A note on fuzzy observables

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Definition

A B-structure is a system $(B, \oplus, \leq, 0, 1)$ where $\oplus$ is a partial binary operation on B, and $\leq$ is a partial ordering with the least element 0 and the greatest element 1.

Example

$$A \odot B = (0,1) \Longrightarrow m(A \oplus B) = m(A) + m(B).$$

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1).$$

$A \odot B = (0,1)$ means that $\mu_A + \mu_B \leq 1$, and $\nu_A + \nu_B \geq 1$.

$$A \oplus B = (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

Example

$(L, \oplus, \leq, 0, 1)$ effect algebra

1. $\oplus$ commutative partial binary operation
2. $\oplus$ associative
3. $\forall a \in L \; \exists! b \in L : a \oplus b = 1$
4. $\exists 1 \oplus a \implies a = 0$.

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Definition

B-state is a mapping $m : B \to [0,1]$ satisfying the following conditions:

1. $m(1) = 1, m(0) = 0$;
2. if $c = a \oplus b$, then $m(c) = m(a) + m(b)$;
3. $b_n \nearrow b \Longrightarrow m(b_n) \nearrow m(b)$.

Definition

B-observable is a mapping $\xi : \mathcal{B}(R) \to B$ satisfying the following conditions:

1. $\xi(R) = 1, \xi(\emptyset) = 0;$
2. if $A, B \in \mathcal{B}(R), A \cap B = \emptyset$, then $\xi(A) \oplus \xi(B)$ exists, and $\xi(A \cup B) = \xi(A) + \xi(B);$
3. $A_n \nearrow A \Longrightarrow \xi(A_n) \nearrow \xi(A).$

Theorem

Assumptions

1. $\exists S$ family of B-states such that
   (i) $(\forall m \in S : m(a) \geq m(b)) \Longrightarrow a \geq b$
   (ii) $(c \geq a, c \geq b, \forall m \in S : m(c) = m(a) + m(b)) \Longrightarrow c = a \oplus b$
2. $\exists \nu : R \times \mathcal{B}(R) \to [0, 1]$
   (i) $\forall x \in R$, the mapping $E \mapsto (x, E)$ is a probability measure
   (ii) $\forall A \in \mathcal{B}(R)$, the mapping $x \mapsto \nu(x, A)$ is measurable
3. ξ is a B-observable
4. $\forall A \in +\mathcal{B}(R)\ \exists \eta(A) \in B : m(\eta(A)) = \int_R \nu(., A) dm \circ \xi$

Assertion

$\eta : \mathcal{B}(R) \to B$ **is a B-observable (so-called fuzzy version of ξ).**