Notes on Intuitionistic Fuzzy Sets
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Divergence measures on intuitionistic fuzzy sets

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Abstract: The basic study of fuzzy sets theory was introduced by Lotfi Zadeh in 1965. Many authors investigated possibilities how two fuzzy sets can be compared and the most common kind of measures used in the mathematical literature are dissimilarity measures. The previous approach to the dissimilarities is too restrictive, because the third axiom in the definition of dissimilarity measure assumes the inclusion relation between fuzzy sets. While there exist many pairs of fuzzy sets, which are incomparable to each other with respect to the inclusion relation. Therefore we need some new concept for measuring a difference between fuzzy sets so that it could be applied for arbitrary fuzzy sets. We focus on the special class of so called local divergences. In the next part we discuss the divergences defined on more general objects, namely intuitionistic fuzzy sets. In this case we define the local property modified to this object. We discuss also the relation of usual divergences between fuzzy sets to the divergences between intuitionistic fuzzy sets.
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1 Introduction

Triangular norms have been introduced into the mathematical literature by Karl Menger in 1942. Triangular norms and conorms are operations which generalize the conjunction and disjunction in fuzzy logic. They were originally used to generalize the triangle inequality from classical metric spaces to probabilistic metric spaces. In the original axioms for triangular norms no

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associativity was required. Theory of continuous t-norms has two rather independent roots, namely, the field of functional equations and the theory of topological semigroups. The full characterization of continuous Archimedean t-norms by means of additive generators has been done after 1960 by Ling and Schweizer and Sklar.

Triangular norms will be mentioned in this section. These functions are useful for modeling a conjunction in fuzzy logic and intersection of fuzzy sets.

The **triangular norm** (t-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(a,b) = T(b,a)$, for all $a, b \in [0,1]$ (commutativity),

(T2) $T(T(a,b),c) = T(a,T(b,c))$, for all $a, b, c \in [0,1]$ (associativity),

(T3) $b \leq c \Rightarrow T(a,b) \leq T(a,c)$, for all $a, b, c \in [0,1]$ (monotonicity),

(T4) $T(a,1) = a$, for all $a \in [0,1]$ (boundary condition).

Therefore, the function T is a monotone, associative and commutative operation defined on $[0,1] \times [0,1]$ with neutral element 1. Some important examples of t-norms, so-called basic t-norms, are the following:

- Minimum t-norm: $T_M(a,b) = \min(a,b)$, for all $a, b \in [0,1]$,
- Product t-norm: $T_P(a,b) = a \cdot b$, for all $a, b \in [0,1]$,
- Łukasiewicz t-norm: $T_L(a,b) = \max(a+b-1,0)$, for all $a, b \in [0,1]$,
- Drastic t-norm:
$$T_D(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1. \\ 0, & \text{otherwise .} \end{cases}$$

For these basic t-norms, it holds that $T_D \leq T_L \leq T_P \leq T_M$. In fact, for any t-norm T, it is fulfilled that $T_D \leq T \leq T_M$.

Moreover, for any t-norm T the following properties are fulfilled:

- $T(a,0) = 0$, for all $a \in [0,1]$,
- $T(a,b) \leq \min\{a,b\}$, for all $a, b \in [0,1]$.

To show these relations we conclude:

- from the monotonicity condition (T3) and the boundary condition (T4) we have that: $0 \leq T(a,0) \leq T(1,0) = 0$, for all $a \in [0,1]$, and therefore $T(a,0) = T(0,a) = 0$,
- using again the conditions (T3) and (T4) we have that: $T(a,b) \leq T(a,1) = a$ and $T(a,b) \leq T(1,b) = b$, simultaneously, for all $a, b \in [0,1]$.

Changing the neutral element from 1 to 0, we obtain the triangular conorm (t-conorm), a function used for modeling a disjunction in fuzzy logic and union of fuzzy sets.

The **triangular conorm** (t-conorm) is a function $S : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(S1) $S(a,b) = S(b,a)$, for all $a, b \in [0,1]$ (commutativity),

(S2) $S(S(a,b),c) = S(a, S(b,c))$, for all $a, b, c \in [0,1]$ (associativity),

(S3) $b \leq c \Rightarrow S(a,b) \leq S(a,c)$, for all $a, b, c \in [0,1]$ (monotonicity),

(S4) $S(a,0) = a$, for all $a \in [0,1]$ (boundary condition).

Thus, a t-conorm has all properties of triangular norms as monotonicity, associativity and commutativity, but it differs only in the neutral element 0.

Similarly, for any t-conorm S the following properties are fulfilled:

- $S(a,1) = 1$, for all $a \in [0,1]$,
- $S(a,b) \geq \max\{a,b\}$, for all $a, b \in [0,1]$.

The t-norm T and t-conorm S are dual if and only if for each $a, b \in [0,1]$ the equation $T(a,b) = 1 - S(1-a, 1-b)$ is fulfilled.

For each previous example of basic t-norm we can consider its dual basic t-conorm as follows:

- Maximum t-conorm: $S_M(a,b) = \max(a,b)$, for all $a, b \in [0,1]$,
- Probabilistic sum: $S_P(a,b) = a + b - a \cdot b$, for all $a, b \in [0,1]$,
- Łukasiewicz t-conorm: $S_L(a,b) = \min(a+b, 1)$, for all $a, b \in [0,1]$,
- Drastic t-conorm:

$$S_D(a,b) = \begin{cases} \max\{a,b\}, & \text{if } \min\{a,b\} = 0. \\ 1, & \text{otherwise }. \end{cases}$$

For the basic t-conorms it holds: $S_D \geq S_L \geq S_P \geq S_M$ and for any t-conorm S it holds: $S_D \geq S \geq S_M$.

Let $A, B \in \mathcal{F}(X)$. Given a t-norm T and a t-conorm S,

- the intersection of A and B with respect to T is defined as the fuzzy set whose membership function is $A \cap_T B(x) = T(A(x), B(x))$, for all $x \in X$;
- the union of A and B with respect to S is defined as the fuzzy set whose membership function is $A \cup_S B(x) = S(A(x), B(x))$, for all $x \in X$.

Thus, we can denote by (X, T, S) the triple formed by the universe with the t-norm and the t-conorm defining the intersection and the union, respectively.

Usually we write shortly $\cup, \cap$ instead of $\cup_S, \cap_T$ if it is clear what triple (X, T, S) is considered.

2 Intuitionistic fuzzy sets (IFSs)

Intuitionistic fuzzy sets (IFSs) have been introduced by K. Atanassov in 1983 [2]. For each point in the universe X a degree of membership and a degree of non-membership are assigned. More formally, Atanassov defined an **intuitionistic fuzzy set** (IF-set) as follows:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where μ_A and ν_A are membership (non-membership) functions $\mu_A, \nu_A : X \to [0, 1]$, such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$ and $\mu_A(x), \nu_A(x)$ are membership and non-membership degrees, respectively, of the element $x \in X$ to the set A. The family of all intuitionistic fuzzy sets defined on the universe X will be denoted by symbol $IFS(X)$.

The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation index. The lack of knowledge on the membership of an element $x \in X$ to the intuitionistic fuzzy set A is expressed by function π.

Clearly, IF-sets are one possible generalizations of the fuzzy sets. Each fuzzy set can be considered as a special case of an IF-set, such that $\nu_A(x) = 1 - \mu_A(x)$ and $\pi_A(x) = 0$. Moreover, each IF-set can be presented as an interval-valued fuzzy set since for each element $x \in X$ the following interval $[\mu_A(x), 1 - \nu_A(x)]$ can be associated.

Let A, B be IF-sets and $T(S)$ be the triangular norm (conorm). Then their union, intersection and complement will be defined in the following way:

(i) union of A and B:

$$A \cup B = \{(x, \mu_{A \cup B}(x), \nu_{A \cap B}(x)) \mid x \in X\},$$

where $\mu_{A \cup B}(x) = S(\mu_A(x), \mu_B(x))$ and $\nu_{A \cap B}(x) = T(\nu_A(x), \nu_B(x))$.

(ii) intersection of A and B:

$$A \cap B = \{(x, \mu_{A \cap B}(x), \nu_{A \cup B}(x)) \mid x \in X\},$$

where $\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A \cup B}(x) = S(\nu_A(x), \nu_B(x))$.

(iii) complement of A:

$$A^c = \{(x, \mu_{A^c}(x), \nu_{A^c}(x)) \mid x \in X\},$$

where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$.

The possible orderings of two IF-sets A and B can be introduced in the following way:

$$A \leq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \text{ for all } x \in X,$$

$$A \preceq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \text{ for all } x \in X,$$

$$A = B \text{ if and only if } A \leq B \text{ and } B \leq A.$$

3 Divergence measures for intuitionistic fuzzy sets (IFS)

In the framework of fuzzy set theory, we can find in the literature several measures of comparison between fuzzy sets. In 1996, Bouchon-Meunier ([3]) tried to define a general measure of comparison for fuzzy sets. Since more measures for comparing fuzzy sets have been introduced (see, among many others, [1, 13, 14]). A nice study about that can be found in [5]. Among all them, the most usual measures of comparison are dissimilarities ([7]). More precisely:

A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a **dissimilarity measure** if it satisfies the following axioms:

(Diss.1) $D(A, A) = 0$ for every $A \in \mathcal{F}(X)$.

(Diss.2) $D(A, B) = D(B, A)$ for every $A, B \in \mathcal{F}(X)$.

(Diss.3) For every $A, B, C \in \mathcal{F}(X)$ such that for $A \subseteq B \subseteq C$, it holds that $D(A, C) \geq \max(D(A, B), D(B, C))$.

There are several examples of dissimilarities. As this definition is not too restrictive, it is possible to define a counterintuitive measure of comparison for which the above axioms hold. The restriction associated to this definition is given by the fact that the requirement in Axiom (Diss.3) is only given for sets such that $A \subseteq B \subseteq C$, but there are a lot of sets which are not comparable with respect to $\subseteq$ and therefore, nothing is required for them. Thus, we need a concept where the restriction about "proximity" are given for any set.

In order to overcome this problem, another measure of comparison between fuzzy sets was proposed in [10], the divergence measure, which satisfies the following natural properties:

- it becomes zero when the two sets coincide,
- it is a non-negative and symmetric function,
- it decreases when the two subsets become "more similar" in some sense.

While it is easy to formulate the first and the second conditions analytically, the third one depends on the formalization of the concept of "more similar". We base our approach on the fact that if we add (in the sense of union) a subset C to both fuzzy subsets A, B, we obtain two subsets which are closer to each other; the same for the intersection. So we propose the following:

Let (X, T, S) be a triple with X a universe and T and S any t-norm and t-conorm, respectively. A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a **divergence measure** with respect to (X, T, S) if and only if for all $A, B \in \mathcal{F}(X)$, D satisfies the following conditions ([9]):

(Div.1) $D(A, A) = 0$;

(Div.2) $D(A, B) = D(B, A)$;

(Div.3) $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$, for all $C \in \mathcal{F}(X)$, where the union and intersection are defined by means of S and T, respectively.

It is clear that a divergence measure is associated to a triple (X, T, S) and a map D can be a divergence measure with respect to one t-norm and need not be with respect to another one. However, when there is not ambiguity, we will say just a divergence measure without specifying the used t-norm and t-conorm. We present some examples of divergence measures.

$$D(A, B) = \begin{cases} 0, & \text{if } A = B. \\ 1, & \text{if } A \neq B. \end{cases}$$

It is easy to see that D is a divergence. The first and the second conditions are trivial.

If $A \neq B$, then $D(A, B) = 1$ and $D(A \cap C, B \cap C) \in \{0, 1\} \leq D(A, B)$. If $A = B$, then $D(A, B) = 0$ and $A \cap C = B \cap C$ and by definition $D(A \cap C, B \cap C) = 0 \leq D(A, B)$. The same holds for the union $D(A \cup C, B \cup C)$. We can conclude that D is a divergence for any (X, T, S).

We consider the triple (X, T_M, S_M), where X is a finite universe. For any pair of fuzzy sets in X we define the function D using the **Hamming distance** as follows:

$$D(A, B) = \sum_{x \in X} \alpha_x \cdot |A(x) - B(x)|,$$

where $\alpha_x \geq 0$ for any $x \in X$ and $\sum_{x \in X} \alpha_x = 1$.

Again, the first and second condition are trivial. Denote $A(x) = a, B(x) = b, C(x) = c$. Without loss of generality, we assume $a \geq b$, then $T(a, c) \geq T(b, c)$ for any T, in particular for T_M. Thus,

- if $c > a \geq b$, then $|T_M(a, c) - T_M(b, c)| = |a - b| \leq |a - b|$,
- if $a \geq c \geq b$, then $|T_M(a, c) - T_M(b, c)| = |c - b| \leq |a - b|$,
- if $a \geq b > c$, then $|T_M(a, c) - T_M(b, c)| = |c - c| = 0 \leq |a - b|$,

and therefore, in all the cases, this inequality holds. From here we have that

$$\begin{aligned} D(A \cap C, B \cap C) &= \sum_{x \in X} \alpha_x \cdot |(A \cap C)(x) - (B \cap C)(x)| \\ &= \sum_{x \in X} \alpha_x \cdot |T_M(A(x), C(x)) - T_M(B(x), C(x))| \\ &\leq \sum_{x \in X} \alpha_x \cdot |A(x) - B(x)| = D(A, B). \end{aligned}$$

Analogously we prove that $D(A \cup C, B \cup C) \leq D(A, B)$ if we consider the maximum t-conorm for defining the union of two fuzzy sets. Thus, D is a divergence.

D is also a divergence if we consider the product t-norm or the Łukasiewicz t-norm and their dual t-conorms for defining the corresponding operations between sets, that is, if we work on (X, T_P, S_P) or (X, T_L, S_L). This is true since

- $|T_P(a, c) - T_P(b, c)| = |a \cdot c - b \cdot c| = c \cdot |a - b| \leq |a - b|$ and $|S_P(a, c) - S_P(b, c)| = |a + c - a \cdot c - (b + c - b \cdot c)| = (1 - c) \cdot |a - b| \leq |a - b|$;

- if $a+c \geq 1$ and $b+c \geq 1$, then $|T_L(a,c) - T_L(b,c)| = |(a+c-1) - (b+c-1)| = |a-b|$ and $|S_L(a,c) - S_L(b,c)| = |1-1| = 0 \leq |a-b|$,
- if $a+c < 1$ and $b+c \geq 1$, then $|T_L(a,c) - T_L(b,c)| = |0 - (b+c-1)| = |b-(1-c)|$, but $a < 1-c \leq b$ and therefore $|T_L(a,c) - T_L(b,c)| \leq |b-a| = |a-b|$. Moreover, $|S_L(a,c) - S_L(b,c)| = |a+c-1| = |(1-c)-a| \leq |b-a|$.
- the case $a+c \geq 1$ and $b+c < 1$ is analogous to the previous one,
- if $a+c < 1$ and $b+c < 1$, then $|T_L(a,c) - T_L(b,c)| = |0-0| = 0 \leq |a-b|$ and $|S_L(a,c) - S_L(b,c)| = |(a+c) - (b+c)| = |a-b|$.

However, this does not hold in general. For instance, if we consider the drastic t-norm, for the case $a = 1,\ b = 0.2,\ c = 0.9$ we have that $|T_D(a,c) - T_D(b,c)| = |T_D(1, 0.9) - T_D(0.2, 0.9)| = |0.9-0| = 0.9 > 0.8 = |a-b|$. Then, $D(A,B) = \sum_{x \in X} \alpha_x \cdot 0.8 = 0.8 > 0.9 = D(A \cap C, B \cap C)$.

As mentioned in the previous section, divergence measures appeared as an alternative to dissimilarities. We could think both concept are related in general, but it is not true.

The concept of divergence measure between two intuitionistic fuzzy sets has been introduced in a similar way. Let $IFS(X)$ denote the set of all intuitionistic fuzzy sets and $A, B, C \in IFS(X)$. The map $D_{IF} : IFS(X) \times IFS(X) \to \mathbb{R}$ is said to be an **IF-divergence measure**, if the following conditions are satisfied:

(1) $D_{IF}(A,A) = 0$,

(2) $D_{IF}(A,B) = D_{IF}(B,A)$,

(3) $\max\{D_{IF}(A \cup C, B \cup C), D_{IF}(A \cap C, B \cap C)\} \leq D(A,B)$.

Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is the finite universe and (X, T_M, S_M) is the triple. We present some examples of IF-divergence measures based on Hamming (D_{HM}) and Hausdorff distance (D_{HD}), respectively:

$$D_{HM}(A,B) = \frac{1}{2n} \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \right),$$

$$D_{HD}(A,B) = \sum_{i=1}^{n} \max\left\{ |\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)| \right\}.$$

If $A = B$, then $|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| = 0$ for all $x \in X$, i.e., $D_{HM} = D_{HD} = 0$. It is easy to see that both measures are symmetric. For (X, T_M, S_M) we have shown that $|\mu_{A \cap C}(x) - \mu_{B \cap C}(x)| = |T_M(\mu_A(x), \mu_C(x)) - T_M(\mu_B(x), \mu_C(x))| \leq |\mu_A(x) - \mu_B(x)|$ for all $x \in X$. The result for non-membership function ν can be obtained in a similar way, i.e., $|\nu_{A \cup C}(x) - \nu_{B \cup C}(x)| \leq |\nu_A(x) - \nu_B(x)|$ for all $x \in X$. Therefore, we get:

$$|\mu_{A \cap C}(x) - \mu_{B \cap C}(x)| + |\nu_{A \cup C}(x) - \nu_{B \cup C}(x)| \leq |\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|,$$

and finally a condition for the divergence based on the Hamming distance holds:

$$D_{HM}(A \cap C, B \cap C) \leq D_{HM}(A,B).$$

Similarly, from the inequality

$$\max\{|\mu_{A\cap C}(x)-\mu_{B\cap C}(x)|,|\nu_{A\cup C}(x)-\nu_{B\cup C}(x)|\}\leq \max\{|\mu_A(x)-\mu_B(x)|,|\nu_A(x)-\nu_B(x)|\},$$

also the condition for Hausdorff distance can be concluded:

$$D_{HD}(A\cap C,B\cap C)\leq D_{HD}(A,B).$$

The remaining conditions $D_{HM}(A\cup C,B\cup C)\leq D_{HM}(A,B)$ and $D_{HD}(A\cup C,B\cup C)\leq D_{HD}(A,B)$ can be shown similarly.

Non-negativity of IF-divergence follows immediately from the following relation:

$$D_{IF}(A,B)\geq D_{IF}(A\cap\varnothing,B\cap\varnothing)=D_{IF}(\varnothing,\varnothing)=0.$$

Many properties of the classical divergence among fuzzy sets can be generalized also for the IF-divergence.

The relation between IF-divergence among intuitionistic fuzzy sets and classical divergence measure among fuzzy sets will be discussed in this section. As we have mentioned, each IF-set $A=\{(x,\mu_A(x),\nu_A(x))\mid x\in X\}$ can be decomposed into two fuzzy sets A_1 and A_2:

$$A_1=\{(x,\mu_{A_1}(x))\mid x\in X\}\,,\ \text{where}\ \mu_{A_1}(x)=\mu_A(x)$$

and

$$A_2=\{(x,\mu_{A_2}(x))\mid x\in X\}\,,\ \text{where}\ \mu_{A_2}(x)=\nu_A(x).$$

Let X be the universe, D_1,D_2 be two divergence measures defined on $\mathcal{F}(X)\times\mathcal{F}(X)$ and A,B be the intuitionistic fuzzy sets. Accordingly the function D_{IF}, defined by

$$D_{IF}(A,B)=f(D_1(\mu_A,\mu_B),D_2(\nu_A,\nu_B)),$$

where $f:[0,\infty)\times[0,\infty)\to[0,\infty)$, fulfills the following conditions:

(1) $f(0,0)=0$ (boundary condition),

(2) $f(t,\cdot)$ and $f(\cdot,t)$ are increasing (monotonicity condition),

is an IF-divergence measure.

We check three conditions for IF-divergence measure. Let $A,B,C\in IFS(X)$.

(1) If $A=B$, then $D_1(A,B)=D_2(A,B)=0$ and $D_{IF}(A,B)$ since boundary condition holds.

(2) $D_{IF}(A,B)=D_{IF}(B,A)$ since D_1,D_2 are both symmetric.

(3) $D_{IF}(A\cap C,B\cap C)=f(D_1(\mu_{A\cap C},\mu_{B\cap C}),D_2(\nu_{A\cup C},\nu_{B\cup C})\leq f(D_1(\mu_A,\mu_B),D_2(\nu_A,\nu_B))$ $=D_{IF}(A,B)$ since the monotonicity condition is fulfilled by f.

Therefore, D_{IF} is an IF-divergence measure.

4 Local divergence measures for IFSs

When we compare two fuzzy sets, it seems natural to suppose that if we only change the value of these sets at one element, the divergence should only depend on what has been changed.

Let D be a **divergence measure** for a triple (X, T, S). Then D has the local property or, briefly, is **local**, if for all $A, B \in \mathcal{F}(X)$ and for all $x \in X$, there exists a map $h_x : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$D(A, B) - D(A \cup \{x\}, B \cup \{x\}) = h_x(A(x), B(x)).$$

A particular case of locality was introduced in [10], but there h_x was fixed for any $x \in X$ and all the elements in the universe were of the same importance. However, based on some application of comparison of multivalued sets, this is not always the case and different maps should be considered. Therefore we introduce a more general definition for locality. We present a general result which holds for any t-norm and any t-conorm.

Let (X, T, S) be a triple with X a finite universe and T and S any t-norm and t-conorm, respectively. Let D be a divergence associated to (X, T, S). D is local if and only if

$$D(A, B) \;=\; \sum_{x \in X} h_x(A(x), B(x)),$$

where $\{h_x\}_{x \in X}$ is a family of maps from $[0,1] \times [0,1]$ into $\mathbb{R}$ such that, for any $x \in X$ and $a, b, c \in [0,1]$, there is:

(i) $h_x(a, a) = 0$, for all $a \in [0,1]$,

(ii) $h_x(a, b) = h_x(b, a)$, for all $a, b \in [0,1]$,

(iii) $h_x(a, b) \geq \max(h_x(S(a, c), S(b, c)), h_x(T(a, c), T(b, c)))$ for all $a, b, c \in [0,1]$.

Locality, as the most important property of divergence measures, has been already discussed. Let A, B be fuzzy sets and D be the local divergence measure. We remark that the difference between $D(A \cup \{x\}, B \cup \{x\})$ and $D(A, B)$ depends only on the element $x \in X$, which has been changed. More formally, the difference $D(A, B) - D(A \cup \{x\}, B \cup \{x\})$ can be expressed by means of a function $h(A(x), B(x))$, since D is a local divergence. Obviously, the function h may not exist in general, and therefore locality is quite restrictive property. The concept of the locality can be introduced also for an IF-divergence, but some generalization is necessary. Suppose that $h : [0,1]^4 \to \mathbb{R}$.

Let D_{IF} be IF-divergence measure. If for all intuitionistic fuzzy sets A, B and for each $x \in X$ the following property is satisfied:

$$D_{IF}(A, B) - D_{IF}(A \cup \{x\}, B \cup \{x\}) = h(\mu_A(x), \nu_A(x), \mu_B(x), \nu_B(x)),$$

then D_{IF} is said to be a **local IF-divergence measure**.

For membership μ_A and non-membership function ν_A it holds that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. We will use the simple notation for ordered pair $(\mu_A(x), \nu_A(x))$ as the elements of the family $\mathcal{I} = \{(a, b) \in [0,1]^2 \mid a + b \leq 1\}$.

A map $D_{IF} : IFS(X) \times IFS(X) \to \mathbb{R}$ is a local IF-divergence measure if and only if there exists a function $h : \mathcal{I}^2 \to \mathbb{R}$, such that for every $A, B \in IFS(X)$:

$$D_{IF}(A, B) = \sum_{x \in X} h(\mu_A(x), \nu_A(x), \mu_B(x), \nu_B(x)),$$

where the function h satisfies the following conditions for every $(a_1, a_2), (b_1, b_2) \in \mathcal{I}$:

(i) $h(a_1, a_2, a_1, a_2) = 0$,

(ii) $h(a_1, a_2, b_1, b_2) = h(b_1, b_2, a_1, a_2)$,

(iii) $h(a_1, a_2, c, b_2) \leq h(a_1, a_2, b_1, b_2)$ for $a_1 \leq c \leq b_1$,
$h(c, a_2, b_1, b_2) \leq h(a_1, a_2, b_1, b_2)$ for $a_1 \leq c \leq b_1$, such that $(c, a_2) \in \mathcal{I}$,
$h(a_1, a_2, b_1, c) \leq h(a_1, a_2, b_1, b_2)$ for $a_2 \leq c \leq b_2$,
$h(a_1, c, b_1, b_2) \leq (a_1, a_2, b_1, b_2)$ for $a_2 \leq c \leq b_2$, such that $(a_1, c) \in \mathcal{I}$,
$h(c, a_2, c, b_2) \leq (a_1, a_2, b_1, b_2)$, such that $(c, a_2), (c, b_2) \in \mathcal{I}$,
$h(a_1, c, b_1, c) \leq (a_1, a_2, b_1, b_2)$, such that $(a_1, c), (b_1, c) \in \mathcal{I}$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and A, B be IF-sets on X. Applying the equation recursively for each element $x \in X$ the following formula can be obtained:

$$\begin{aligned} D_{IF}(A, B) &= D_{IF}(A \cup \{x_1\}, B \cup \{x_1\}) + h(\mu_A(x_1), \nu_A(x_1), \mu_B(x_1), \nu_B(x_1)) \\ &= D_{IF}(A \cup \{x_1\} \cup \{x_2\}, B \cup \{x_1\} \cup \{x_2\}) + \sum_{i=1}^{2} h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) \\ &= \cdots = D_{IF}(X, X) + \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)). \end{aligned}$$

Now, we verify three conditions that need to be satisfied by the function h:

(i) Let $A \in IFS(X)$ be defined as follows: $\mu_A(x_i) = a_1, \nu_A(x_i) = a_2$ for every $i = 1, \ldots, n$, i.e., $(a_1, a_2) \in \mathcal{I}$. Further, the local IF-divergence can be expressed in the following way:

$$\begin{aligned} D_{IF}(A, A) &= \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_A(x_i), \nu_A(x_i)) \\ &= \sum_{i=1}^{n} h(a_1, a_2, a_1, a_2) = n \cdot h(a_1, a_2, a_1, a_2), \end{aligned}$$

and therefore $D_{IF}(A, A) = 0 \Leftrightarrow h(a_1, a_2, a_1, a_2) = 0$.

(ii) Let $A, B \in IFS(X)$ be defined as follows: $\mu_A(x_i) = a_1, \nu_A(x_i) = a_2, \mu_B(x_i) = b_1$, $\nu_B(x_i) = b_2$ for every $i = 1, \ldots, n$, i.e., $(a_1, a_2), (b_1, b_2) \in \mathcal{I}$. We express the local IF-divergence:

$$\begin{aligned} D_{IF}(A, B) &= \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) \\ &= \sum_{i=1}^{n} h(a_1, a_2, b_1, b_2) = n \cdot h(a_1, a_2, b_1, b_2), \end{aligned}$$

and

$$D_{IF}(B,A) = \sum_{i=1}^{n} h(\mu_B(x_i), \nu_B(x_i), \mu_A(x_i), \nu_A(x_i))$$
$$= \sum_{i=1}^{n} h(b_1, b_2, a_1, a_2) = n \cdot h(b_1, b_2, a_1, a_2).$$

Finally, we get $D_{IF}(A,B) = D_{IF}(B,A) \Leftrightarrow h(a_1,a_2,b_1,b_2) = h(b_1,b_2,a_1,a_2)$.

(iii) Let $A, B \in IFS(X)$ be defined as follows: $\mu_A(x_i) = a_1, \nu_A(x_i) = a_2, \mu_B(x_i) = b_1$, $\nu_B(x_i) = b_2$, such that $(a_1,a_2),(b_1,b_2) \in \mathcal{I}$, for every $i = 1,\ldots,n$. Only the case for $T = T_M$ and $S = S_M$ will be considered.

(a) Intersection:
Let $C \in IFS(X)$ be defined as follows: $\mu_C(x_i) = c, \nu_C(x_i) = 0$, such that $a_1 \leq c \leq b_1$ and $(c,0) \in \mathcal{I}$, for every $i = 1,\ldots,n$.

$$\mu_{A\cap C}(x_i) = T(\mu_A(x_i), \mu_C(x_i)) = T(a_1, c) = a_1 = \mu_A(x_i)$$
$$\nu_{A\cup C}(x_i) = S(\nu_A(x_i), \nu_C(x_i)) = S(a_2, 0) = a_2 = \nu_A(x_i),$$

and therefore $A \cap C = A$.

$$\mu_{B\cap C}(x_i) = T(\mu_B(x_i), \mu_C(x_i)) = T(b_1, c) = c = \mu_C(x_i)$$
$$\nu_{B\cup C}(x_i) = S(\nu_B(x_i), \nu_C(x_i)) = S(b_2, 0) = b_2 = \nu_B(x_i).$$

The following relation can be concluded $D_{IF}(A\cap C, B\cap C) = D_{IF}(A, B\cap C) \leq D_{IF}(A,B)$ since the third condition of IF-divergence holds, i.e.:

$$D_{IF}(A,B) = \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$$
$$= \sum_{i=1}^{n} h(a_1, a_2, b_1, b_2) = n \cdot h(a_1, a_2, b_1, b_2),$$

and

$$D_{IF}(A\cap C, B\cap C) = D_{IF}(A, B\cap C)$$
$$= \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_C(x_i), \nu_B(x_i))$$
$$= \sum_{i=1}^{n} h(a_1, a_2, c, b_2) = n \cdot h(a_1, a_2, c, b_2).$$

Finally, we get $D_{IF}(A\cap C, B\cap C) \leq D_{IF}(A,B) \Leftrightarrow h(a_1,a_2,c,b_2) \leq h(a_1,a_2,b_1,b_2)$.

(b) Union:
Let $C \in IFS(X)$ be defined as follows: $\mu_C(x_i) = c, \nu_C(x_i) = \max\{a_2, b_2\}$, such that $a_1 \leq c \leq b_1$ and $(c,a_2),(c,b_2) \in \mathcal{I}$, for every $i = 1,\ldots,n$.

$$\mu_{A\cup C}(x_i) = S(\mu_A(x_i), \mu_C(x_i)) = S(a_1, c) = c = \mu_C(x_i),$$
$$\nu_{A\cap C}(x_i) = T(\nu_A(x_i), \nu_C(x_i)) = T(a_2, \max\{a_2, b_2\}) = a_2 = \nu_A(x_i),$$

$$\mu_{B\cup C}(x_i) = S(\mu_B(x_i), \mu_C(x_i)) = S(b_1, c) = b_1 = \mu_B(x_i),$$
$$\nu_{B\cap C}(x_i) = T(\nu_B(x_i), \nu_C(x_i)) = T(b_2, \max\{a_2, b_2\}) = b_2 = \nu_B(x_i),$$

and therefore $B \cup C = B$.
The following relation can be concluded $D_{IF}(A \cup C, B \cup C) = D_{IF}(A \cup C, B) \leq D_{IF}(A, B)$ since the third condition of IF-divergence holds, i.e.:

$$D_{IF}(A, B) = \sum_{i=1}^{n} h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$$
$$= \sum_{i=1}^{n} h(a_1, a_2, b_1, b_2) = n \cdot h(a_1, a_2, b_1, b_2),$$

and

$$D_{IF}(A \cup C, B \cup C) = D_{IF}(A \cup C, B) = \sum_{i=1}^{n} h(\mu_C(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$$
$$= \sum_{i=1}^{n} h(c, a_2, b_1, b_2) = n \cdot h(c, a_2, b_1, b_2).$$

Finally, we get $D_{IF}(A\cup C, B\cup C) \leq D_{IF}(A, B) \Leftrightarrow h(c, a_2, b_1, b_2) \leq h(a_1, a_2, b_1, b_2)$.

The remaining parts can be proved in a similar way.

More details about local IF-divergence measure can be found in [8].

5 Entropy measures for intuitionistic fuzzy sets

Let $A \in \mathcal{F}(X)$. The closest crisp set to A, which will be denoted by N_A, is

$$N_A(x) = \begin{cases} 1, & \text{if } A(x) \geq \frac{1}{2}. \\ 0, & \text{if } A(x) < \frac{1}{2}. \end{cases}$$

This concept will be useful later, as well as the concept of equilibrium.
$E \in \mathcal{F}(X)$ is the **equilibrium set** if and only if $E(x) = \frac{1}{2}$ for all $x \in X$.

Accounting the grade of impreciseness, E is the maximal fuzzy set, as the ambiguity of its membership or non-membership is the largest possible. We can define the partial order respecting this aspect in the following way.

Let $A, B \in \mathcal{F}(X)$. A is said to **be sharper than** B, which is denoted by $A \prec B$, if and only if $|A(x) - 1/2| \geq |B(x) - 1/2|$, for all $x \in X$.

The fuzziness of a fuzzy set can be measured by means of the fuzziness measures or entropies. In all the cases, the previous concepts of "sharper than", equilibrium set and complement set are necessary.

A map $f : \mathcal{F}(X) \to \mathbb{R}$ is a **fuzziness measure** if for all $A, B \in \mathcal{F}(X)$, f satisfies the following conditions:

(f1) $f(A) = 0$ if A is crisp,

(f2) $f(A) = f(A^c)$,

(f3) $A \prec B \Rightarrow f(A) \leq f(B)$,

(f4) $f(E)$ is the maximum value of f, where E is the equilibrium fuzzy set.

An entropy measure for IF-sets is based on the axioms given by De Luca and Termini. The mapping $E : IFS(X) \to [0, 1]$, for which the following conditions are satisfied:

(1) $E(A) = 0$ if and only if $A \subseteq X$, i.e., $\mu_A(x) \in \{0, 1\}$ for all $x \in X$,

(2) $E(A) = 1$ if and only if $\mu_A(x) = \nu_A(x) = \pi_A(x) = \frac{1}{3}$ for all $x \in X$,

(3) $E(A) = E(A^c)$,

(4) $E(A) \leq E(B)$ if and only if $\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x)$ for $\mu_B(x) \leq \nu_B(x)$
or $\mu_A(x) \geq \mu_B(x), \nu_A(x) \leq \nu_B(x)$ for $\mu_B(x) \geq \nu_B(x)$,

is said to be an **entropy measure** on IFS.

However, in the literature (see also [12]) other approach to define the entropy measure can be found. Some authors propose to replace the last axiom by another one:

(4a) $E(A) \leq E(B)$ if and only if
$\mu_A(x) \leq \mu_B(x), \nu_A(x) \leq \nu_B(x)$ for $\max\{\mu_B(x), \nu_B(x)\} \leq \frac{1}{3}$
or
$\mu_A(x) \geq \mu_B(x), \nu_A(x) \geq \nu_B(x)$ for $\min\{\mu_B(x), \nu_B(x)\} \geq \frac{1}{3}$.

We give one example of entropy measure for intuitionistic fuzzy set A defined on the finite universe $X = \{x_1, \ldots, x_n\}$ as follows (see also [4], [6] and [12] for more examples):

$$E(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)}{\max\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)}.$$

Now, we check all the conditions that must be satisfied by the measure E:

(1) If the set A is crisp, then $\mu_A(x) \in \{0, 1\}$ and for non-membership function we have $\nu_A(x) = 0$ if $\mu_A(x) = 1$ and $\nu_A(x) = 1$ if $\mu_A(x) = 0$ for all $x \in X$. In both cases $\pi_A(x) = 0$. Therefore: $\min\{\mu_A(x), \nu_A(x)\} = 0$ and $\max\{\mu_A(x), \nu_A(x)\} = 1$ and for $E(A)$ we have:

$$E(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)}{\max\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)} = \frac{1}{n} \sum_{i=1}^{n} \frac{0+0}{1+0} = \frac{1}{n} \sum_{i=1}^{n} 0 = \frac{0}{n} = 0.$$

(2) Since the boundary condition $\mu_A(x) = \nu_A(x) = \pi_A(x) = \frac{1}{3}$ for all $x \in X$ is considered, we obtain the following relation:

$$E(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)}{\max\{\mu_A(x_i), \nu_A(x_i)\} + \pi_A(x_i)} = \frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{3} + \frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{n}{n} = 1.$$

(3) For the intuitionistic fuzzy set A and its complement A^c the following relations hold:

$$\min\left\{\mu_A^c(x),\nu_A^c(x)\right\}=\min\left\{\nu_A(x),\mu_A(x)\right\},\max\left\{\mu_A^c(x),\nu_A^c(x)\right\}=\max\left\{\nu_A(x),\mu_A(x)\right\}$$
and $\pi_{A^c}(x)=\pi_A(x)$ for all $x\in X$.

Therefore $E(A)=E(A^c)$.

(4) If $\mu_A(x)\le\mu_B(x),\nu_A(x)\ge\nu_B(x),\mu_B(x)\le\nu_B(x)$, then $\mu_A(x)\le\nu_A(x)$ for all $x\in X$,

$$E(A)=\frac{\min\left\{\mu_A(x),\nu_A(x)\right\}+\pi_A(x)}{\max\left\{\mu_A(x),\nu_A(x)\right\}+\pi_A(x)}=\frac{\mu_A(x)+\pi_A(x)}{\nu_A(x)+\pi_A(x)}=\frac{1-\nu_A(x)}{1-\mu_A(x)},$$

and

$$E(B)=\frac{\min\left\{\mu_B(x),\nu_B(x)\right\}+\pi_B(x)}{\max\left\{\mu_B(x),\nu_B(x)\right\}+\pi_B(x)}=\frac{\mu_B(x)+\pi_B(x)}{\nu_B(x)+\pi_B(x)}=\frac{1-\nu_B(x)}{1-\mu_B(x)}.$$

We obtain the following inequality,

$$\frac{1-\nu_A(x)}{1-\mu_A(x)}\le\frac{1-\nu_B(x)}{1-\mu_B(x)},$$

since $1-\mu_A(x)\ge1-\mu_B(x)$ and $1-\nu_A(x)\le1-\nu_B(x)$ for all $x\in X$, and therefore $E(A)\le E(B)$.

If $\mu_A(x)\ge\mu_B(x),\nu_A(x)\le\nu_B(x),\mu_B(x)\ge\nu_B(x)$, then $\mu_A(x)\ge\nu_A(x)$ for all $x\in X$,

$$E(A)=\frac{\min\left\{\mu_A(x),\nu_A(x)\right\}+\pi_A(x)}{\max\left\{\mu_A(x),\nu_A(x)\right\}+\pi_A(x)}=\frac{\nu_A(x)+\pi_A(x)}{\mu_A(x)+\pi_A(x)}=\frac{1-\mu_A(x)}{1-\nu_A(x)},$$

and

$$E(B)=\frac{\min\left\{\mu_B(x),\nu_B(x)\right\}+\pi_B(x)}{\max\left\{\mu_B(x),\nu_B(x)\right\}+\pi_B(x)}=\frac{\nu_B(x)+\pi_B(x)}{\mu_B(x)+\pi_B(x)}=\frac{1-\mu_B(x)}{1-\nu_B(x)}.$$

Finally,

$$\frac{1-\mu_A(x)}{1-\nu_A(x)}\le\frac{1-\mu_B(x)}{1-\nu_B(x)},$$

since $1-\mu_A(x)\le1-\mu_B(x)$ and $1-\nu_A(x)\ge1-\nu_B(x)$ for all $x\in X$, and therefore $E(A)\le E(B)$ in this case as well.

So, the map E is an entropy measure on IFS.

6 Conclusions

We have extended the results for dissimilarity measures, divergence measures, local divergences and entropy measures from fuzzy sets into the more general objects as intuitionistic fuzzy sets. Our aim is to continue with a deeper study of the related problems, e.g. local property of divergence measures. Some examples and possible applications of this approach will be discussed in the future work.

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