Notes on Intuitionistic Fuzzy Sets
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A note about almost uniform convergence on D-poset of intuitionistic fuzzy sets

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This paper is dedicated to Prof. Krassimir T. Atanassov on the occasion of his 70th birthday

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Abstract: The aim of this contribution is studying the almost uniform convergence on D-poset of intuitionistic fuzzy sets. We prove the connection between almost everywhere convergence of random variables in Kolmogorov probability space and almost uniform convergence of observables in the mentioned D-poset. We define a product operation on D-poset of intuitionistic fuzzy sets and prove the existence of a joint observable, too.
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1 Introduction

In paper [5] we defined the almost uniform convergence for a sequence of intuitionistic fuzzy observables $(x_n)_1^\infty$ and we proved a variation of Egorov's theorem. We showed the connection between almost everywhere convergence of random variables in Kolmogorov probability space and almost uniform convergence of intuitionistic fuzzy observables, too. Recall that the intuitionistic fuzzy observable x is a mapping from Borel sets $\mathcal{B}(R)$ to the family of all IF-events on $(\Omega, \mathcal{S})$ denoted by $\mathcal{F}$. An intuitionistic fuzzy event is an intuitionistic fuzzy set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable and $\mu_A + \nu_A \leq 1_\Omega$ (see [2, 3, 9]). Remark that in year 2023 we celebrated the 40-th anniversary of the invention of the concept and theory of intuitionistic fuzzy sets by K. T. Atanassov in the paper [1]. In paper [4] we studied a more general situation. We formulated three definitions of almost uniform convergence in the MV-algebra of intutionistic fuzzy sets $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$. In the third definition of almost uniform convergence, we used a partial binary operation $\ominus$ defined by

$$\mathbf{A} \ominus \mathbf{B} = ((\mu_A - \mu_B) \vee 0_\Omega, (\nu_A - \nu_B + 1_\Omega) \wedge 1_\Omega)$$

for $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{M}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{M}$ and $\mathbf{B} \leq \mathbf{A}$. As $\mathbf{A} \ominus \mathbf{B} = \mathbf{A} \odot \neg\mathbf{B}$, then MV-algebra of intuitionistic fuzzy sets $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ is a D-poset. Recall that the binary operations $\oplus$ and $\odot$ are given by

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \end{aligned}$$

and an unary operation $\neg$ is given by the formula $\neg\mathbf{A} = (1_\Omega - \mu_A, 1_\Omega - \nu_A)$ for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}$. Therefore, some procedures useful in probability theory on MV-algebras are applicable in a more general structure, the so called D-poset. Recall that D-posets include not only MV-algebras, but also orthomodular lattices.

In this contribution, we will study the D-poset of intuitionistic fuzzy sets and an almost uniform convergence of observables in this structure. We will prove the connection between almost everywhere convergence of random variables in the Kolmogorov probability space and almost uniform convergence of observables, too. We are inspired by paper [8], where B. Riečan formulated an almost uniform convergence for general D-posets. Remark that in a whole text we use a notation "IF" in short as the phrase "intuitionistic fuzzy".

2 D-poset of intuitionistic fuzzy sets

In this section, we study the basic notions from probability theory on D-poset of IF-sets. Recall that the notion of D-poset was introduced by F. Chovanec and F. Kôpka, see the paper [7].

Definition 2.1 ([7]). *Let D be a partially ordered set with the greatest element 1_D and with a partial binary operation $\ominus$ such that $b \ominus a$ is defined if and only if $a \leq b$ and satisfying the following conditions:*

(i) if $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$;

(ii) if $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$;

A structure $(D, \leq, \ominus, 1_D)$ is called a poset with a difference, i.e., D-poset.

Example 2.1. *Let* $(\Omega, \mathcal{S})$ *be a measurable space,* $\mathcal{D}$ *be the family of all pairs* $\mathbf{A} = (\mu_A, \nu_A)$*, where* $\mu_A, \nu_A : \Omega \to [0,1]$ *are* $\mathcal{S}$*-measurable functions. Let* $\leq$ *be a partial ordering on* $\mathcal{D}$ *such that* $\mathbf{A} \leq \mathbf{B}$ *if and only if* $\mu_A \leq \mu_B, \nu_A \geq \nu_B$ *for each* $\mathbf{A}, \mathbf{B} \in \mathcal{D}$*. A partial binary operation* $\ominus$ *defined by the formula*

$$\mathbf{B} \ominus \mathbf{A} = (\mu_B - \mu_A, \nu_B - \nu_A + 1_\Omega)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{D}$*,* $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{D}$*,* $\mathbf{A} \leq \mathbf{B}$ *is a difference on* $\mathcal{D}$*. The greatest element in* $\mathcal{D}$ *is* $(1_\Omega, 0_\Omega)$*. Then the system* $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ *is a D-poset.*

Example 2.2 ([7])**.** *Let* X *be a non-empty set and let* $\mathcal{F}$ *be the family of all real functions* $f : X \to [0,1]$*. Let* $\leq$ *be a partial ordering on* $\mathcal{F}$ *such that* $f \leq g$ *if and only if* $f(t) \leq g(t)$ *for each* $t \in X$*. Let* $\phi : [0,1] \to [0,\infty)$ *be a strongly increasing continuous function such that* $\phi(0) = 0$*. A partial binary operation* $\ominus$ *defined by the formula*

$$(g \ominus f)(t) = \phi^{-1}\big(\phi(g(t)) - \phi(f(t))\big)$$

for each $f, g \in \mathcal{F}$*,* $f \leq g$*,* $t \in X$*, is a difference on* $\mathcal{F}$*. Then the system* $(\mathcal{F}, \leq, \ominus, 1_X)$ *is a D-poset (i.e., an D-poset of fuzzy sets).*

***Remark* 2.1.** *It is evident that the element* $1_D \ominus 1_D$ *is the least element in a D-poset* D*, i.e.,* $0_D = 1_D \ominus 1_D$*, see [10]. Therefore,* $(0_\Omega, 1_\Omega)$ *is the least element in the D-poset* $\mathcal{D}$ *constructed in Example 2.1. Really,* $(1_\Omega, 0_\Omega) \ominus (1_\Omega, 0_\Omega) = (0_\Omega, 1_\Omega)$*.*

Now we introduce the notion of a state and an observable in a D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$, constructed in Example 2.1.

Definition 2.2. *Let* $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ *be the D-poset constructed in Example 2.1. A state on a D-poset* $\mathcal{D}$ *is a mapping* $m : \mathcal{D} \to [0,1]$ *satisfying the following conditions:*

(i) $m((1_\Omega, 0_\Omega)) = 1$*;*

(ii) if $\mathbf{A}, \mathbf{B} \in \mathcal{D}$*,* $\mathbf{A} \leq \mathbf{B}$*, then* $m(\mathbf{B} \ominus \mathbf{A}) = m(\mathbf{B}) - m(\mathbf{A})$*;*

(iii) if $\mathbf{A}, \mathbf{A}_i \in \mathcal{D}$*,* $i \in N$*,* $\mathbf{A}_i \nearrow \mathbf{A}$*, then* $m(\mathbf{A}_i) \nearrow m(\mathbf{A})$*.*

An n*-dimensional observable on a D-poset* $\mathcal{D}$ *is a mapping* $x : \mathcal{B}(R^n) \to \mathcal{D}$ *satisfying the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$*;*

(ii) if $A, B \in \mathcal{B}(R^n)$*,* $A \subset B$*, then* $x(A) \leq x(B)$ *and* $x(B \setminus A) = x(B) \ominus x(A)$*;*

(iii) if $A, A_i \in \mathcal{B}(R^n)$*,* $i \in N$*,* $A_i \nearrow A$*, then* $x(A_i) \nearrow x(A)$*.*

When $n = 1$*, we say that* x *is an observable.*

Since an observable $x : \mathcal{B}(R) \to \mathcal{D}$ corresponds to a random variable $\xi : \Omega \to R$, the joint observable corresponds to a random vector $T = (\xi, \eta)$. Similarly, the joint observable can be defined as a morphism $h : \mathcal{B}(R^2) \to \mathcal{D}$. First, we need to define product operation on a D-poset $\mathcal{D}$ of IF-sets. In paper [6], F. Kôpka introduced a meet function as a generalized product on D-poset D. The meet function covers a product on the MV-algebras, on the D-posets and on the effect algebras.

Definition 2.3. *We say that a commutative and an associative binary operation $\bullet$ on a D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ is product if it satisfying the following conditions for every $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}$:*

(i) $(1_\Omega, 0_\Omega) \bullet \mathbf{A} = \mathbf{A}$;

(ii) if $\mathbf{A} \leq \mathbf{B}$, *then* $\mathbf{C} \bullet \mathbf{A} \leq \mathbf{C} \bullet \mathbf{B}$ *and* $\mathbf{C} \bullet (\mathbf{B} \ominus \mathbf{A}) = \mathbf{C} \bullet \mathbf{B} \ominus \mathbf{C} \bullet \mathbf{A}$.

Now we show an example of product operation on a D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ constructed in Example 2.1.

Theorem 2.1. *The operation $*$ defined by*

$$\mathbf{A} * \mathbf{B} = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big) = \big(\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B\big).$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{D}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{D}$ is product operation on a D-poset $\mathcal{D}$. There the operation $\cdot$ is a classical multiplication.

Proof. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}$. Evidently, the operation $*$ is commutative and associative. Moreover,

(i) $(1_\Omega, 0_\Omega) * \mathbf{A} = (1_\Omega \cdot \mu_A, 0_\Omega + \nu_A - 0_\Omega \cdot \nu_A) = (\mu_A, \nu_A) = \mathbf{A}$.

(ii) If $\mathbf{A} \leq \mathbf{B}$, i.e., $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$, then we have $\mu_C \cdot \mu_A \leq \mu_C \cdot \mu_B$ and

$$1_\Omega - (1_\Omega - \nu_C) \cdot (1_\Omega - \nu_A) \geq 1_\Omega - (1_\Omega - \nu_C) \cdot (1_\Omega - \nu_B).$$

But

$$\begin{aligned}
\mathbf{C} * \mathbf{A} &= \big(\mu_C \cdot \mu_A, 1_\Omega - (1_\Omega - \nu_C) \cdot (1_\Omega - \nu_A)\big) = && (1)\\
&= \big(\mu_C \cdot \mu_A, \nu_C + \nu_A - \nu_C \cdot \nu_A\big), \\
\mathbf{C} * \mathbf{B} &= \big(\mu_C \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_C) \cdot (1_\Omega - \nu_B)\big) = && (2)\\
&= \big(\mu_C \cdot \mu_B, \nu_C + \nu_B - \nu_C \cdot \nu_B\big), \\
\mathbf{C} * (\mathbf{B} \ominus \mathbf{A}) &= (\mu_C \cdot \mu_B - \mu_C \cdot \mu_A, \nu_B - \nu_A - \nu_C \cdot \nu_B + \nu_C \cdot \nu_A + 1_\Omega), && (3)\\
\mathbf{C} * \mathbf{B} \ominus \mathbf{C} * \mathbf{A} &= (\mu_C \cdot \mu_B - \mu_C \cdot \mu_A, \nu_B - \nu_A - \nu_C \cdot \nu_B + \nu_C \cdot \nu_A + 1_\Omega). && (4)
\end{aligned}$$

Finally, we have $\mathbf{C} * \mathbf{A} \leq \mathbf{C} * \mathbf{B}$ by (1), (2) and $\mathbf{C} * (\mathbf{B} \ominus \mathbf{A}) = \mathbf{C} * \mathbf{B} \ominus \mathbf{C} * \mathbf{A}$ by (3), (4). □

Definition 2.4. *Let $x, y : \mathcal{B}(R) \to \mathcal{D}$ be two observables of a D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ constructed in Example 2.1. The joint observable of the observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{D}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$;

(ii) if $A, B \in \mathcal{B}(R^2)$, $A \subset B$, then $h(A) \le h(B)$ and $h(B \setminus A) = h(B) \ominus h(A)$;

(iii) if $A, A_i \in \mathcal{B}(R^2)$, $i \in N$, $A_i \nearrow A$, then $h(A_i) \nearrow h(A)$.

(iv) $h(C \times D) = x(C) * y(D)$ for each $C, D \in \mathcal{B}(R)$.

Theorem 2.2. *For each two observables $x, y : \mathcal{B}(R) \to \mathcal{D}$ there exists their joint observable.*

Proof. Put $x(A) = (x^\flat(A), 1 - x^\sharp(A))$, $y(B) = (y^\flat(B), 1 - y^\sharp(B))$ for each $A \in \mathcal{B}(R)$. Then $x^\flat, x^\sharp, y^\flat, y^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T}$ is the family of all $\mathcal{S}$-measurable functions from Ω to [0,1]. We want to construct $h(K) = (h^\flat(K), 1 - h^\sharp(K))$.

Fix $\omega \in \Omega$ and put $\mu(A) = x^\flat(A)(\omega)$, $\nu(B) = y^\flat(B)(\omega)$. It is not difficult to prove that $\mu, \nu : \mathcal{B}(R) \to [0,1]$ are probability measures. Let $\mu \times \nu : \mathcal{B}(R^2) \to [0,1]$ be the product of measures and define $h^\flat(K)(\omega) = \mu \times \nu(K)$. Then $h^\flat : \mathcal{B}(R^2) \to \mathcal{T}$.

If $C, D \in \mathcal{B}(R)$, then $h^\flat(C \times D)(\omega) = \mu \times \nu(C \times D) = \mu(C) \cdot \nu(D) = x^\flat(C)(\omega) \cdot y^\flat(D)(\omega)$, hence $h^\flat(C \times D) = x^\flat(C) \cdot y^\flat(D)$. Similarly, $h^\sharp : \mathcal{B}(R^2) \to \mathcal{T}$ can be constructed so that $h^\sharp(C \times D) = x^\sharp(C) \cdot y^\sharp(D)$.

Put $h(A) = (h^\flat(A), 1 - h^\sharp(A))$, for $A \in \mathcal{B}(R^2)$. By Theorem 2.1 we have

$$\begin{aligned} x(C) * y(D) &= (x^\flat(C), 1 - x^\sharp(C)) * (y^\flat(D), 1 - y^\sharp(D)) = \\ &= (x^\flat(C) \cdot y^\flat(D), 1 - x^\sharp(C) \cdot y^\sharp(D)) = \\ &= (h^\flat(C \times D), 1 - h^\sharp(C \times D)) = h(C \times D). \end{aligned}$$

for each $C, D \in \mathcal{B}(R)$. □

The following theorem makes a statement about the finite compatibility of a sequence of observables in the D-poset of intuitionistic fuzzy sets.

Theorem 2.3. *Let $(\mathcal{D}, \le, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1. Then each sequence $(x_n)_1^\infty$ of observables is finitely compatible in the following sense:*

For every finite, non-empty set $J \subset N$ there exists a mapping $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{D}$ such that the following conditions hold:

(i) $h_J(R^{|J|}) = (1_\Omega, 0_\Omega)$;

(ii) if $A, B \in \mathcal{B}(R^{|J|})$, $A \subset B$, then $h_J(A) \le h_J(B)$ and $h_J(B \setminus A) = h_J(B) \ominus h_J(A)$;

(iii) if $A, A_i \in \mathcal{B}(R^{|J|})$, $i \in N$, $A_i \nearrow A$, then $h_J(A_i) \nearrow h_J(A)$.

(iv) if $J_1 \subset J_2$, then $h_{J_2}\big(\pi_{J_2,\, J_1}^{-1}(A)\big) = h_{J_1}(A)$ for each $A \in \mathcal{B}(R^{|J_1|})$, where $\pi_{J_2,\, J_1} : R^{|J_2|} \to R^{|J_1|}$ is the projection;

(v) if $J = \{t_1, \dots, t_k\}$ and $A_1, \dots, A_k \in \mathcal{B}(R)$, then

$$h_J(A_1 \times \cdots \times A_k) = x_{t_1}(A_1) * \cdots * x_{t_k}(A_k).$$

Proof. Using Theorem 2.2 to each $n \in N$, there exists the mapping $h_n : \mathcal{B}(R^n) \to \mathcal{D}$ called a joint observable such that the conditions (i), (ii), (iii) hold and for each $A_1, \dots, A_k \in \mathcal{B}(R)$

$$h_n(A_1 \times \cdots \times A_k) = x_1(A_1) * \cdots * x_n(A_n).$$

If $J = \{t_1, \dots, t_k\}$, then put $h_J = h_{t_k} \circ \pi_{I,\,J}^{-1}$, where $I = \{1, \dots, t_k\}$. □

Now we define the notion of compatibility of observables in the D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$.

Definition 2.5. *Let $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1. We say that a sequence $(x_n)_1^\infty$ of observables on D-poset $\mathcal{D}$ is compatible, if there exists an observable $x : \mathcal{B}(R) \to \mathcal{D}$ and a sequence $(f_n)_1^\infty$ of Borel measurable functions $f_n : R \to R$ such that*

$$x_n = x \circ f_n^{-1}$$

for each $n \in N$.

3 Almost uniform convergence in D-poset of IF-sets

In this section, we will study an almost uniform convergence in D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ constructed in Example 2.1. Every such MV-algebra M with a partial binary operation $\ominus$ given by formula the $b \ominus a = b \odot \neg a$ is a D-poset (see [10]). We will try to apply the results for MV-algebras to more general structures.

Now we formulate two definitions of almost uniform convergence for observables in D-poset of IF-sets.

Definition 3.1. *Let $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1 and m be a state on D-poset $\mathcal{D}$. We say that the sequence $(y_n)_1^\infty$ of the observables converges m-almost uniformly to 0, if*

$$\begin{aligned} &\forall \alpha > 0 \;\; \exists\, \mathbf{A} \in \mathcal{D} :\; m(\mathbf{A}) > 1 - \alpha, \\ &\forall \beta > 0 \;\; \exists\, k \in N \;\; \forall n \geq k :\; \mathbf{A} \leq y_n\big((-\beta, \beta)\big). \end{aligned}$$

Definition 3.2. *Let $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1 and m be a state on D-poset $\mathcal{D}$. We say that the sequence $(y_n)_1^\infty$ of the observables converges m-almost uniformly to 0, if*

$\forall \alpha > 0 \;\exists\, \mathbf{B} \in \mathcal{D} :\; m(\mathbf{B}) > 1 - \alpha,$

$\forall \beta > 0 \;\exists\, k \in N \;\forall n \geq k \;\exists\, \mathbf{C}_n \in \mathcal{D},\; m(\mathbf{C}_n) < \alpha,\; \mathbf{C}_n \leq \mathbf{C}_{n+1} \leq \mathbf{B} :\; \mathbf{B} \ominus \mathbf{C}_n \leq y_n\big((-\beta, \beta)\big).$

In the following two theorems, we work with an almost everywhere and an almost uniform convergence of random variables. We used Egorov's theorem for random variables: *Let $(\Omega, \mathcal{S}, P)$ be a probability space and $(\xi_n)_1^\infty$ be a sequence of random variables. If a sequence $(\xi_n)_1^\infty$ converges P-almost everywhere to 0, then the sequence $(\xi_n)_1^\infty$ converges almost uniformly to 0.* Remark that the sequence of random variables $(\xi_n)_1^\infty$ converges to 0 almost uniformly on A, if for every $\alpha > 0$ there exists a measurable set A such that $P(A) > 1 - \alpha$ and such that for every $\beta > 0$ there exists k such that $A \subset \{t \in \Omega : |\xi_n(t)| < \beta\}$ for every $n \geq k$.

Theorem 3.1. *Let $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1 and m be a state on D-poset $\mathcal{D}$. Let $(y_n)_1^\infty$ be a sequence of compatible observables in D-poset $\mathcal{D}$, i.e., $y_n = y \circ f_n^{-1}$, where $f_n : R \to R$ are Borel measurable functions. If a sequence $(f_n)_1^\infty$ of Borel functions converges m_y-almost everywhere to 0, then the sequence $(y_n)_1^\infty$ of observables converges m-almost uniformly to 0.*

Proof. Remark that $m_y : \mathcal{B}(R) \to [0,1]$ defined by $m_y(A) = m\big(y(A)\big)$ is a probability measure. Since the sequence $(f_n)_1^\infty$ of Borel functions converges m_y-almost everywhere to 0, then by Egorov's theorem the sequence $(f_n)_1^\infty$ converges m_y-almost uniformly to 0. It means that for every $\alpha > 0$ there exists a set $A \in \mathcal{B}(R)$ such that $m_y(A) > 1 - \alpha$ and such that for every $\beta > 0$ there exists k such that $A \subset f_n^{-1}\big((-\beta, \beta)\big)$ for every $n \geq k$.

Put $\mathbf{A} = y(A)$. Then $m(\mathbf{A}) = m\big(y(A)\big) = m_y(A) > 1 - \alpha$ and for $n \geq k$ we have

$$\mathbf{A} = y(A) \leq y \circ f_n^{-1}\big((-\beta, \beta)\big) = y_n\big((-\beta, \beta)\big).$$

Therefore, the sequence $(y_n)_1^\infty$ of compatible observables in the D-poset $\mathcal{D}$ converges m-almost uniformly to 0. □

We can define a function of several observables in D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ with the help of a joint observable.

Definition 3.3. *Let $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ be the D-poset constructed in Example 2.1. Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{D}$ be observables, h_n their joint observable and $g_n : R^n \to R$ a Borel measurable function. Then the observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{D}$ is defined by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

Now we explain the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$, where R^N is a space of all sequences $(t_i)_1^\infty$ of real numbers. As a cylinder we understand a set $C \subset R^N$ given by

$$C = \{(t_i)_1^\infty \in R^N : (t_1, \ldots, t_n) \in A\},$$

where $n \in N$ and $A \in \mathcal{B}(R^n)$. By $\mathcal{C}$ we denote the family of all cylinders in R^N and by $\sigma(\mathcal{C})$ the corresponding σ-algebra generated by $\mathcal{C}$. Such a cylinder C can be expressed in the form $C = \pi_n^{-1}(A)$, where a mapping $\pi_n : R^N \to R^n$ is the n-th coordinate random vector given by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$, then

$$\mathcal{C} = \{\pi_n^{-1}(A) \mid n \in N, A \in \mathcal{B}(R^n)\}.$$

Therefore, there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\pi_n^{-1}(A)) = P_n(A) = m\big(h_n(A)\big)$$

for each $A \in \mathcal{B}(R^n)$, where $h_n : \mathcal{B}(R) \to \mathcal{D}$ is a joint observable of observables $x_1, \ldots, x_n$ on D-poset $\mathcal{D}$. Hence, we can define the random variable $\xi_n : R^N \to R$ with respect to $\sigma(\mathcal{C})$ by $\xi_n((t_i)_1^\infty) = t_n$. Next we show two proofs of the following theorem, using both definitions of almost uniform convergence on D-poset of IF-sets $\mathcal{D}$.

Theorem 3.2. *Let $(x_i)_1^\infty$ be a sequence of observables in the D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ with product $*$ and m be a state. Let $(\xi_n)_1^\infty$ be the sequence of corresponding random variables defined on Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$. Let $(g_n)_1^\infty$ be a sequence of Borel measurable functions, $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1, \ldots, \xi_n)\big)_1^\infty$ converges P-almost everywhere to 0, then the sequence $\big(g_n(x_1, \ldots, x_n)\big)_1^\infty$ converges m-almost uniformly to 0.*

First Proof. Let $h_n : \mathcal{B}(R^n) \to \mathcal{D}$ be the joint observable of $x_1, \ldots, x_n$ and $\pi_n : R^N \to R^n$ be the n-th coordinate random vector defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$. Hence the observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{D}$ is given by $y_n = h_n \circ g_n^{-1}$ and the random variable $\eta_n = g_n(\xi_1, \ldots, \xi_n) : R^N \to R$ is defined by $\eta_n = g_n \circ \pi_n$.

Let the sequence $(\eta_n)_1^\infty$ converges P-almost everywhere to 0. By the Egorov's theorem the sequence $(\eta_n)_1^\infty$ converges P-almost uniformly to 0 in the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$. Then, by definition, for every $\alpha > 0$ there exists $A \in \sigma(\mathcal{C})$ such that $P(A) > 1 - \alpha$ and such that for every $\beta > 0$ there exists k such that $A \subset \eta_n^{-1}\big((-\beta, \beta)\big)$ for every $n \geq k$. Since $A \in \sigma(\mathcal{C})$, then there exist $n \in N$ and $B \in \mathcal{B}(R^n)$ such that $A = \pi_n^{-1}(B)$. But $A \subset \eta_n^{-1}\big((-\beta, \beta)\big)$. Therefore,

$$\pi_n^{-1}(B) \subset \{(t_i)_1^\infty \in R^N : (t_1, \ldots, t_n) \in g_n^{-1}\big((-\beta, \beta)\big)\};$$

i.e.,

$$\pi_n^{-1}(B) \subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big).$$

Put $\mathbf{A} = h_n(B)$. Then,

$$m(\mathbf{A}) = m\big(h_n(B)\big) = P\big(\pi_n^{-1}(B)\big) = P(A) > 1 - \alpha$$

and

$$\begin{aligned} \pi_n^{-1}(B) &\subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ h_n\Big(\pi_n^{-1}(B)\Big) &\leq h_n\Big(\pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big)\Big), \\ h_n(B) &\leq h_n\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ \mathbf{A} &\leq y_n\big((-\beta, \beta)\big). \end{aligned}$$

Hence, the sequence $(y_n)_1^\infty$ converges m-almost uniformly to 0 in the D-poset $\mathcal{D}$. This completes the first proof. □

Second Proof. Let $h_n : \mathcal{B}(R^n) \to \mathcal{D}$ be the joint observable of $x_1, \ldots, x_n$ and $\pi_n : R^N \to R^n$ be the n-th coordinate random vector defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$. Hence the observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{D}$ is given by $y_n = h_n \circ g_n^{-1}$ and the random variable $\eta_n = g_n(\xi_1, \ldots, \xi_n) : R^N \to R$ is defined by $\eta_n = g_n \circ \pi_n$.

Let the sequence $(\eta_n)_1^\infty$ converges P-almost everywhere to 0. By the Egorov's theorem the sequence $(\eta_n)_1^\infty$ converges P-almost uniformly to 0 in the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$. Then, by definition, for every $\alpha > 0$ there exists $A \in \sigma(\mathcal{C})$ such that $P(A) > 1 - \frac{\alpha}{2}$ and such that for every $\beta > 0$ there exists k such that $A \subset \eta_n^{-1}\big((-\beta, \beta)\big)$ for every $n \geq k$. By the approximation theorem there exists $B \in \sigma(\mathcal{C})$ such that $P(A \triangle B) < \frac{\alpha}{2}$. Evidently,

$$P(B) \geq P(A) - P(A \setminus B) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha.$$

Fix n and put

$$B_n = \{(t_j)_1^\infty \in B : \exists i, i \leq n, \big|\eta_i((t_j)_1^\infty)\big| \geq \beta\}.$$

Since $B, B_n \in \sigma(\mathcal{C})$, then there exist $n \in N$ and $C, C_n \in \mathcal{B}(R^n)$ such that $B = \pi_n^{-1}(C)$, $B_n = \pi_n^{-1}(C_n)$. As $B \setminus B_n \subset \eta_n^{-1}\big((-\beta, \beta)\big)$, then we have

$$\pi_n^{-1}(C) \setminus \pi_n^{-1}(C_n) \subset \{(t_j)_1^\infty \in R^N : (t_1, \ldots, t_n) \in g_n^{-1}\big((-\beta, \beta)\big)\}$$

i.e.,

$$\pi_n^{-1}(C) \setminus \pi_n^{-1}(C_n) \subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big).$$

Put $\mathbf{B} = h_n(C)$, $\mathbf{C}_n = h_n(C_n)$. Then,

$$m(\mathbf{B}) = m\big(h_n(C)\big) = P\big(\pi_n^{-1}(C)\big) = P(B) > 1 - \alpha$$

and

$$m(\mathbf{C}_n) = m\big(h_n(C_n)\big) = P\big(\pi_n^{-1}(C_n)\big) = P(B_n) \leq P(B \setminus A) < \alpha.$$

Denote

$$\begin{aligned} \mathbf{C}_n &= h_n(C_n) = h_{n+1}\big(\pi_{n+1,n}^{-1}(C_n)\big), \\ \mathbf{C}_{n+1} &= h_{n+1}(C_{n+1}) = h_{n+1}\big(\pi_{n+1,n+1}^{-1}(C_{n+1})\big), \\ \mathbf{B} &= h_n(C) = h_{n+1}\big(\pi_{n+1,n}^{-1}(C)\big). \end{aligned}$$

Then

$$\begin{gathered} B_n \subset B_{n+1} \subset B \\ \pi_n^{-1}(C_n) \subset \pi_{n+1}^{-1}(C_{n+1}) \subset \pi_n^{-1}(C), \\ \pi_{n+1}^{-1}\Big(\pi_{n+1,n}^{-1}(C_n)\Big) \subset \pi_{n+1}^{-1}\Big(\pi_{n+1,n+1}^{-1}(C_{n+1})\Big) \subset \pi_{n+1}^{-1}\Big(\pi_{n+1,n}^{-1}(C)\Big), \\ \pi_{n+1,n}^{-1}(C_n) \subset \pi_{n+1,n+1}^{-1}(C_{n+1}) \subset \pi_{n+1,n}^{-1}(C), \\ h_{n+1}\Big(\pi_{n+1,n}^{-1}(C_n)\Big) \leq h_{n+1}\Big(\pi_{n+1,n+1}^{-1}(C_{n+1})\Big) \leq h_{n+1}\Big(\pi_{n+1,n}^{-1}(C)\Big), \\ h_n(C_n) \leq h_{n+1}(C_{n+1}) \leq h_n(C), \\ \mathbf{C}_n \leq \mathbf{C}_{n+1} \leq \mathbf{B} \end{gathered}$$

and

$$\begin{aligned} B \setminus B_n &\subset \eta_n^{-1}\big((-\beta, \beta)\big), \\ \pi_n^{-1}(C) \setminus \pi_n^{-1}(C_n) &\subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ h_n\Big(\pi_n^{-1}(C) \setminus \pi_n^{-1}(C_n)\Big) &\leq h_n\bigg(\pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big)\bigg), \\ h_n\big(\pi_n^{-1}(C)\big) \ominus h_n\big(\pi_n^{-1}(C_n)\big) &\leq h_n\bigg(\pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big)\bigg), \\ h_n(C) \ominus h_n(C_n) &\leq h_n\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ \mathbf{B} \ominus \mathbf{C}_n &\leq y_n\big((-\beta, \beta)\big). \end{aligned}$$

Hence, the sequence $(y_n)_1^\infty$ converges m-almost uniformly to 0 in D-poset $\mathcal{D}$. □

4 Conclusion

The paper is concerned in the probability theory on the D-poset $(\mathcal{D}, \leq, \ominus, (1_\Omega, 0_\Omega))$ constructed in Example 2.1. We defined product operation and a joint observable on D-poset $\mathcal{D}$ and proved its existence. We formulated two definitions of m-almost uniformly convergence for a sequence of observables in the D-poset $\mathcal{D}$. We showed the connection between an almost everywhere convergence of random variables in Kolmogorov probability space and an almost uniformly convergence of observables in D-poset $\mathcal{D}$.

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