzy group was introduced by Rosenfled [48] and Anthony and Sherwood [2] gave the definition of fuzzy groups. Norms were introduced in the framework of probabilistic metric spaces. However, they are widely applied in several other fields, e.g., in fuzzy set theory, fuzzy logic, and their applications. By using norms,



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the author investigated some properties of fuzzy algebraic structures [8–47]. In this paper, we define Q-intuitionistic fuzzy subgroups of a group with respect to norms (*t*-norms and *t*-conorms) and investigate properties of them.

2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer to [1,3,5-7].

Definition 2.1. A group is a non-empty set G on which there is a binary operation $(a, b) \rightarrow ab$ such that:

- (1) if a and b belong to G, then ab is also in G (closure),
- (2) a(bc) = (ab)c for all $a, b, c \in G$ (associativity),
- (3) there is an element $e \in G$ such that ae = ea = a for all $a \in G$ (identity),
- (4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., ab = ba for all $a, b \in G$.

Remark 2.2. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \rightarrow a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \rightarrow ab$ for which the identity is denoted by e.

Proposition 2.3. Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

- (1) H is a subgroup of G.
- (2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 2.4. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into [0, 1]. The set of all fuzzy subsets of G is called the [0, 1]-power set of G and is denoted $[0, 1]^G$.

Definition 2.5. For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.6. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, *IFS*) in X such that $\mu_A, \nu_A \in [0, 1]^X$ and for all $x \in X$ we have $(\mu_A(x) + \nu_A(x)) \in [0, 1]$. In particular, \emptyset_X and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0, 1)$ and $U_X(x) = (1, 0)$, respectively. We will denote the set of all *IFSs* in X as *IFS*(X).

Definition 2.7. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be *IFSs* in X. Then:

(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) Equality: A = B iff $A \subseteq B$ and $B \subseteq A$.

Definition 2.8. A *t*-norm *T* is a function $T : [0,1] \times [0,1] \rightarrow [0,1]$ having the following four properties:

- (T1) T(x, 1) = x (neutral element)
- (T2) $T(x, y) \le T(x, z)$ if $y \le z$ (monotonicity)
- (T3) T(x, y) = T(y, x) (commutativity)
- (T4) T(x, T(y, z)) = T(T(x, y), z) (associativity),

for all $x, y, z \in [0, 1]$.

Corollary 2.9. Let T be a t-norm. Then for all $x \in [0, 1]$,

- (1) T(x,0) = 0.
- (2) T(0,0) = 0.

Example 2.10. (1) Standard intersection t-norm $T_m(x, y) = \min\{x, y\}$.

- (2) Bounded sum *t*-norm $T_b(x, y) = \max\{0, x + y 1\}.$
- (3) Algebraic product *t*-norm $T_p(x, y) = xy$.
- (4) Drastic *t*-norm

$$T_D(x,y) = \begin{cases} y, & \text{if } x = 1\\ x, & \text{if } y = 1\\ 0, & \text{otherwise} \end{cases}$$

(5) Nilpotent minimum *t*-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\}, & \text{if } x+y > 1\\ 0, & \text{otherwise.} \end{cases}$$

(6) Hamacher product *t*-norm

$$T_{H_0}(x,y) = \begin{cases} 0, & \text{if } x = y = 0\\ \frac{xy}{x+y-xy}, & \text{otherwise.} \end{cases}$$

The drastic *t*-norm is the pointwise smallest *t*-norm and the minimum is the pointwise largest *t*-norm:

$$T_D(x,y) \le T(x,y) \le T_{\min}(x,y)$$

for all $x, y \in [0, 1]$.

Lemma 2.11. Let T be a t-norm. Then

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.12. A *t*-conorm *C* is a function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ having the following four properties:

- $(C1) \quad C(x,0) = x$
- (C2) $C(x, y) \leq C(x, z)$ if $y \leq z$
- (C3) C(x,y) = C(y,x)
- (C4) C(x, C(y, z)) = C(C(x, y), z),

for all $x, y, z \in [0, 1]$.

Corollary 2.13. Let C be a C-conorm. Then for all $x \in [0, 1]$,

- (1) C(x,1) = 1.
- (2) C(0,0) = 0.

Example 2.14. (1) Standard union *t*-conorm $C_m(x, y) = \max\{x, y\}$.

- (2) Bounded sum *t*-conorm $C_b(x, y) = \min\{1, x + y\}$.
- (3) Algebraic sum *t*-conorm $C_p(x, y) = x + y xy$.
- (4) Drastic *t*-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0\\ x & \text{if } y = 0\\ 1 & \text{otherwise} \end{cases}$$

dual to the drastic *t*-norm.

(5) Nilpotent maximum t-conorm, dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x+y < 1\\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity)

$$C_{H_2}(x,y) = \frac{x+y}{1+xy}$$

is a dual to one of the Hamacher *t*-norms. Note that all *t*-conorms are bounded by the maximum and the drastic t-conorm:

$$C_{\max}(x,y) \le C(x,y) \le C_D(x,y)$$

for any *t*-conorm *C* and all $x, y \in [0, 1]$.

Recall that *t*-norm *T* (respectively, *t*-conorm *C*) is idempotent if for all $x \in [0, 1]$, T(x, x) = x (respectively, C(x, x) = x).

Lemma 2.15. Let C be a t-conorm. Then

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

3 Main results

Definition 3.1. Let (G, .) be a group and Q be a non empty set. An intuitionistic fuzzy set $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ is said to be a Q-intuitionistic fuzzy subgroup of G with respect to norms (*t*-norm T and *t*-conorm C) if the following conditions are satisfied:

(1)
$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q))$$
$$\supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))),$$
$$A(x^{-1},q) = (\mu_A(x^{-1},q), \nu_A(x^{-1},q))$$
$$\supseteq A(x,q)$$

$$= (\mu_A(x,q),\nu_A(x,q))$$

which means:

(a)
$$\mu_A(xy,q) \ge T(\mu_A(x,q),\mu_A(y,q))$$

- (b) $\nu_A(xy,q) \le C(\nu_A(x,q),\nu_A(y,q)),$
- (c) $\mu_A(x^{-1}, q) \ge \mu_A(x, q),$
- (d) $\nu_A(x^{-1}, q) \le \nu_A(x, q),$

for all $x, y \in G$ and $q \in Q$. Throughout this paper the set of all Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C) will be denoted by QIFSN(G).

Lemma 3.2. The conditions (2) of Definition 3.1 imply that

$$A(x^{-1},q) = A(x,q)$$

for all $x \in G$ and $q \in Q$.

Proof. Let $x \in G$ and $q \in Q$. As $A(x^{-1},q) \supseteq A(x,q)$ so $\mu_A(x^{-1},q) \ge \mu_A(x,q)$, and $\nu_A(x^{-1},q) \le \nu_A(x,q)$. Then

$$\mu_A(x,q) = \mu_A((x^{-1})^{-1},q) \ge \mu_A(x^{-1},q) \ge \mu_A(x,q)$$

and

$$\nu_A(x,q) = \nu_A((x^{-1})^{-1},q) \le \nu_A(x^{-1},q) \le \nu_A(x,q)$$

and then $\mu_A(x,q) = \mu_A(x^{-1},q)$ and $\nu_A(x,q) = \nu_A(x^{-1},q)$. Thus

$$A(x^{-1},q) = (\mu_A(x^{-1},q),\nu_A(x^{-1},q)) = (\mu_A(x,q),\nu_A(x,q)) = A(x,q).$$

Proposition 3.3. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ such that T and C be idempotent. Then

$$A(e_G,q) \supseteq A(x,q)$$

for all $x \in G$ and $q \in Q$.

Proof. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $x \in G$ and $q \in Q$. Then

$$\mu_A(e_G, q) = \mu_A(xx^{-1}, q)$$

$$\geq T(\mu_A(x, q), \mu_A(x^{-1}, q))$$

$$= T(\mu_A(x, q), \mu_A(x, q)) = \mu_A(x, q)$$

and so

$$\mu_A(e_G, q) \ge \mu_A(x, q). \tag{a}$$

Also

$$\nu_A(e_G, q) = \nu_A(xx^{-1}, q)$$

$$\leq C(\nu_A(x, q), \nu_A(x^{-1}, q))$$

$$= C(\nu_A(x, q), \nu_A(x, q))$$

$$= \nu_A(x, q)$$

and then

$$\nu_A(e_G, q) \le \nu_A(x, q). \tag{b}$$

Thus from (a) and (b) we have that

$$A(e_G, q) = (\mu_A(e_G, q), \nu_A(e_G, q)) \supseteq (\mu_A(x, q), \nu_A(x, q)) = A(x, q).$$

Proposition 3.4. Let
$$A = (\mu_A, \nu_A) \in QIFSN(G)$$
. If T and C are idempotent and $A(xy^{-1}, q) = A(e_G, q)$,

then

$$A(x,q) = A(y,q)$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\mu_{A}(x,q) = \mu_{A}(xy^{-1}y,q)$$

$$\geq T(\mu_{A}(xy^{-1},q),\mu_{A}(y,q))$$

$$= T(\mu_{A}(e_{G},q),\mu_{A}(y,q))$$

$$\geq T(\mu_{A}(y,q),\mu_{A}(y,q))$$

$$= \mu_{A}(y,q) = \mu_{A}(yx^{-1}x,q)$$

$$\geq T(\mu_{A}(yx^{-1},q),\mu_{A}(x,q))$$

$$= T(\mu_{A}((xy^{-1})^{-1},q),\mu_{A}(x,q))$$

$$= T(\mu_{A}(e_{G},q),\mu_{A}(x,q))$$

$$\geq T(\mu_{A}(x,q),\mu_{A}(x,q))$$

$$\geq T(\mu_{A}(x,q),\mu_{A}(x,q))$$

$$= \mu_{A}(x,q),$$

thus

$$\mu_A(x,q) = \mu_A(y,q). \tag{a}$$

Also

$$\begin{aligned}
\nu_A(x,q) &= \nu_A(xy^{-1}y,q) \\
&\leq C(\nu_A(xy^{-1},q),\nu_A(y,q)) \\
&= C(\nu_A(e_G,q),\nu_A(y,q)) \\
&\leq C(\nu_A(y,q),\nu_A(y,q)) \\
&= \nu_A(y,q) &= \nu_A(yx^{-1}x,q) \\
&\leq C(\mu_A(yx^{-1},q),\nu_A(x,q)) \\
&= C(\nu_A((xy^{-1})^{-1},q),\nu_A(x,q)) \\
&= C(\nu_A(e_G,q),\nu_A(x,q)) \\
&\leq C(\nu_A(x,q),\nu_A(x,q)) \\
&\leq C(\nu_A(x,q),\nu_A(x,q)) \\
&= \nu_A(x,q),
\end{aligned}$$

then

$$\nu_A(x,q) = \nu_A(y,q). \tag{b}$$

Therefore

$$A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = (\mu_A(y,q), \nu_A(y,q)) = A(y,q).$$

Proposition 3.5. Let T and C be idempotent. Then

$$A = (\mu_A, \nu_A) \in QIFSN(G)$$

if and only if

$$A(xy^{-1}, q) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q)))$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let
$$A = (\mu_A, \nu_A) \in QIFSN(G)$$
 and $x, y \in G, q \in Q$. Then
 $\mu_A(xy^{-1}, q) \ge T(\mu_A(x, q), \mu_A(y^{-1}, q))$
 $\ge T(\mu_A(x, q), \mu_A(y, q))$

and

$$\nu_A(xy^{-1}, q) \le C(\nu_A(x, q), \nu_A(y^{-1}, q)) \\\le C(\nu_A(x, q), \nu_A(y, q)),$$

and then

$$A(xy^{-1},q) = (\mu_A(xy^{-1},q),\nu_A(xy^{-1},q))$$

$$\supseteq A(T(\mu_A(x,q),\mu_A(y,q)), C(\nu_A(x,q),\nu_A(y,q))).$$

Conversely, let

$$A(xy^{-1}, q) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))).$$

Then

$$\mu_A(x^{-1}, q) = \mu_A(e_G x^{-1}, q)$$

$$\geq T(\mu_A(e_G, q), \mu_A(x, q))$$

$$\geq T(\mu_A(x, q), \mu_A(x, q))$$

$$= \mu_A(x, q)$$

and

$$\nu_A(x^{-1}, q) = \nu_A(e_G x^{-1}, q)$$

$$\leq C(\nu_A(e_G, q), \nu_A(x, q))$$

$$\leq C(\mu_A(x, q), \nu_A(x, q))$$

$$= \nu_A(x, q)$$

and then

$$A(x^{-1},q) \supseteq A(x,q). \tag{1}$$

Also

$$\mu_A(xy,q) = \mu_A(x(y^{-1})^{-1},q) \ge T(\mu_A(x,q),\mu_A(y^{-1},q)) \ge T(\mu_A(x,q),\mu_A(y,q))$$

and

$$\nu_A(xy,q) = \nu_A(x(y^{-1})^{-1},q) \le C(\nu_A(x,q),\nu_A(y^{-1},q)) \le C(\nu_A(x,q),\nu_A(y,q))$$

Hence

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) \supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))).$$
(2)

Therefore from (1) and (2) we get that

$$A = (\mu_A, \nu_A) \in QIFSN(G).$$

Proposition 3.6. Let $\mu_A, \nu_A \in [0, 1]^{G \times Q}$ such that

$$A(e_G, q) = (1, 0)$$

and

$$A(xy^{-1}, q) \supseteq (T(\mu(x, q), \mu(y, q)), C(\mu(x, q), \mu(y, q)))$$

for all $x, y \in G$ and $q \in Q$. Then

$$A = (\mu_A, \nu_A) \in QIFSN(G).$$

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\mu_A(x^{-1},q) = \mu_A(e_G x^{-1},q) \ge T(\mu_A(e_G,q),\mu_A(x,q)) = T(1,\mu_A(x,q)) = \mu_A(x,q)$$

and

$$\nu_A(x^{-1},q) = \nu_A(e_G x^{-1},q) \le C(\nu_A(e_G,q),\nu_A(x,q)) = C(0,\nu_A(x,q)) = \nu_A(x,q)$$

and so

$$A(x^{-1},q) = (\mu_A(x^{-1},q),\nu_A(x^{-1},q)) \supseteq (\mu_A(x,q),\nu_A(x,q)) = A(x,q).$$
(1)

Also

$$\mu_A(xy,q) = \mu_A(x((y)^{-1})^{-1},q) \ge T(\mu_A(x,q),\mu_A(y^{-1},q)) \ge T(\mu_A(x,q),\mu_A(y,q))$$

and

$$\nu_A(xy,q) = \nu_A(x((y)^{-1})^{-1},q) \le C(\nu_A(x,q),\nu_A(y^{-1},q)) \le C(\nu_A(x,q),\nu_A(y,q))$$

and so

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) \supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))).$$
(2)

Therefore from (1) and (2) we will have

$$A = (\mu_A, \nu_A) \in QIFSN(G).$$

Proposition 3.7. If $A = (\mu_A, \nu_A) \in QIFSN(G)$, then

$$H = \{ x \in G \mid A(x,q) = (1,0) \; \forall q \in Q \}$$

is a subgroup of G.

Proof. Let $x, y \in H$ and $q \in Q$. Then $\mu_A(x,q) = \mu_A(y,q) = 1$ and $\nu_A(x,q) = \nu_A(y,q) = 0$. Since $A = (\mu_A, \nu_A) \in QIFSN(G)$, so

$$\mu_A(xy^{-1},q) \ge T(\mu_A(x,q),\mu_A(y,q)) = T(1,1) = 1$$

and this implies that $\mu_A(xy^{-1}, q) = 1$. Also

$$\nu_A(xy^{-1},q) \le C(\nu_A(x,q),\nu_A(y,q)) = C(0,0) = 0$$

and so $\nu_A(xy^{-1},q) = 0$. Then

$$A(xy^{-1},q) = (\mu_A(xy^{-1},q),\nu_A(xy^{-1},q)) = (1,0)$$

and then $xy^{-1} \in H$ so from Proposition 2.3 we obtain that H will be a subgroup of G.

Proposition 3.8. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ such that T and C be idempotent. Then

$$H = \{ x \in G \mid A(x,q) = A(e_G,q) \quad \forall q \in Q \}$$

is a subgroup of G.

Proof. Let $x, y \in H$ and $q \in Q$ then $\mu_A(x, q) = \mu_A(y, q) = \mu_A(e_G, q)$ and $\nu_A(x, q) = \nu_A(y, q) = \nu_A(e_G, q)$. Since $A = (\mu_A, \nu_A) \in QIFSN(G)$ so

$$\mu_A(xy^{-1}, q) \ge T(\mu_A(x, q), \mu_A(y, q))$$

= $T(\mu_A(e_G, q), \mu_A(e_G, q))$
= $\mu_A(e_G, q)$
 $\ge \mu_A(xy^{-1}, q)$

and so $\mu_A(xy^{-1}, q) = \mu_A(e_G, q)$.

Also

$$\nu_A(xy^{-1},q) \le C(\nu_A(x,q),\nu_A(y,q))$$

= $C(\nu_A(e_G,q),\nu_A(e_G,q))$
= $\nu_A(e_G,q)$
 $\le \nu_A(xy^{-1},q),$

then $\nu_A(xy^{-1},q) = \nu_A(e_G,q)$. Therefore,

$$A(xy^{-1},q) = (\mu_A(xy^{-1},q),\nu_A(xy^{-1},q)) = (\mu_A(e_G,q),\nu_A(e_G,q)) = A(e_G,q).$$

Thus $xy^{-1} \in H$ and Proposition 2.3 give us that H is a subgroup of G.

Proposition 3.9. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $A(xy^{-1}, q) = (1, 0)$ then

$$A(x,q) = A(y,q)$$

for all $x, y \in G$ and $q \in Q$.

Proof. Assume that $x, y \in G, q \in Q$. As $A \in QIFST(G)$, so

$$\mu_A(x,q) = \mu_A(xy^{-1}y,q)$$

$$\geq T(\mu_A(xy^{-1},q),\mu_A(y,q)) = T(1,\mu_A(y,q)) = \mu_A(y,q)$$

$$= \mu_A(y^{-1},q) = \mu_A(x^{-1}xy^{-1},q)$$

$$\geq T(\mu_A(x^{-1},q),\mu_A(xy^{-1},q)) = T(\mu_A(x^{-1},q),1)$$

$$= \mu_A(x^{-1},q) = \mu_A(x,q),$$

hence

$$\mu_A(x,q) = \mu_A(y,q).$$

Also

$$\nu_A(x,q) = \nu_A(xy^{-1}y,q)
\leq C(\nu_A(xy^{-1},q),\nu_A(y,q)) = C(0,\nu_A(y,q)) = \nu_A(y,q)
= \nu_A(y^{-1},q) = \nu_A(x^{-1}xy^{-1},q)
\leq C(\nu_A(x^{-1},q),\nu_A(xy^{-1},q)) = C(\nu_A(x^{-1},q),0)
= \nu_A(x^{-1},q) = \nu_A(x,q),$$

thus

$$\nu_A(x,q) = \nu_A(y,q).$$

Now

$$A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = (\mu_A(y,q), \nu_A(y,q)) = A(y,q).$$

Proposition 3.10. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$. Then

$$A(xy,q) = A(yx,q)$$

if and only if

$$A(x,q) = A(y^{-1}xy,q)$$

for all $x, y \in G$ and $q \in Q$.

 $\textit{Proof.} \ \ \text{Let} \ x,y \in G, q \in Q \ \text{and} \ A(xy,q) = A(yx,q). \ \text{Then}$

$$\mu_A(y^{-1}xy,q) = \mu_A(y^{-1}(xy),q) = \mu_A(xyy^{-1},q) = \mu_A(xe_G,q) = \mu_A(x,q)$$

and

$$\nu_A(y^{-1}xy,q) = \nu_A(y^{-1}(xy),q) = \nu_A(xyy^{-1},q) = \nu_A(xe_G,q) = \nu_A(x,q)$$

and so

$$A(y^{-1}xy,q) = (\mu_A(y^{-1}xy,q),\nu_A(y^{-1}xy,q)) = (\mu_A(x,q),\nu_A(x,q)) = A(x,q).$$

Conversely, let $A(x,q) = A(y^{-1}xy,q)$ then

$$\mu_A(xy,q) = \mu_A(x(yx)x^{-1},q) = \mu_A(yx,q)$$

and

$$\nu_A(xy,q) = \nu_A(x(yx)x^{-1},q) = \nu_A(yx,q)$$

and then

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) = (\mu_A(yx,q), \nu_A(yx,q)) = A(yx,q).$$

Proposition 3.11. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$. If

$$A(xy^{-1}, q) = (0, 1),$$

then either

or

$$A(y,q) = (0,1)$$

A(x,q) = (0,1)

for all $x, y \in G$ and $q \in Q$.

Proof. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ then for all $x, y \in G$ and $q \in Q$ we obtain that

$$0 = \mu_A(xy^{-1}, q) \ge T(\mu_A(x, q), \mu_A(y, q))$$

and then either $\mu_A(x,q) = 0$ or $\mu_A(y,q) = 0$. Also

$$1 = \nu_A(xy^{-1}, q) \le C(\nu_A(x, q), \nu_A(y, q))$$

and then either $\nu_A(x,q) = 1$ or $\mu_A(y,q) = 1$. Therefore either

$$A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = (0,1)$$

or

$$A(y,q) = (\mu_A(y,q), \nu_A(x,q)) = (0,1).$$

Proposition 3.12. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $x, y \in G, q \in Q$. If T and C be *idempotent and* $A(x, q) \neq A(y, q)$, *then*

$$A(xy,q) = (T(\mu_A(x,q),\mu_A(y,q)), C(\nu_A(x,q),\nu_A(y,q))).$$

Proof. Let $A(x,q) \supset A(y,q)$ then $\mu_A(x,q) > \mu_A(y,q)$ and $\nu_A(x,q) < \nu_A(y,q)$. As $\mu_A(x,q) > \mu_A(y,q)$ for all $x, y \in G$ and $q \in Q$ so $\mu_A(x,q) > \mu_A(xy,q)$ and so

$$\mu_A(y,q) = T(\mu_A(x,q),\mu_A(y,q))$$

and

$$\mu_A(xy,q) = T(\mu_A(x,q),\mu_A(xy,q)).$$

Now

$$\mu_{A}(xy,q) \geq T(\mu_{A}(x,q),\mu_{A}(y,q)) = \mu_{A}(y,q) = \mu_{A}(x^{-1}xy,q) \geq T(\mu_{A}(x^{-1},q),\mu_{A}(xy,q)) = T(\mu_{A}(x,q),\mu_{A}(xy,q)) = \mu_{A}(xy,q),$$

and so

$$\mu_A(xy,q) = \mu_A(y,q) = T(\mu_A(x,q),\mu_A(y,q))$$

Also since $\nu_A(x,q) < \nu_A(y,q)$ for all $x, y \in G$ and $q \in Q$, so $\nu_A(x,q) < \nu_A(xy,q)$, and then

$$\nu_A(y,q) = C(\nu_A(x,q),\nu_A(y,q))$$

and

$$\nu_A(xy,q) = C(\nu_A(x,q),\nu_A(xy,q)).$$

Now

$$\nu_A(xy,q) \leq C(\nu_A(x,q),\nu_A(y,q))$$

= $\nu_A(y,q)$
= $\nu_A(x^{-1}xy,q)$
 $\leq C(\nu_A(x^{-1},q),\nu_A(xy,q))$
= $C(\nu_A(x,q),\nu_A(xy,q))$
= $\nu_A(xy,q),$

and thus

$$\nu_A(xy,q) = \nu_A(y,q) = C(\nu_A(x,q),\nu_A(y,q))$$

Therefore

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) = (T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))). \quad \Box$$

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