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Measurable entropy of intuitionistic fuzzy dynamical systems

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Abstract:
We introduce the concept of intuitionistic fuzzy dynamical system by using the definition of intuitionistic fuzzy measure given in a recent paper. Also, we define the measurable entropy of intuitionistic fuzzy dynamical systems.

Keywords:
Intuitionistic fuzzy set, intuitionistic fuzzy measure, intuitionistic fuzzy partition, intuitionistic fuzzy dynamical system, entropy.

1 Introduction

The papers [10] and [11] can be considered as a starting point for an ergodic theory of fuzzy dynamical systems. In [7], [12]-[15] this theory is continued and developed. For example, in the very recent article [14] a general scheme for the introduction of the entropy of dynamical systems is proposed. In this paper we will prove that for intuitionistic fuzzy dynamical systems defined in the natural way this general scheme is inapplicable, but we can introduce the entropy of intuitionistic fuzzy dynamical systems with classical methods. Moreover, the entropy of (classical) dynamical systems (see, for example, [16], p. 73) and the entropy of fuzzy dynamical systems (see [11]) are obtained as particular cases.

In Section 2 we recall the concept of intuitionistic fuzzy measure introduced in [6] in a more general context. Section 3 contains some useful properties of the product of intuitionistic fuzzy sets and of the operation introduced on intuitionistic fuzzy sets with the help of the triangular norm defined by

$t_P(x,y) = xy$ and triangular conorm defined by $s_P(x,y) = x + y - xy$. The definition and the basic properties of finite intuitionistic fuzzy partitions are included in the Section 4. The last section is reserved for the introduction of the measurable entropy of intuitionistic fuzzy dynamical systems.

2 Intuitionistic fuzzy f-algebras and intuitionistic fuzzy f-measures

A triangular norm (conorm) is a function $f : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, monotone in each component and which satisfies the boundary condition $f(x,1) = x, (f(x,0) = x), \forall x \in [0,1]$. The associate triangular conorm (norm) is the function $f^c : [0,1] \times [0,1] \to [0,1]$ defined by $f^c(x,y) = 1 - f(1-x, 1-y), \forall x, y \in [0,1]$.

Let $X \neq \emptyset$ be a given set and we denote by $IFS(X)$ the set of all the intuitionistic fuzzy sets in X (see [1]).

The following expressions are defined in [1], [8] for all $A, B \in IFS(X)$, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$:

$$A \subseteq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \text{ for all } x \in X$$

$$A = B \text{ if and only if } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x) \text{ for all } x \in X$$

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$$

$$A\tilde{f}B = \{\langle x, f(\mu_A(x), \mu_B(x)), f^c(\nu_A(x), \nu_B(x))\rangle : x \in X\}$$

where f is a triangular norm or a triangular conorm.

If f is the triangular norm $f(x,y) = xy$ we obtain the product of intuitionistic fuzzy sets A and B (see [2]),

$$A \cdot B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle : x \in X\}$$

If f is the triangular conorm $f(x,y) = \min(x+y, 1)$ or the (associate) triangular norm $f(x,y) = \max(x+y-1, 0)$ we obtain the sum of intuitionistic fuzzy sets A and B,

$$A \oplus B = \{\langle x, \min(\mu_A(x) + \mu_B(x), 1), \max(\nu_A(x) + \nu_B(x) - 1, 0)\rangle : x \in X\}$$

and the conjunction of intuitionistic fuzzy sets A and B

$$A\&B = \{\langle x, \max(\mu_A(x) + \mu_B(x) - 1, 0), \min(\nu_A(x) + \nu_B(x), 1)\rangle : x \in X\},$$

respectively.

In [6] the limit of a sequence of intuitionistic fuzzy sets is defined and with the help of this concept the operations on $IFS(X)$ induced by triangular norms and triangular conorms are extended at countable case. Also, the disjointness of a family of intuitionistic fuzzy sets is given in the following way.

Definition 1 *([6])Let f be a triangular norm. A finite family $\{A_1, ..., A_n\} \subseteq IFS(X)$ is said to be f-disjoint if and only if for each $k \in \{1, ..., n\}$ we have*

$$\left(\widetilde{f^c}_{i \neq k} A_i\right) \tilde{f} A_k = \tilde{0}_X, \text{ where } \tilde{0}_X = \{\langle x, 0, 1\rangle : x \in X\}.$$

An infinite family $(A_n)_{n \in \mathbf{N}} \subseteq IFS(X)$ is called f-disjoint if and only if for each $n \in \mathbf{N}$ the finite family $\{A_1, ..., A_n\}$ is f-disjoint.

Remark 1 *Obviously each subfamily of an f-disjoint family is f-disjoint, too. Therefore $(A_n)_{n \in \mathbf{N}}$ f-disjoint implies $A_i f A_j = \tilde{0}_X, \forall i, j \in \mathbf{N}, i \neq j$.*

Similarly with the fuzzy situation (see [3] and [9]) we introduce the concept of intuitionistic fuzzy measure.

Definition 2 *(see [6]) Let f be a triangular norm. A subfamily $\mathcal{I} \subseteq IFS(X)$ which satisfies*

(i) $\tilde{1}_X \in \mathcal{I}$, *where* $\tilde{1}_X = \{\langle x, 1, 0\rangle : x \in X\}$;
(ii) $A \in \mathcal{I}$ *implies* $A^c \in \mathcal{I}$;
(iii) $A, B \in \mathcal{I}$ *implies* $A \cdot B \in \mathcal{I}$;
(iv) $(A_n)_{n \in \mathbf{N}} \subseteq \mathcal{I}$ *implies* $\widetilde{f^c}_{n \in \mathbf{N}} A_n \in \mathcal{I}$

will be called an intuitionistic fuzzy f-algebra on X. The pair $(X, \mathcal{I})$ will be called an intuitionistic fuzzy f-measurable space.

Definition 3 *([6]) A function $m : \mathcal{I} \to \mathbf{R}_+, \mathcal{I}$ being an intuitionistic fuzzy f-algebra on X, which satisfies the following conditions:*

(i) $m(\tilde{0}_X) = 0$;
(ii) $A, B \in \mathcal{I}$ *implies* $m\left(A\tilde{f}B\right) + m\left(A\widetilde{f^c}B\right) = m(A) + m(B)$;
(iii) $(A_n)_{n \in \mathbf{N}} \subseteq \mathcal{I}, A_n \subseteq A_{n+1}, \forall n \in \mathbf{N}$ *and* $\lim_{n \to \infty} A_n \in \mathcal{I}$ *implies*

$$\lim_{n \to \infty} m(A_n) = m\left(\lim_{n \to \infty} A_n\right),$$

is called an intuitionistic fuzzy f-measure on $\mathcal{I}$. The triplet $(X, \mathcal{I}, m)$ is called an intuitionistic fuzzy f-measure space.

For short, an intuitionistic fuzzy &-measure will be called intuitionistic fuzzy measure.

Remark 2 *If* $X = \{x_1, ..., x_N\}$ *and* $\mathcal{I} = IFS(X)$ *then the function* $m : \mathcal{I} \to \overline{\mathbf{R}}_+$ *defined by*

$$m(A) = \sum_{i=1}^{N} (1 - \mu_A(x_i) - \nu_A(x_i)),$$

where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, *is an intuitionistic fuzzy* f*-measure, for every triangular norm* f *which verifies* $f(x,y) + f^c(x,y) = x + y, \forall x, y \in [0,1]$. *Because* $m\left(\widetilde{0}_X\right) = m\left(\widetilde{1}_X\right) = 0$ *and* $m(\{\langle x, \alpha, \beta\rangle : x \in X\}) > 0, \forall \alpha, \beta \geq 0, \alpha + \beta < 1$ *we obtain that* m *is non-monotone.*

3 Properties of the product of intuitionistic fuzzy sets

In [3] some properties of the product of fuzzy sets are proved. The product of intuitionistic fuzzy sets has the analogous properties. We will need these properties in the sequel.

Theorem 4 *If* $\{A_1, ..., A_n\} \subseteq IFS(X)$ *is a &-disjoint family then*

$$(i) \quad C \cdot \left(\bigoplus_{k=1}^{n} A_k\right) = \bigoplus_{k=1}^{n} (C \cdot A_k)$$

$$(ii) \quad C \cdot \left(\mathop{\&}_{k=1}^{n} A_k\right) = \mathop{\&}_{k=1}^{n} (C \cdot A_k)$$

for every $C \in IFS(X)$.

Proof. Because the operations $\oplus$ and & are associative we prove the equalities only in the particular case $n = 2$.

(*i*) Let $A, B \in IFS(X), A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ and $A\&B = \widetilde{0}_X$, that is $\mu_{A\&B}(x) = 0, \forall x \in X$ and $\nu_{A\&B}(x) = 1, \forall x \in X$. This means that $\mu_A(x) + \mu_B(x) \leq 1$ and $\nu_A(x) + \nu_B(x) \geq 1$ for every $x \in X$.

Let $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in X\}$. We get

$$\begin{aligned} \mu_{C\cdot(A\oplus B)}(x) &= \mu_C(x)\mu_{A\oplus B}(x) = \mu_C(x)\min(\mu_A(x) + \mu_B(x), 1) = \\ &= \mu_C(x)\mu_A(x) + \mu_C(x)\mu_B(x), \forall x \in X \end{aligned}$$

and

$$\begin{aligned}\mu_{(C\cdot A)\oplus(C\cdot B)}(x) &= \min\left(\mu_C(x)\mu_A(x)+\mu_C(x)\mu_B(x),1\right) = \\ &= \mu_C(x)\mu_A(x)+\mu_C(x)\mu_B(x), \forall x\in X.\end{aligned}$$

We prove the equality between the degrees of non-membership

$$\begin{aligned}&\nu_{C\cdot(A\oplus B)}(x) = \\ &= \nu_C(x)+\nu_{A\oplus B}(x)-\nu_C(x)\nu_{A\oplus B}(x) = \\ &= \nu_C(x)+\nu_A(x)+\nu_B(x)-1-\nu_C(x)\left(\nu_A(x)+\nu_B(x)-1\right) = \\ &= 2\nu_C(x)+\nu_A(x)+\nu_B(x)-\nu_C(x)\nu_A(x)-\nu_C(x)\nu_B(x)-1, \forall x\in X,\end{aligned}$$

and

$$\begin{aligned}&\nu_{(C\cdot A)\oplus(C\cdot B)}(x) = \\ &= \max\left(\nu_C(x)\left(2-\nu_A(x)-\nu_B(x)\right)+\nu_A(x)+\nu_B(x)-1,0\right) = \\ &= 2\nu_C(x)+\nu_A(x)+\nu_B(x)-\nu_C(x)\nu_A(x)-\nu_C(x)\nu_B(x)-1, \forall x\in X,\end{aligned}$$

because $\nu_A(x)\le 1, \nu_B(x)\le 1$ and $\nu_A(x)+\nu_B(x)\ge 1, \forall x\in X$.

(ii) Analogously with (i) ■

Theorem 5 *If the family* $\{A_1,...,A_n\}\subseteq IFS(X)$ *is &-disjoint then* $\{A\cdot A_1,...,A\cdot A_n\}$ *is &-disjoint, for every* $A\in IFS(X)$.

Proof. Let $A\in IFS(X)$ and $k\in\{1,...,n\}$. By using the previous theorem we obtain

$$\begin{aligned}\left(\bigoplus_{i\neq k}(A\cdot A_i)\right)\&(A\cdot A_k) &= \left(A\cdot\left(\bigoplus_{i\neq k}A_i\right)\right)\&(A\cdot A_k) = \\ &= A\cdot\left(\left(\bigoplus_{i\neq k}A_i\right)\&A_k\right) = A\cdot\widetilde{0}_X = \widetilde{0}_X \; ■\end{aligned}$$

4 Finite intuitionistic fuzzy partitions

In this section we introduce the concept of intuitionistic fuzzy partition analogous with [10] in the case of fuzzy sets.

Definition 6 *A finite and &-disjoint family $P = \{A_1, ..., A_p\} \subseteq IFS(X)$ is a finite intuitionistic fuzzy partition of X if and only if*

$$\overset{p}{\underset{i=1}{\oplus}} A_i = \widetilde{1}_X.$$

We will denote by $IP(X)$ the class of all finite intuitionistic fuzzy partitions of X.

Definition 7 *Let $P, Q \in IP(X), P = \{A_1, ..., A_p\}$ and $Q = \{B_1, ..., B_q\}$. The partition Q is a refinement of P (we denote by $P \prec Q$) if and only if the following conditions hold*

(i) $q \geq p$;

(ii) There is a partition $\{I(1), ..., I(p)\}$ of the set $\{1, ...q\}$ such that

$$A_i = \underset{j \in I(i)}{\oplus} B_j \text{ and } A_i \& \left(\underset{j \notin I(i)}{\oplus} B_j \right) = \widetilde{0}_X, \forall i \in \{1, ..., p\}.$$

Let $P, Q \in IP(X), P = \{A_1, ..., A_p\}$ and $Q = \{B_1, ..., B_q\}$. The algebraic join $P \nabla Q$ of P and Q is defined by

$$P \nabla Q = \{A_i \cdot B_j : i \in \{1, ..., p\}, j \in \{1, ..., q\}\},$$

with lexicographic ordering.

Theorem 8 *If $P, Q \in IP(X)$ then $P \nabla Q \in IP(X), P \prec P \nabla Q$ and $Q \prec P \nabla Q$.*

Proof. From Theorem 4 and Definition 6 we obtain

$$\begin{aligned} \underset{\substack{i \in \{1, ..., p\} \\ j \in \{1, ..., q\}}}{\oplus} (A_i \cdot B_j) &= \overset{p}{\underset{i=1}{\oplus}} \left(\overset{q}{\underset{j=1}{\oplus}} (A_i \cdot B_j) \right) = \overset{p}{\underset{i=1}{\oplus}} \left(A_i \cdot \left(\overset{q}{\underset{j=1}{\oplus}} B_j \right) \right) = \\ &= \overset{p}{\underset{i=1}{\oplus}} \left(A_i \cdot \widetilde{1}_X \right) = \overset{p}{\underset{i=1}{\oplus}} A_i = \widetilde{1}_X. \end{aligned}$$

We prove that $\{A_i \cdot B_j : i \in \{1, ..., p\}, j \in \{1, ..., q\}\}$ is a &-disjoint family. If $r \in \{1, ..., p\}, s \in \{1, ..., q\}$ then

$$\begin{aligned} &\left(\underset{i \neq r, j \neq s}{\oplus} (A_i \cdot B_j) \right) \& A_r \cdot B_s = \left(\underset{i \neq r}{\oplus} A_i \right) \cdot \left(\underset{j \neq s}{\oplus} B_j \right) \& (A_r \cdot B_s) \subseteq \\ &\subseteq \left(\underset{i \neq r}{\oplus} A_i \right) \cdot \left(\overset{q}{\underset{j=1}{\oplus}} B_j \right) \& (A_r \cdot B_s) \subseteq \left(\underset{i \neq r}{\oplus} A_i \right) \& A_r = \widetilde{0}_X. \end{aligned}$$

Therefore $P \bigtriangledown Q \in IP(X)$.

The condition (i) in Definition 7 is verified because $pq \geq p$. For to prove (ii), we choose $I(i) = \{(i,1), ..., (i,q)\}, \forall i \in \{1, ..., p\}$. It is evident that $\{I(1), ..., I(p)\}$ is a partition of the set$\{(1,1), (1,2), ..., (p,q)\}$ and in addition

$$A_i = A_i \cdot \tilde{1}_X = A_i \cdot \left(\overset{q}{\underset{j=1}{\oplus}} B_j \right) = \overset{q}{\underset{j=1}{\oplus}} (A_i \cdot B_j) = \underset{(i,j)\in I(i)}{\oplus} (A_i \cdot B_j)$$

and

$$\begin{aligned} A_i \& \left(\underset{(i,j)\notin I(i)}{\oplus} A_i \cdot B_j \right) &= A_i \& \left(\underset{(k,j)\neq(i,j)}{\oplus} (A_k \cdot B_j) \right) = \\ &= A_i \& \left(\left(\underset{k\neq i}{\oplus} A_k \right) \left(\overset{q}{\underset{j=1}{\oplus}} B_j \right) \right) = A_i \& \left(\underset{k\neq i}{\oplus} A_k \right) = \tilde{0}_X, \end{aligned}$$

therefore $P \prec P \bigtriangledown Q$ ■

We consider $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\} \in IFS(X)$ and a function $T : X \nrightarrow X$. Let $T^n, n \geq 1$, be the n iterate of T. The inverse image of A by T^n is $T^{-n}A \in IFS(X)$ defined as

$$T^{-n}A = \left\{ \left\langle x, T^{-n}\mu_A(x), T^{-n}\nu_A(x) \right\rangle : x \in X \right\},$$

where $T^{-n}\mu_A(x) = \mu_A(T^n(x))$ and $T^{-n}\nu_A(x) = \nu_A(T^n(x))$ (see [4]).

Theorem 9 *If f is a triangular norm or a triangular conorm and $T : X \rightarrow X$ then*

$$\left(T^{-n}A\right) \tilde{f} \left(T^{-n}B\right) = T^{-n}\left(A\tilde{f}B\right), \forall A, B \in IFS(X).$$

Proof. By using the similar relation obtained for fuzzy sets (see [3], Proposition 3.3) we have

$$\begin{aligned} &\left(T^{-n}A\right) \tilde{f} \left(T^{-n}B\right) = \\ &= \left\{ \left\langle x, f\left(T^{-n}\mu_A(x), T^{-n}\mu_B(x)\right), f^c\left(T^{-n}\nu_A(x), T^{-n}\nu_B(x)\right) \right\rangle : x \in X \right\} = \\ &= \left\{ \left\langle x, T^{-n}\left(f\left(\mu_A(x), \mu_B(x)\right)\right), T^{-n}\left(f^c\left(\nu_A(x), \nu_B(x)\right)\right) \right\rangle : x \in X \right\} = \\ &= \left\{ \left\langle x, T^{-n}\mu_{A\tilde{f}B}(x), T^{-n}\nu_{A\tilde{f}B}(x) \right\rangle : x \in X \right\} = \\ &= T^{-n}\left(A\tilde{f}B\right) \end{aligned}$$

if $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle : x \in X\}$ ■

The extension to the finite case of this result implies the following theorems.

Theorem 10 *If* $P=\{A_1,...,A_p\}\in IP(X)$ *and* $T:X\to X$ *then* $T^{-n}P=\{T^{-n}A_1,...,T^{-n}A_p\}\in IP(X),\forall n\in\mathbf{N},n\geq 1.$

Proof. By the hypothesis, $\overset{p}{\underset{i=1}{\oplus}} A_i=\widetilde{1}_X$ and $A_k\&\left(\underset{i\neq k}{\oplus} A_i\right)=\widetilde{0}_X,\forall k\in$ $\in\{1,...,p\}$. The previous theorem implies

$$\overset{p}{\underset{i=1}{\oplus}}\left(T^{-n}A_i\right)=T^{-n}\left(\overset{p}{\underset{i=1}{\oplus}} A_i\right)=T^{-n}(\widetilde{1}_X)=\widetilde{1}_X$$

and

$$\begin{aligned} T^{-n}A_k\&\left(\underset{i\neq k}{\oplus} T^{-n}A_i\right) &= T^{-n}A_k\&T^{-n}\left(\underset{i\neq k}{\oplus} A_i\right)= \\ &= T^{-n}\left(A_k\&\left(\underset{i\neq k}{\oplus} A_i\right)\right)=T^{-n}\left(\widetilde{0}_X\right)=\widetilde{0}_X, \end{aligned}$$

for every $k\in\{1,...,p\}$ ■

Theorem 11 *Let* $T:X\to X$ *and* $P,Q\in IP(X)$. *We have*

(*i*) $T^{-n}(P\nabla Q)=T^{-n}P\nabla T^{-n}Q,\forall n\in\mathbf{N},n\geq 1;$
(*ii*) *If* $P\prec Q$ *then* $T^{-n}P\prec T^{-n}Q,\forall n\in\mathbf{N},n\geq 1.$

Proof. Let $P=\{A_1,...,A_p\}$ and $Q=\{B_1,...,B_q\}$.

(*i*) Theorem 9 implies $T^{-n}(A\cdot B)=(T^{-n}A)\cdot(T^{-n}B)$, for every $A,B\in IFS(X)$ and $n\geq 1$. We get

$$\begin{aligned} &T^{-n}(P\bigtriangledown Q)= \\ &= \left\{T^{-n}(A_1\cdot B_1),T^{-n}(A_1\cdot B_2)...,T^{-n}(A_p\cdot B_q)\right\}= \\ &= \left\{\left(T^{-n}A_1\right)\cdot\left(T^{-n}B_1\right),\left(T^{-n}A_1\right)\cdot\left(T^{-n}B_2\right)...,\left(T^{-n}A_p\right)\cdot\left(T^{-n}B_q\right)\right\}= \\ &= T^{-n}P\bigtriangledown T^{-n}Q. \end{aligned}$$

(*ii*) Because $P\prec Q$ there exists a partition $\{I(1),...,I(p)\}$ of $\{1,...,q\}$ such that

$$A_i=\underset{j\in I(i)}{\oplus} B_j \text{ and } A_i\&\left(\underset{j\notin I(i)}{\oplus} B_j\right)=\widetilde{0}_X,$$

for every $i\in\{1,...,p\}$. By using again Theorem 9 we obtain

$$\left(T^{-n}A_i\right)\&\left(\underset{j\notin I(i)}{\oplus} T^{-n}B_j\right)=T^{-n}\left(A_i\&\left(\underset{j\notin I(i)}{\oplus} B_j\right)\right)=T^{-n}\left(\widetilde{0}_X\right)=\widetilde{0}_X,$$

and

$$\left(T^{-n}A_i\right)=T^{-n}\left(\underset{j\in I(i)}{\oplus} B_j\right)=\underset{j\in I(i)}{\oplus}\left(T^{-n}B_j\right),\forall i\in\{1,...,p\},$$

therefore $T^{-n}P\prec T^{-n}Q$ ■

5 Entropy of intuitionistic fuzzy dynamical systems

In [14] a general scheme for the introduction of the entropy of dynamical systems is proposed. One of necessary properties of the measure which is not verified for intuitionistic fuzzy measures in the sense of Definition 3 is

$$m(A) = m\left(\tilde{1}_X\right) \text{ implies } m(A \cdot B) = m(B).$$

Indeed, if m is that in Remark 2 and $A = \{\langle x, \alpha, 1-\alpha\rangle : x \in X\}, \alpha \in (0,1), B = \left\{\left\langle x, \frac{1}{4}, \frac{1}{4}\right\rangle : x \in X\right\}$ then $m(A) = m\left(\tilde{1}_X\right) = 0$ but $m(A \cdot B) = \frac{N\alpha}{2} \neq \frac{N}{2} = m(B)$.

In this section we will introduce the concept of intuitionistic fuzzy dynamical system and we will define the entropy of an intuitionistic fuzzy dynamical system following the classical method (see for example [16]).

Definition 12 *The function $T : X \to X$ is called an intuitionistic fuzzy measure preserving transformation of an intuitionistic fuzzy measure space $(X, \mathcal{I}, m)$ if and only if the following conditions are fulfilled:*

(i) *T is $\mathcal{I}$-measurable, that is $T^{-1}A \in \mathcal{I}$ for every $A \in \mathcal{I}$;*
(ii) *$m(T^{-1}A) = m(A)$ for every $A \in \mathcal{I}$.*

Definition 13 *An intuitionistic fuzzy dynamical system is a complex $(X, \mathcal{I}, m, T)$, where $(X, \mathcal{I}, m)$ is an intuitionistic fuzzy measure space and $T : X \to X$ is an intuitionistic fuzzy measure preserving transformation of $(X, \mathcal{I}, m)$.*

Firstly, we introduce the entropy of a finite $\mathcal{I}$-measurable intuitionistic fuzzy partition.

Definition 14 *Let $(X, \mathcal{I}, m)$ be an intuitionistic fuzzy measure space. The intuitionistic fuzzy partition $P = \{A_1, ..., A_p\}$ so that $A_i \in \mathcal{I}$ for every $i \in \{1, ..., p\}$ is said to be $\mathcal{I}$-measurable.*

$IP_{\mathcal{I}}(X)$ denotes the family of all finite $\mathcal{I}$-measurable intuitionistic fuzzy partitions of X.

It is evident that $P \in IP_{\mathcal{I}}(X)$ and $T : X \to X$ $\mathcal{I}$-measurable imply $T^{-n}P \in IP_{\mathcal{I}}(X), \forall n \in \mathbf{N}, n \geq 1$.

We consider $(X, \mathcal{I}, m)$ an intuitionistic fuzzy measure space with $m(A) \leq 1$ for every $A \in \mathcal{I}$ and $m(X) = 1$.

Definition 15 *Let* $P = \{A_1, ..., A_p\} \in IP_{\mathcal{I}}(X)$. *We define the entropy of* P *by*

$$H(P) = -\sum_{i=1}^{p} \varphi\left(m\left(A_i\right)\right)$$

where $\varphi : [0, \infty] \to \mathbf{R}$ *is the Shannon's function* $\varphi(x) = \begin{cases} x \log_2 x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

For to introduce the entropy of intuitionistic fuzzy dynamical systems we will prove some properties of the entropy of intuitionistic fuzzy partitions.

Theorem 16 *Let* $P, Q \in IP_{\mathcal{I}}(X)$ *and* T *be an intuitionistic fuzzy measure preserving transformation. Then*

(*i*) $H(P) \geq 0$;
(*ii*) *If* $P \prec Q$ *then* $H(P) \leq H(Q)$;
(*iii*) $H(P \nabla Q) \leq H(P) + H(Q)$;
(*iv*) $H(T^{-n}P) = H(P), \forall n \in \mathbf{N}, n \geq 1$.

Proof. (*i*) It is evident.

(*ii*) Let $P, Q \in IP_{\mathcal{I}}(X), P = \{A_1, ..., A_p\}$ and $Q = \{B_1, ..., B_q\}$. The hypothesis implies the existence of a partition $\{I(1), ..., I(p)\}$ of $\{1, ...q\}$ such that

$$A_i = \bigoplus_{j \in I(i)} B_j \text{ and } A_i \& \left(\bigoplus_{j \notin I(i)} B_j\right) = \tilde{0}_X,$$

for every $i \in \{1, ..., p\}$. This means that

$$\begin{aligned} H(P) &= \\ &= -\sum_{i=1}^{p} m(A_i) \log m(A_i) = -\sum_{i=1}^{p} m\left(\bigoplus_{j \in I(i)} B_j\right) \log m\left(\bigoplus_{j \in I(i)} B_j\right) = \\ &= -\sum_{i=1}^{p} \sum_{j \in I(i)} m(B_j) \log \sum_{j \in I(i)} m(B_j) \leq -\sum_{i=1}^{p} \sum_{j \in I(i)} m(B_j) \log m(B_j) = \\ &= -\sum_{j=1}^{q} m(B_j) \log m(B_j) = H(Q). \end{aligned}$$

(*iii*) Let $P, Q \in IP_{\mathcal{I}}(X)$ as above. We obtain successively

$$\begin{aligned} H(P \nabla Q) &= -\sum_{i=1}^{p} \sum_{j=1}^{q} m(A_i \cdot B_j) \log m(A_i \cdot B_j) = \\ &= -\sum_{i=1}^{p} m(A_i) \log m(A_i) - \sum_{i=1}^{p} \sum_{j=1}^{q} m(A_i \cdot B_j) \log \frac{m(A_i \cdot B_j)}{m(A_i)} = \end{aligned}$$

(because $\sum_{j=1}^{q} m(A_i \cdot B_j) = m(A_i), \forall i \in \{1,...,p\}$)

$$= -\sum_{i=1}^{p} m(A_i)\log m(A_i) - \sum_{j=1}^{q}\sum_{i=1}^{p} m(A_i)\frac{m(A_i \cdot B_j)}{m(A_i)} \log \frac{m(A_i \cdot B_j)}{m(A_i)} \leq$$

(because the function φ is concave and $\sum_{i=1}^{p} m(A_i) = 1$)

$$\leq -\sum_{i=1}^{p} m(A_i)\log m(A_i) -$$

$$-\sum_{j=1}^{q}\left(\sum_{i=1}^{p} m(A_i)\frac{m(A_i \cdot B_j)}{m(A_i)}\right)\left(\log \sum_{i=1}^{p} m(A_i)\frac{m(A_i \cdot B_j)}{m(A_i)}\right) =$$

(because $\sum_{i=1}^{p} m(A_i \cdot B_j) = m\left(\bigoplus_{i=1}^{p} (A_i \cdot B_j)\right) = m\left(B_j\right), \forall j \in \{1,...,q\}$)

$= H(P)+H(Q)$.

(*iv*) The Definition 12 implies that $m\left(T^{-n}A\right) = m(A), \forall A \in \mathcal{I}, \forall n \geq 1$. For every $P \in IP_{\mathcal{I}}(X)$, $P = \{A_1,...,A_p\}$ we have

$$\begin{aligned} H\left(T^{-n}P\right) &= -\sum_{i=1}^{p} m(T^{-n}A_i)\log m(T^{-n}A_i) = \\ &= -\sum_{i=1}^{p} m(A_i)\log m(A_i) = H(P) \; \blacksquare \end{aligned}$$

Remark 3 *If we define the conditional entropy we can give other properties of the entropy of intuitionistic fuzzy partitions (see [11] for the fuzzy case).*

Theorem 17 *For every $P \in IP_{\mathcal{I}}(X)$ there exists* $\lim_{n\to\infty} \frac{1}{n} H\left(\mathop{\nabla}_{i=0}^{n-1} T^{-i}P\right)$.

Proof. Indeed, putting $a_n = H\left(\mathop{\nabla}_{i=0}^{n-1} T^{-i}P\right)$, the properties (*iii*) and (*iv*) in Theorem 16 imply the inequality $a_{n+m} \leq a_n + a_m, \forall n, m \in \mathbf{N}, n, m \geq 1$. Because $a_n \geq 0, \forall n \in \mathbf{N}, n \geq 1$ (by the same theorem) we obtain the existence of the limit $\lim_{n\to\infty} \frac{1}{n} a_n$ (see [16], p.79) $\blacksquare$

The last result permits to introduce the measurable entropy of intuitionistic fuzzy dynamical systems.

Definition 18 *Let $(X, \mathcal{I}, m, T)$ be an intuitionistic fuzzy dynamical system. For every $P \in IP_{\mathcal{I}}(X)$ we define*

$$h(P,T) = \lim_{n\to\infty} \frac{1}{n} H\left(\mathop{\nabla}_{i=0}^{n-1} T^{-i}P\right)$$

and then the entropy of the intuitionistic fuzzy dynamical system by

$$H(T) = \sup\left\{h(P,T) : P \in IP_{\mathcal{I}}(X)\right\}.$$

Remark 4 *We can deduce the basic properties of the entropy of intuitionistic fuzzy dynamical systems (for fuzzy case, see [11]). For example, the entropy is an isomorphism invariant for intuitionistic fuzzy dynamical systems.*

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