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Statistical Estimation on MV-algebras

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Abstract: The aim of this paper is determining the point and interval estimation of the mean value of the observable from the set of all interval $(-\infty, t)$ to the MV-algebra.

1 MV-algebras

By the Mundici theorem ([8]) MV-algebra can be characterized by the help of l-groups.

1.1 ***Definition.*** *An l-group is and algebraic system*

$$(G, +, \leq)$$

such that

$(G, +)$ *is and Abelian group*
$(G, \leq)$ *is a partially ordered set being a lattice*
$a \leq b \Longrightarrow a + c \leq b + c$ *for any a, b, c in G.*

1.2 ***Definition.*** *An MV-algebra is an algebraic system*

$$(M, \oplus, \odot, \leq, 0, u)$$

where

$M = [0, u]$ *is an interval in an l-group* $G = (G, +, \leq)$
0 *is the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$)*
u *is the strong unit of G (i.e. to any $a \in G$ there exists $n \in N$ such that $a \leq u + u + ... + u$(n-times))*
$a \oplus b = (a + b) \wedge 1,$
$a \odot b = (a + b - 1) \vee 0.$

1.3 ***Definition.*** *An state on an MV-algebra M is a mapping $m : M \to [0, 1]$ satisfying the following conditions:*

(i) $m\,(u) = 1,\; m\,(0) = 0$;

(ii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$;

(iii) $a_n \searrow a \Longrightarrow m(a_n) \searrow m(a)$.

1.4 ***Definition.*** *Let* $\mathcal{J} = \{(-\infty, t); t \in R\}$. *An observable on* M *is any mapping* $x : \mathcal{J} \to M$ *satisfying the conditions:*

(i) $t_n \nearrow \infty \Longrightarrow x((-\infty, t_n)) \nearrow u$;

(ii) $t_n \searrow -\infty \Longrightarrow x((-\infty, t_n \searrow 0$;

(iii) $t_n \nearrow t \Longrightarrow x((-\infty, t_n)) \nearrow x((-\infty, t_n)) \nearrow x((-\infty, t))$.

1.5 ***Theorem.****[5] Let* $m : M \to [0, 1]$ *be a state,* $x : \mathcal{J} \to M$ *be an observable. Define* $F : R \to [0, 1]$ *by the formula*

$$F(t) = m(x((-\infty, t))), t \in R$$

Then F *has the following properties:*

(i) F *is non-decreasing;*

(ii) $\lim_{t\to\infty} F(t) = 1$;

(iii) $\lim_{t\to-\infty} F(t) = 0$;

(iv) F *is left continuous in any point* $t \in R$.

Proof. is straightforward.

1.6 ***Definition.*** *An observable* $x : \mathcal{J} \to M$ *is called to be integrable if there exists*

$$E(x) = \int_R t dF(t),$$

where $F : R \to [0, 1]$ *is distribution function of the observable* x. *The observable* x *is square integrable, if there exists*

$$\int_R t^2 dF(t).$$

2 MV-algebras with Product

2.1 ***Definition.*** *An MV-algebra with product is a pair*$(M, .)$, *where* M *is an MV-algebra and . is a commutative and associative binary operation on* M *satisfying the following conditions:*

(i) $u \,.\, a = a$ *for any* $a \in M$

(ii) $a \,.((b - c) \vee 0) = (a.b - a.c) \vee 0$ *for any* $a, b, c \in M$

2.2 *Theorem.* *Let M be a σ-complete MV-algebra with product, $\mathcal{M} = \{\Delta_t^n; t \in R\}, x_1, ..., x_n : \mathcal{J} \longrightarrow M$ be observables, where $\Delta_t^n = \{(u_1, ..., u_n); \sum_{i=1}^n u_i < t\}$. Then there exists a mapping $h_n : \mathcal{M} \to M$ such that the mapping $z : \mathcal{J} \longrightarrow M$ defined by*

$$z((-\infty, t)) = h_n(\Delta_t^n),$$

is an observable.

Proof. See [5], Theorem 2.3.

2.3 *Definition.* *Let M be a σ-complete MV-algebra with product, $x_1, ..., x_n : \mathcal{J} \to M$ be observables. Then its sum is defined by the formula*

$$\left(\sum_{i=1}^n x_i\right)(-\infty, t) = h_n\left(\Delta_t^n\right) = h_n\left(g_n^{-1}\left((-\infty, t)\right)\right)$$

$$\sum_{i=1}^n x_i = h_n \circ g_n^{-1}$$

where $g : R^n \to R$, $g\left(m_1, ..., m_n\right) = m_1 + ... + m_n$.

2.4 *Definition.* *Observables $x_1, ..., x_n$ are independent, if for any $t_1, ..., t_n \in R$*

$$m\left(h_n\left((-\infty, t_1) \times (-\infty, t_2) \times ... \times (-\infty, t_n)\right)\right) =$$

$$= m\left(x_1\left((-\infty, t_1)\right)\right) \cdot m\left(x_1\left((-\infty, t_2)\right)\right) \cdot ... \cdot m\left(x_1\left((-\infty, t_n)\right)\right).$$

2.5 *Definition.* *An observable $x : \mathcal{J} \longrightarrow M$ is called strong, if*

$$[a, b) \cap [c, d) = \emptyset \Longrightarrow \left(x\left([a, b)\right).\alpha\right) \wedge \left(x\left([c, d)\right).\beta\right) = 0$$

for any $\alpha, \beta \in M$.

2.6 *Definition.* *A state $m : M \longrightarrow \langle 0, 1\rangle$ is called σ-additive, if*

$$m\left(\bigvee_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m\left(A_n\right)$$

whenever $A_n \cap A_m = 0$ $(n \neq m)$.

3 Applications

3.1 *Definition.* *Let M be a σ-complete MV-algebra with product, $x_1, ..., x_n$ be independent observables. Then we define*

$$\left(\frac{1}{n}\sum_{i=1}^n x_i - a\right)\left((-\infty, t)\right) = \left(\sum_{i=1}^n x_i\right)\left((-\infty, (t + a)\, n)\right) =$$

$$= h_n\left(g_n^{-1}\left((-\infty, (t + a)\, n)\right)\right)$$

3.2 *Theorem.* *Let M be a σ-complete MV-algebra with product, $m : M \to [0,1]$ be a σ-additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable strong observables.Let $E[x_1] = E[x_2] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$. Then for any $t \in R$*

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}((-\infty, t))\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}} du.$$

Proof. See [2], Theorem 3.3.

We shall write $x \sim N(a, \sigma^2)$, if

$$m(x(-\infty, t)) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u-a^2}{2\sigma^2}} du.$$

for any $t \in R$. If $a = 0, \sigma = 1$, then $m(x(-\infty, t))$ is denoted by $\Phi(t)$.

3.3 *Theorem.* *Let M be a σ-complete MV-algebra with product, $m : M \to [0,1]$ be a σ-additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable strong observables.Let $E[x_1] = E[x_2] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then for any $t \in R$*

$$\lim_{n\to\infty} m\left(\frac{1}{n}\sum_{i=1}^{n} x_i - E(\bar{x})\right)\left(-\infty, \frac{t\sigma}{\sqrt{n}}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}} du.$$

Proof. $E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum \underbrace{E(x_i)}_{a} = \frac{1}{n} \cdot n \cdot a = a$,

hence we see that $\bar{x}$ is the point estimation of the a.

3.4 *Definition.* *Let $x : \mathcal{J} \to M$ be an observable, $E(x) = a, \sigma^2(x) = \sigma$. $(y_i)_{i=1}^{\infty}$ be a seguence of observable estimation. The sequence $(y_i)_{i=1}^{\infty}$ is an interval estimation of a, if there exist $\delta > 0$ such that*

$$\lim_{n\to\infty} m(y_n - \delta < a < y_n + \delta) = 0.$$

The number $\alpha = 1 - m(y_n - \delta < a < y_n + \delta)$ is called significance level.

3.5 *Theorem.* *Let all assumption of Theorem 3.3 be satisfied and $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\alpha = 2(1 - \Phi(\delta))$, i.e.*

$$\lim_{n\to\infty} m\left(\frac{\bar{x}_n - a}{\frac{\sigma}{\sqrt{n}}}(-\delta, \delta)\right) = 1 - \alpha.$$

Then $(\bar{x}_n)_n$ is an interval estimation of the mean value $a = E(x_i)$.

Proof. Evidently
$m(\bar{x}_n - \delta < a < \bar{x}_n + \delta) = m((\bar{x}_n - a)(R\setminus(-\delta, \delta))) =$
$= m\left(\left(\frac{\bar{x}_n - a}{\frac{\sigma}{\sqrt{n}}}\right)\left(R\setminus\left(-\frac{\delta\sigma}{\sqrt{n}}, \frac{\delta\sigma}{\sqrt{n}}\right)\right)\right) = 1 - m\left(\left(\frac{\bar{x}_n - a}{\frac{\sigma}{\sqrt{n}}}\right)\left(\left(-\frac{\delta\sigma}{\sqrt{n}}, \frac{\delta\sigma}{\sqrt{n}}\right)\right)\right)$

Since
$(-\delta,\delta) = (-\infty,\delta) - (-\infty,-\delta)$

and
$\lim_{n\to\infty} m\left(\frac{\bar{x}_n - a}{\frac{\sigma}{\sqrt{n}}}(-\infty,\delta)\right) = \Phi(\delta)$

we obtain
$\lim_{n\to\infty} m\left(\frac{\bar{x}_n - a}{\frac{\sigma}{\sqrt{n}}}(-\delta,\delta)\right) = \Phi(\delta) - \underbrace{\Phi(-\delta)}_{1-\Phi(\delta)} = 2\Phi(\delta) - 1 = 1 - 2(1-\Phi(\delta)) = 1 - \alpha$

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