

# Norms over $Q$ -intuitionistic fuzzy subgroups of a group

Rasul Rasuli

Department of Mathematics, Payame Noor University (PNU)  
Tehran, Iran  
e-mail: rasulirasul@yahoo.com

**Received:** 13 July 2021

**Revised:** 27 July 2021

**Accepted:** 10 August 2022

**Online First:** 13 March 2023

**Abstract:** In this work, by using norms ( $t$ -norms and  $t$ -conorms),  $Q$ -intuitionistic fuzzy subgroups of a group are defined and investigated some of their properties and structured characteristics.

**Keywords:** Group theory, Fuzzy set theory, Norms, Intuitionistic fuzzy set.

**2020 Mathematics Subject Classification:** 82D25, 03E72, 47A30.

## 1 Introduction

Undoubtedly the notion of fuzzy set theory initiated by Zadeh [50] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair  $\{0, 1\}$  any more but on the continuous interval  $[0, 1]$  was born. As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by Atanassov [3, 4]. The concept of fuzzy group was introduced by Rosenfeld [48] and Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on  $t$ -norm. Solairaju and Nagarajan [49] introduced the notion of  $Q$ -fuzzy groups. Norms were introduced in the framework of probabilistic metric spaces. However, they are widely applied in several other fields, e.g., in fuzzy set theory, fuzzy logic, and their applications. By using norms,



the author investigated some properties of fuzzy algebraic structures [8–47]. In this paper, we define  $Q$ -intuitionistic fuzzy subgroups of a group with respect to norms ( $t$ -norms and  $t$ -conorms) and investigate properties of them.

## 2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For more details we refer to [1, 3, 5–7].

**Definition 2.1.** A group is a non-empty set  $G$  on which there is a binary operation  $(a, b) \rightarrow ab$  such that:

- (1) if  $a$  and  $b$  belong to  $G$ , then  $ab$  is also in  $G$  (closure),
- (2)  $a(bc) = (ab)c$  for all  $a, b, c \in G$  (associativity),
- (3) there is an element  $e \in G$  such that  $ae = ea = a$  for all  $a \in G$  (identity),
- (4) if  $a \in G$ , then there is an element  $a^{-1} \in G$  such that  $aa^{-1} = a^{-1}a = e$  (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group  $G$  is called abelian if the binary operation is commutative, i.e.,  $ab = ba$  for all  $a, b \in G$ .

**Remark 2.2.** There are two standard notations for the binary group operation: either the additive notation, that is  $(a, b) \rightarrow a + b$  in which case the identity is denoted by 0, or the multiplicative notation, that is  $(a, b) \rightarrow ab$  for which the identity is denoted by  $e$ .

**Proposition 2.3.** Let  $G$  be a group. Let  $H$  be a non-empty subset of  $G$ . The following are equivalent:

- (1)  $H$  is a subgroup of  $G$ .
- (2)  $x, y \in H$  implies  $xy^{-1} \in H$  for all  $x, y$ .

**Definition 2.4.** Let  $G$  be an arbitrary group with a multiplicative binary operation and identity  $e$ . A fuzzy subset of  $G$ , we mean a function from  $G$  into  $[0, 1]$ . The set of all fuzzy subsets of  $G$  is called the  $[0, 1]$ -power set of  $G$  and is denoted  $[0, 1]^G$ .

**Definition 2.5.** For sets  $X, Y$  and  $Z$ ,  $f = (f_1, f_2) : X \rightarrow Y \times Z$  is called a complex mapping if  $f_1 : X \rightarrow Y$  and  $f_2 : X \rightarrow Z$  are mappings.

**Definition 2.6.** Let  $X$  be a nonempty set. A complex mapping  $A = (\mu_A, \nu_A) : X \rightarrow [0, 1] \times [0, 1]$  is called an intuitionistic fuzzy set (in short,  $IFS$ ) in  $X$  such that  $\mu_A, \nu_A \in [0, 1]^X$  and for all  $x \in X$  we have  $(\mu_A(x) + \nu_A(x)) \in [0, 1]$ . In particular,  $\emptyset_X$  and  $U_X$  denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in  $X$  defined by  $\emptyset_X(x) = (0, 1)$  and  $U_X(x) = (1, 0)$ , respectively. We will denote the set of all  $IFS$ s in  $X$  as  $IFS(X)$ .

**Definition 2.7.** Let  $X$  be a nonempty set and let  $A = (\mu_A, \nu_A)$  and  $B = (\mu_B, \nu_B)$  be  $IFS$ s in  $X$ . Then:

- (1) Inclusion:  $A \subseteq B$  iff  $\mu_A \leq \mu_B$  and  $\nu_A \geq \nu_B$ .
- (2) Equality:  $A = B$  iff  $A \subseteq B$  and  $B \subseteq A$ .

**Definition 2.8.** A  $t$ -norm  $T$  is a function  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  having the following four properties:

- (T1)  $T(x, 1) = x$  (neutral element)
  - (T2)  $T(x, y) \leq T(x, z)$  if  $y \leq z$  (monotonicity)
  - (T3)  $T(x, y) = T(y, x)$  (commutativity)
  - (T4)  $T(x, T(y, z)) = T(T(x, y), z)$  (associativity),
- for all  $x, y, z \in [0, 1]$ .

**Corollary 2.9.** Let  $T$  be a  $t$ -norm. Then for all  $x \in [0, 1]$ ,

- (1)  $T(x, 0) = 0$ .
- (2)  $T(0, 0) = 0$ .

**Example 2.10.** (1) Standard intersection  $t$ -norm  $T_m(x, y) = \min\{x, y\}$ .

(2) Bounded sum  $t$ -norm  $T_b(x, y) = \max\{0, x + y - 1\}$ .

(3) Algebraic product  $t$ -norm  $T_p(x, y) = xy$ .

(4) Drastic  $t$ -norm

$$T_D(x, y) = \begin{cases} y, & \text{if } x = 1 \\ x, & \text{if } y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum  $t$ -norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\}, & \text{if } x + y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

(6) Hamacher product  $t$ -norm

$$T_{H_0}(x, y) = \begin{cases} 0, & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy}, & \text{otherwise.} \end{cases}$$

The drastic  $t$ -norm is the pointwise smallest  $t$ -norm and the minimum is the pointwise largest  $t$ -norm:

$$T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$$

for all  $x, y \in [0, 1]$ .

**Lemma 2.11.** Let  $T$  be a  $t$ -norm. Then

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all  $x, y, w, z \in [0, 1]$ .

**Definition 2.12.** A  $t$ -conorm  $C$  is a function  $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$  having the following four properties:

- (C1)  $C(x, 0) = x$
  - (C2)  $C(x, y) \leq C(x, z)$  if  $y \leq z$
  - (C3)  $C(x, y) = C(y, x)$
  - (C4)  $C(x, C(y, z)) = C(C(x, y), z)$ ,
- for all  $x, y, z \in [0, 1]$ .

**Corollary 2.13.** Let  $C$  be a  $C$ -conorm. Then for all  $x \in [0, 1]$ ,

- (1)  $C(x, 1) = 1$ .
- (2)  $C(0, 0) = 0$ .

**Example 2.14.** (1) Standard union  $t$ -conorm  $C_m(x, y) = \max\{x, y\}$ .

(2) Bounded sum  $t$ -conorm  $C_b(x, y) = \min\{1, x + y\}$ .

(3) Algebraic sum  $t$ -conorm  $C_p(x, y) = x + y - xy$ .

(4) Drastic  $t$ -conorm

$$C_D(x, y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic  $t$ -norm.

(5) Nilpotent maximum  $t$ -conorm, dual to the nilpotent minimum  $T$ -norm:

$$C_{nM}(x, y) = \begin{cases} \max\{x, y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity)

$$C_{H_2}(x, y) = \frac{x + y}{1 + xy}$$

is a dual to one of the Hamacher  $t$ -norms. Note that all  $t$ -conorms are bounded by the maximum and the drastic  $t$ -conorm:

$$C_{\max}(x, y) \leq C(x, y) \leq C_D(x, y)$$

for any  $t$ -conorm  $C$  and all  $x, y \in [0, 1]$ .

Recall that  $t$ -norm  $T$  (respectively,  $t$ -conorm  $C$ ) is idempotent if for all  $x \in [0, 1]$ ,  $T(x, x) = x$  (respectively,  $C(x, x) = x$ ).

**Lemma 2.15.** Let  $C$  be a  $t$ -conorm. Then

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z)),$$

for all  $x, y, w, z \in [0, 1]$ .

### 3 Main results

**Definition 3.1.** Let  $(G, \cdot)$  be a group and  $Q$  be a non empty set. An intuitionistic fuzzy set  $A = (\mu_A, \nu_A) \in IFS(G \times Q)$  is said to be a  $Q$ -intuitionistic fuzzy subgroup of  $G$  with respect to norms ( $t$ -norm  $T$  and  $t$ -conorm  $C$ ) if the following conditions are satisfied:

- (1) 
$$A(xy, q) = (\mu_A(xy, q), \nu_A(xy, q))$$
$$\supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))),$$
- (2) 
$$A(x^{-1}, q) = (\mu_A(x^{-1}, q), \nu_A(x^{-1}, q))$$
$$\supseteq A(x, q)$$
$$= (\mu_A(x, q), \nu_A(x, q))$$

which means:

- (a)  $\mu_A(xy, q) \geq T(\mu_A(x, q), \mu_A(y, q)),$
- (b)  $\nu_A(xy, q) \leq C(\nu_A(x, q), \nu_A(y, q)),$
- (c)  $\mu_A(x^{-1}, q) \geq \mu_A(x, q),$
- (d)  $\nu_A(x^{-1}, q) \leq \nu_A(x, q),$

for all  $x, y \in G$  and  $q \in Q$ . Throughout this paper the set of all  $Q$ -intuitionistic fuzzy subgroups of  $G$  with respect to norms ( $t$ -norm  $T$  and  $t$ -conorm  $C$ ) will be denoted by  $QIFSN(G)$ .

**Lemma 3.2.** *The conditions (2) of Definition 3.1 imply that*

$$A(x^{-1}, q) = A(x, q)$$

for all  $x \in G$  and  $q \in Q$ .

*Proof.* Let  $x \in G$  and  $q \in Q$ . As  $A(x^{-1}, q) \supseteq A(x, q)$  so  $\mu_A(x^{-1}, q) \geq \mu_A(x, q)$ , and  $\nu_A(x^{-1}, q) \leq \nu_A(x, q)$ . Then

$$\mu_A(x, q) = \mu_A((x^{-1})^{-1}, q) \geq \mu_A(x^{-1}, q) \geq \mu_A(x, q)$$

and

$$\nu_A(x, q) = \nu_A((x^{-1})^{-1}, q) \leq \nu_A(x^{-1}, q) \leq \nu_A(x, q)$$

and then  $\mu_A(x, q) = \mu_A(x^{-1}, q)$  and  $\nu_A(x, q) = \nu_A(x^{-1}, q)$ . Thus

$$A(x^{-1}, q) = (\mu_A(x^{-1}, q), \nu_A(x^{-1}, q)) = (\mu_A(x, q), \nu_A(x, q)) = A(x, q). \quad \square$$

**Proposition 3.3.** *Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  such that  $T$  and  $C$  be idempotent. Then*

$$A(e_G, q) \supseteq A(x, q)$$

for all  $x \in G$  and  $q \in Q$ .

*Proof.* Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  and  $x \in G$  and  $q \in Q$ . Then

$$\begin{aligned}\mu_A(e_G, q) &= \mu_A(xx^{-1}, q) \\ &\geq T(\mu_A(x, q), \mu_A(x^{-1}, q)) \\ &= T(\mu_A(x, q), \mu_A(x, q)) = \mu_A(x, q)\end{aligned}$$

and so

$$\mu_A(e_G, q) \geq \mu_A(x, q). \quad (a)$$

Also

$$\begin{aligned}\nu_A(e_G, q) &= \nu_A(xx^{-1}, q) \\ &\leq C(\nu_A(x, q), \nu_A(x^{-1}, q)) \\ &= C(\nu_A(x, q), \nu_A(x, q)) \\ &= \nu_A(x, q)\end{aligned}$$

and then

$$\nu_A(e_G, q) \leq \nu_A(x, q). \quad (b)$$

Thus from (a) and (b) we have that

$$A(e_G, q) = (\mu_A(e_G, q), \nu_A(e_G, q)) \supseteq (\mu_A(x, q), \nu_A(x, q)) = A(x, q). \quad \square$$

**Proposition 3.4.** Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$ . If  $T$  and  $C$  are idempotent and

$$A(xy^{-1}, q) = A(e_G, q),$$

then

$$A(x, q) = A(y, q)$$

for all  $x, y \in G$  and  $q \in Q$ .

*Proof.* Let  $x, y \in G$  and  $q \in Q$ . Then

$$\begin{aligned}\mu_A(x, q) &= \mu_A(xy^{-1}y, q) \\ &\geq T(\mu_A(xy^{-1}, q), \mu_A(y, q)) \\ &= T(\mu_A(e_G, q), \mu_A(y, q)) \\ &\geq T(\mu_A(y, q), \mu_A(y, q)) \\ &= \mu_A(y, q) = \mu_A(yx^{-1}x, q) \\ &\geq T(\mu_A(yx^{-1}, q), \mu_A(x, q)) \\ &= T(\mu_A((xy^{-1})^{-1}, q), \mu_A(x, q)) \\ &= T(\mu_A(xy^{-1}, q), \mu_A(x, q)) \\ &= T(\mu_A(e_G, q), \mu_A(x, q)) \\ &\geq T(\mu_A(x, q), \mu_A(x, q)) \\ &= \mu_A(x, q),\end{aligned}$$

thus

$$\mu_A(x, q) = \mu_A(y, q). \quad (a)$$

Also

$$\begin{aligned}
\nu_A(x, q) &= \nu_A(xy^{-1}y, q) \\
&\leq C(\nu_A(xy^{-1}, q), \nu_A(y, q)) \\
&= C(\nu_A(e_G, q), \nu_A(y, q)) \\
&\leq C(\nu_A(y, q), \nu_A(y, q)) \\
&= \nu_A(y, q) = \nu_A(yx^{-1}x, q) \\
&\leq C(\mu_A(yx^{-1}, q), \nu_A(x, q)) \\
&= C(\nu_A((xy^{-1})^{-1}, q), \nu_A(x, q)) \\
&= C(\nu_A(xy^{-1}, q), \nu_A(x, q)) \\
&= C(\nu_A(e_G, q), \nu_A(x, q)) \\
&\leq C(\nu_A(x, q), \nu_A(x, q)) \\
&= \nu_A(x, q),
\end{aligned}$$

then

$$\nu_A(x, q) = \nu_A(y, q). \quad (\text{b})$$

Therefore

$$A(x, q) = (\mu_A(x, q), \nu_A(x, q)) = (\mu_A(y, q), \nu_A(y, q)) = A(y, q). \quad \square$$

**Proposition 3.5.** *Let  $T$  and  $C$  be idempotent. Then*

$$A = (\mu_A, \nu_A) \in QIFSN(G)$$

*if and only if*

$$A(xy^{-1}, q) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q)))$$

*for all  $x, y \in G$  and  $q \in Q$ .*

*Proof.* Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  and  $x, y \in G, q \in Q$ . Then

$$\begin{aligned}
\mu_A(xy^{-1}, q) &\geq T(\mu_A(x, q), \mu_A(y^{-1}, q)) \\
&\geq T(\mu_A(x, q), \mu_A(y, q))
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(xy^{-1}, q) &\leq C(\nu_A(x, q), \nu_A(y^{-1}, q)) \\
&\leq C(\nu_A(x, q), \nu_A(y, q)),
\end{aligned}$$

and then

$$\begin{aligned}
A(xy^{-1}, q) &= (\mu_A(xy^{-1}, q), \nu_A(xy^{-1}, q)) \\
&\supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))).
\end{aligned}$$

Conversely, let

$$A(xy^{-1}, q) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))).$$

Then

$$\begin{aligned}
\mu_A(x^{-1}, q) &= \mu_A(e_G x^{-1}, q) \\
&\geq T(\mu_A(e_G, q), \mu_A(x, q)) \\
&\geq T(\mu_A(x, q), \mu_A(x, q)) \\
&= \mu_A(x, q)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x^{-1}, q) &= \nu_A(e_G x^{-1}, q) \\
&\leq C(\nu_A(e_G, q), \nu_A(x, q)) \\
&\leq C(\mu_A(x, q), \nu_A(x, q)) \\
&= \nu_A(x, q)
\end{aligned}$$

and then

$$A(x^{-1}, q) \supseteq A(x, q). \quad (1)$$

Also

$$\mu_A(xy, q) = \mu_A(x(y^{-1})^{-1}, q) \geq T(\mu_A(x, q), \mu_A(y^{-1}, q)) \geq T(\mu_A(x, q), \mu_A(y, q))$$

and

$$\nu_A(xy, q) = \nu_A(x(y^{-1})^{-1}, q) \leq C(\nu_A(x, q), \nu_A(y^{-1}, q)) \leq C(\nu_A(x, q), \nu_A(y, q))$$

Hence

$$A(xy, q) = (\mu_A(xy, q), \nu_A(xy, q)) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))). \quad (2)$$

Therefore from (1) and (2) we get that

$$A = (\mu_A, \nu_A) \in QIFSN(G). \quad \square$$

**Proposition 3.6.** Let  $\mu_A, \nu_A \in [0, 1]^{G \times Q}$  such that

$$A(e_G, q) = (1, 0)$$

and

$$A(xy^{-1}, q) \supseteq (T(\mu(x, q), \mu(y, q)), C(\mu(x, q), \mu(y, q)))$$

for all  $x, y \in G$  and  $q \in Q$ . Then

$$A = (\mu_A, \nu_A) \in QIFSN(G).$$

*Proof.* Let  $x, y \in G$  and  $q \in Q$ . Then

$$\mu_A(x^{-1}, q) = \mu_A(e_G x^{-1}, q) \geq T(\mu_A(e_G, q), \mu_A(x, q)) = T(1, \mu_A(x, q)) = \mu_A(x, q)$$

and

$$\nu_A(x^{-1}, q) = \nu_A(e_G x^{-1}, q) \leq C(\nu_A(e_G, q), \nu_A(x, q)) = C(0, \nu_A(x, q)) = \nu_A(x, q)$$

and so

$$A(x^{-1}, q) = (\mu_A(x^{-1}, q), \nu_A(x^{-1}, q)) \supseteq (\mu_A(x, q), \nu_A(x, q)) = A(x, q). \quad (1)$$



Also

$$\mu_A(xy, q) = \mu_A(x((y)^{-1})^{-1}, q) \geq T(\mu_A(x, q), \mu_A(y^{-1}, q)) \geq T(\mu_A(x, q), \mu_A(y, q))$$

and

$$\nu_A(xy, q) = \nu_A(x((y)^{-1})^{-1}, q) \leq C(\nu_A(x, q), \nu_A(y^{-1}, q)) \leq C(\nu_A(x, q), \nu_A(y, q))$$

and so

$$A(xy, q) = (\mu_A(xy, q), \nu_A(xy, q)) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))). \quad (2)$$

Therefore from (1) and (2) we will have

$$A = (\mu_A, \nu_A) \in QIFSN(G). \quad \square$$

**Proposition 3.7.** *If  $A = (\mu_A, \nu_A) \in QIFSN(G)$ , then*

$$H = \{x \in G \mid A(x, q) = (1, 0) \quad \forall q \in Q\}$$

*is a subgroup of  $G$ .*

*Proof.* Let  $x, y \in H$  and  $q \in Q$ . Then  $\mu_A(x, q) = \mu_A(y, q) = 1$  and  $\nu_A(x, q) = \nu_A(y, q) = 0$ . Since  $A = (\mu_A, \nu_A) \in QIFSN(G)$ , so

$$\mu_A(xy^{-1}, q) \geq T(\mu_A(x, q), \mu_A(y, q)) = T(1, 1) = 1$$

and this implies that  $\mu_A(xy^{-1}, q) = 1$ . Also

$$\nu_A(xy^{-1}, q) \leq C(\nu_A(x, q), \nu_A(y, q)) = C(0, 0) = 0$$

and so  $\nu_A(xy^{-1}, q) = 0$ . Then

$$A(xy^{-1}, q) = (\mu_A(xy^{-1}, q), \nu_A(xy^{-1}, q)) = (1, 0)$$

and then  $xy^{-1} \in H$  so from Proposition 2.3 we obtain that  $H$  will be a subgroup of  $G$ .  $\square$

**Proposition 3.8.** *Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  such that  $T$  and  $C$  be idempotent. Then*

$$H = \{x \in G \mid A(x, q) = A(e_G, q) \quad \forall q \in Q\}$$

*is a subgroup of  $G$ .*

*Proof.* Let  $x, y \in H$  and  $q \in Q$  then  $\mu_A(x, q) = \mu_A(y, q) = \mu_A(e_G, q)$  and  $\nu_A(x, q) = \nu_A(y, q) = \nu_A(e_G, q)$ . Since  $A = (\mu_A, \nu_A) \in QIFSN(G)$  so

$$\begin{aligned} \mu_A(xy^{-1}, q) &\geq T(\mu_A(x, q), \mu_A(y, q)) \\ &= T(\mu_A(e_G, q), \mu_A(e_G, q)) \\ &= \mu_A(e_G, q) \\ &\geq \mu_A(xy^{-1}, q) \end{aligned}$$

and so  $\mu_A(xy^{-1}, q) = \mu_A(e_G, q)$ .

Also

$$\begin{aligned}
\nu_A(xy^{-1}, q) &\leq C(\nu_A(x, q), \nu_A(y, q)) \\
&= C(\nu_A(e_G, q), \nu_A(e_G, q)) \\
&= \nu_A(e_G, q) \\
&\leq \nu_A(xy^{-1}, q),
\end{aligned}$$

then  $\nu_A(xy^{-1}, q) = \nu_A(e_G, q)$ . Therefore,

$$A(xy^{-1}, q) = (\mu_A(xy^{-1}, q), \nu_A(xy^{-1}, q)) = (\mu_A(e_G, q), \nu_A(e_G, q)) = A(e_G, q).$$

Thus  $xy^{-1} \in H$  and Proposition 2.3 give us that  $H$  is a subgroup of  $G$ . □

**Proposition 3.9.** *Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  and  $A(xy^{-1}, q) = (1, 0)$  then*

$$A(x, q) = A(y, q)$$

for all  $x, y \in G$  and  $q \in Q$ .

*Proof.* Assume that  $x, y \in G, q \in Q$ . As  $A \in QIFST(G)$ , so

$$\begin{aligned}
\mu_A(x, q) &= \mu_A(xy^{-1}y, q) \\
&\geq T(\mu_A(xy^{-1}, q), \mu_A(y, q)) = T(1, \mu_A(y, q)) = \mu_A(y, q) \\
&= \mu_A(y^{-1}, q) = \mu_A(x^{-1}xy^{-1}, q) \\
&\geq T(\mu_A(x^{-1}, q), \mu_A(xy^{-1}, q)) = T(\mu_A(x^{-1}, q), 1) \\
&= \mu_A(x^{-1}, q) = \mu_A(x, q),
\end{aligned}$$

hence

$$\mu_A(x, q) = \mu_A(y, q).$$

Also

$$\begin{aligned}
\nu_A(x, q) &= \nu_A(xy^{-1}y, q) \\
&\leq C(\nu_A(xy^{-1}, q), \nu_A(y, q)) = C(0, \nu_A(y, q)) = \nu_A(y, q) \\
&= \nu_A(y^{-1}, q) = \nu_A(x^{-1}xy^{-1}, q) \\
&\leq C(\nu_A(x^{-1}, q), \nu_A(xy^{-1}, q)) = C(\nu_A(x^{-1}, q), 0) \\
&= \nu_A(x^{-1}, q) = \nu_A(x, q),
\end{aligned}$$

thus

$$\nu_A(x, q) = \nu_A(y, q).$$

Now

$$A(x, q) = (\mu_A(x, q), \nu_A(x, q)) = (\mu_A(y, q), \nu_A(y, q)) = A(y, q). \quad \square$$

**Proposition 3.10.** Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$ . Then

$$A(xy, q) = A(yx, q)$$

if and only if

$$A(x, q) = A(y^{-1}xy, q)$$

for all  $x, y \in G$  and  $q \in Q$ .

*Proof.* Let  $x, y \in G, q \in Q$  and  $A(xy, q) = A(yx, q)$ . Then

$$\mu_A(y^{-1}xy, q) = \mu_A(y^{-1}(xy), q) = \mu_A(xyy^{-1}, q) = \mu_A(xe_G, q) = \mu_A(x, q)$$

and

$$\nu_A(y^{-1}xy, q) = \nu_A(y^{-1}(xy), q) = \nu_A(xyy^{-1}, q) = \nu_A(xe_G, q) = \nu_A(x, q)$$

and so

$$A(y^{-1}xy, q) = (\mu_A(y^{-1}xy, q), \nu_A(y^{-1}xy, q)) = (\mu_A(x, q), \nu_A(x, q)) = A(x, q).$$

Conversely, let  $A(x, q) = A(y^{-1}xy, q)$  then

$$\mu_A(xy, q) = \mu_A(x(yx)x^{-1}, q) = \mu_A(yx, q)$$

and

$$\nu_A(xy, q) = \nu_A(x(yx)x^{-1}, q) = \nu_A(yx, q)$$

and then

$$A(xy, q) = (\mu_A(xy, q), \nu_A(xy, q)) = (\mu_A(yx, q), \nu_A(yx, q)) = A(yx, q). \quad \square$$

**Proposition 3.11.** Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$ . If

$$A(xy^{-1}, q) = (0, 1),$$

then either

$$A(x, q) = (0, 1)$$

or

$$A(y, q) = (0, 1)$$

for all  $x, y \in G$  and  $q \in Q$ .

*Proof.* Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  then for all  $x, y \in G$  and  $q \in Q$  we obtain that

$$0 = \mu_A(xy^{-1}, q) \geq T(\mu_A(x, q), \mu_A(y, q))$$

and then either  $\mu_A(x, q) = 0$  or  $\mu_A(y, q) = 0$ . Also

$$1 = \nu_A(xy^{-1}, q) \leq C(\nu_A(x, q), \nu_A(y, q))$$

and then either  $\nu_A(x, q) = 1$  or  $\mu_A(y, q) = 1$ . Therefore either

$$A(x, q) = (\mu_A(x, q), \nu_A(x, q)) = (0, 1)$$

or

$$A(y, q) = (\mu_A(y, q), \nu_A(y, q)) = (0, 1). \quad \square$$

**Proposition 3.12.** *Let  $A = (\mu_A, \nu_A) \in QIFSN(G)$  and  $x, y \in G, q \in Q$ . If  $T$  and  $C$  be idempotent and  $A(x, q) \neq A(y, q)$ , then*

$$A(xy, q) = (T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))).$$

*Proof.* Let  $A(x, q) \supset A(y, q)$  then  $\mu_A(x, q) > \mu_A(y, q)$  and  $\nu_A(x, q) < \nu_A(y, q)$ .

As  $\mu_A(x, q) > \mu_A(y, q)$  for all  $x, y \in G$  and  $q \in Q$  so  $\mu_A(x, q) > \mu_A(xy, q)$  and so

$$\mu_A(y, q) = T(\mu_A(x, q), \mu_A(y, q))$$

and

$$\mu_A(xy, q) = T(\mu_A(x, q), \mu_A(xy, q)).$$

Now

$$\begin{aligned} \mu_A(xy, q) &\geq T(\mu_A(x, q), \mu_A(y, q)) \\ &= \mu_A(y, q) \\ &= \mu_A(x^{-1}xy, q) \\ &\geq T(\mu_A(x^{-1}, q), \mu_A(xy, q)) \\ &= T(\mu_A(x, q), \mu_A(xy, q)) \\ &= \mu_A(xy, q), \end{aligned}$$

and so

$$\mu_A(xy, q) = \mu_A(y, q) = T(\mu_A(x, q), \mu_A(y, q)).$$

Also since  $\nu_A(x, q) < \nu_A(y, q)$  for all  $x, y \in G$  and  $q \in Q$ , so  $\nu_A(x, q) < \nu_A(xy, q)$ , and then

$$\nu_A(y, q) = C(\nu_A(x, q), \nu_A(y, q))$$

and

$$\nu_A(xy, q) = C(\nu_A(x, q), \nu_A(xy, q)).$$

Now

$$\begin{aligned} \nu_A(xy, q) &\leq C(\nu_A(x, q), \nu_A(y, q)) \\ &= \nu_A(y, q) \\ &= \nu_A(x^{-1}xy, q) \\ &\leq C(\nu_A(x^{-1}, q), \nu_A(xy, q)) \\ &= C(\nu_A(x, q), \nu_A(xy, q)) \\ &= \nu_A(xy, q), \end{aligned}$$

and thus

$$\nu_A(xy, q) = \nu_A(y, q) = C(\nu_A(x, q), \nu_A(y, q)).$$

Therefore

$$A(xy, q) = (\mu_A(xy, q), \nu_A(xy, q)) = (T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q))). \quad \square$$

## Acknowledgements

It is our pleasant duty to thank referees for their useful suggestions which helped us to improve our manuscript.

## References

- [1] Abu Osman, M. T. (1987). On some products of fuzzy subgroups. *Fuzzy Sets and Systems*, 24, 79–86.
- [2] Anthony, J. M., & Sherwood, H. (1977). Fuzzy groups redefined. *Journal of Mathematical Analysis and Application*, 69, 124–130.
- [3] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.
- [4] Atanassov, K. T. (1994). New operations defined over the intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 61, 137–142.
- [5] Buckley, J. J., & Eslami, E. (2002). *An Introduction to Fuzzy Logic and Fuzzy Sets*. Springer-Verlag Berlin Heidelberg GmbH.
- [6] Hungerford, T. (2003). *Algebra*. Graduate Texts in Mathematics. Springer.
- [7] Malik, D. S., & Mordeson, J. N. (1995). *Fuzzy Commutative Algebra*. World Science Publishing Co.Pte.Ltd.
- [8] Rasuli, R. (2016). Fuzzy ideals of subtraction semigroups with respect to a  $t$ -norm and a  $t$ -conorm. *The Journal of Fuzzy Mathematics Los Angeles*, 24(4), 881–892.
- [9] Rasuli, R. (2016). Fuzzy modules over a  $t$ -norm. *International Journal of Open Problems in Computer Science and Mathematics*, 9(3), 12–18.
- [10] Rasuli, R. (2016). Fuzzy subrings over a  $t$ -norm. *The Journal of Fuzzy Mathematics Los Angeles*, 24(4), 995–1000.
- [11] Rasuli, R. (2016). Norms over intuitionistic fuzzy subrings and ideals of a ring. *Notes on Intuitionistic Fuzzy Sets*, 22(5), 72–83.
- [12] Rasuli, R. (2017). Norms over fuzzy Lie algebra. *Journal of New Theory*, 15, 32–38.
- [13] Rasuli, R. (2017). Fuzzy subgroups on direct product of groups over a  $t$ -norm. *Journal of Fuzzy Set Valued Analysis*, 3, 96–101.
- [14] Rasuli, R. (2017). Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms. *Journal of New Theory*, 18, 39–52.
- [15] Rasuli, R. (2017). *Intuitionistic Fuzzy Subrings and Ideals of a Ring under Norms*. LAP LAMBERT Academic Publishing, ISBN: 978-620-2-06926-7.

- [16] Rasuli, R. (2018). Characterization of  $Q$ -fuzzy subrings (anti  $Q$ -fuzzy subrings) with respect to a  $T$ -norm ( $T$ -conorms). *Journal of Information and Optimization Science*, 31, 1–11.
- [17] Rasuli, R. (2018).  $T$ -Fuzzy submodules of  $R \times M$ . *Journal of New Theory*, 22, 92–102.
- [18] Rasuli, R. (2018). Fuzzy subgroups over a  $T$ -norm. *Journal of Information and Optimization Science*. 39, 1757–1765.
- [19] Rasuli, R. (2018). Fuzzy sub-vector spaces and sub-bivector spaces under  $t$ -norms. *General Letters in Mathematics*, 5(1), 47–57.
- [20] Rasuli, R. (2019). Anti fuzzy submodules over a  $t$ -conorm and some of their properties. *The Journal of Fuzzy Mathematics Los Angeles*, 27, 229–236.
- [21] Rasuli, R. (2019). Artinian and Noetherian fuzzy rings. *International Journal of Open Problems in Computer Science and Mathematics*, 12(1), 86–92.
- [22] Rasuli, R. (2019). Fuzzy equivalence relation, fuzzy congruence relation and fuzzy normal subgroups on group  $G$  over  $t$ -norms. *Asian Journal of Fuzzy and Applied Mathematics*, 7, 14–28.
- [23] Rasuli, R. (2019). Norms over anti fuzzy  $G$ -submodules. *MathLAB Journal*, 2, 56–64.
- [24] Rasuli, R. (2019). Norms over bifuzzy bi-ideals with operators in semigroups. *Notes on Intuitionistic Fuzzy Sets*, 25(1), 1–11.
- [25] Rasuli, R. (2019). Norms over basic operations on intuitionistic fuzzy sets. *The Journal of Fuzzy Mathematics Los Angeles*, 27(3), 561–582.
- [26] Rasuli, R. (2019).  $T$ -fuzzy bi-ideals in semirings. *Earthline Journal of Mathematical Sciences*, 27(1), 241–263.
- [27] Rasuli, R. (2019). Norms over intuitionistic fuzzy vector spaces. *Algebra Letters*, 1(1), 1–19.
- [28] Rasuli, R. (2019). Some results of anti fuzzy subrings over  $t$ -conorms. *MathLAB Journal*, 1(4), 25–32.
- [29] Rasuli, R. (2020). Anti fuzzy equivalence relation on rings with respect to  $t$ -conorm  $C$ . *Earthline Journal of Mathematical Sciences*, 3(1), 1–19.
- [30] Rasuli, R. (2020). Anti fuzzy subbigroups of bigroups under  $t$ -conorms. *The Journal of Fuzzy Mathematics Los Angeles*, 28(1), 181–200.
- [31] Rasuli, R. (2020).  $t$ -Norms over fuzzy multigroups. *Earthline Journal of Mathematical Sciences*, 3(2), 207–228.
- [32] Rasuli, R. (2020). Anti  $Q$ -fuzzy subgroups under  $t$ -conorms. *Earthline Journal of Mathematical Sciences*, 4(1), 13–28.

- [33] Rasuli, R. (2020). Anti fuzzy congruence on product lattices with respect to  $S$ -norms. *The Second National Congress on Mathematics and Statistics*, Conbad Kavous University, 2020.
- [34] Rasuli, R. (2020). Direct product of fuzzy multigroups under  $t$ -norms. *Open Journal of Discrete Applied Mathematics*, 3(1), 75–85.
- [35] Rasuli, R. (2020). Level subsets and translations of  $QFST(G)$ . *MathLAB Journal*, 5(1), 1–11.
- [36] Rasuli, R. (2020). Conorms over anti fuzzy vector spaces. *Open Journal of Mathematical Sciences*, 4, 158–167.
- [37] Rasuli, R. (2020). Intuitionistic fuzzy subgroups with respect to norms  $(T, S)$ . *Engineering and Applied Science Letters (EASL)*, 3(2), 40–53.
- [38] Rasuli, R. (2020). Anti complex fuzzy subgroups under  $s$ -norms. *Engineering and Applied Science Letters (EASL)*, 3(4), 1–10.
- [39] Rasuli, R., & Moatamedi Nezhad, M. M. Characterization of fuzzy modules and anti fuzzy modules under norms. *The First International Conference on Basic Sciences*, Tehran, 21 October 2020.
- [40] Rasuli, R., & Moatamedi Nezhad, M. M. (2020). Fuzzy subrings and anti fuzzy subrings under norms. *The First International Conference on Basic Sciences*, Tehran, 21 October 2020.
- [41] Rasuli, R. (2021). Anti  $Q$ -fuzzy translations of anti  $Q$ -soft subgroups. *The Third National Conference on Management and Fuzzy Systems*, University of Eyvanekey, March 2021.
- [42] Rasuli, R. (2021). Conorms over conjugates and generalized characteristics of anti  $Q$ -fuzzy subgroups. *The Third National Conference on Management and Fuzzy Systems*, University of Eyvanekey, March 2021.
- [43] Rasuli, R. (2021). Fuzzy congruence on product lattices under  $T$ -norms. *Journal of Information and Optimization Sciences*, 42(2), 333–343.
- [44] Rasuli, R. (2021). Intuitionistic fuzzy congruences on product lattices under norms. *Journal of Interdisciplinary Mathematics*, 24(2), 1–24.
- [45] Rasuli, R. (2021). Conorms over level subsets and translations of anti  $Q$ -fuzzy subgroups. *International Journal of Mathematics and Computation*, 32(2), 55–67.
- [46] Rasuli, R., & Naraghi, H.  $T$ -Norms over  $Q$ -fuzzy subgroups of group. *Jordan Journal of Mathematics and Statistics*, 12, 1–13.
- [47] Rasuli, R., Moatamedi Nezhad, M., & Naraghi, H. (2020). Characterization of  $TF(G)$  and direct product of it. *The First National Conference on Soft Computing and Cognitive Science*, Conbad Kavous University, July 2020.

- [48] Rosenfled, A. (1971). Fuzzy groups. *Journal of Mathematical Analysis and Application*, 35, 512–517.
- [49] Solairaju, A., & Nagarajan, R. (2009). A new structure and constructions of  $Q$ -fuzzy group. *Advances in Fuzzy Mathematics*, 4, 23–29.
- [50] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.