Notes on Intuitionistic Fuzzy Sets
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$\mathbf{m}$-almost everywhere convergence of intuitionistic fuzzy observables induced by Borel measurable function

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Abstract: In paper [4] we studied the upper and the lower limits of sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state $\mathbf{m}$ for a definition the notion of almost everywhere convergence. We compared two concepts of $\mathbf{m}$-almost everywhere convergence. The aim of this paper is to show the connection between almost everywhere convergence in classical probability space induced by Kolmogorov construction and $\mathbf{m}$-almost everywhere convergence in intuitionistic fuzzy space. We studied the sequence of intuitionistic fuzzy observables induced by Borel measurable function.
Keywords: Intuitionistic fuzzy event, Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Joint intuitionistic fuzzy observable, Product, Upper limit, Lower limit, $\mathbf{m}$-almost everywhere convergence, Function of several intuitionistic fuzzy observables, Borel measurable function, Kolmogorov construction.
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1 Introduction

In [1–3] K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [7] B. Riečan defined the intuitionistic fuzzy state on the family of intuitionistic fuzzy events

$$\mathcal{F} = \{(\mu_A, \nu_A)\,; \mu_A + \nu_A \leq 1_\Omega\},$$

where μ_A, ν_A are $\mathcal{S}$-measurable functions, $\mu_A, \nu_A : \Omega \to [0,1]$, as a mapping $\mathbf{m}$ from the family $\mathcal{F}$ to the set R by the formula

$$\mathbf{m}\big((\mu_A, \nu_A)\big) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big),$$

where $P : \mathcal{S} \to [0,1]$ is a probability measure and $\alpha \in [0,1]$.

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state $\mathbf{m}$ for a definition the notion of almost everywhere convergence. We compared two concepts of $\mathbf{m}$-almost everywhere convergence.

In this paper we study the $\mathbf{m}$-almost everywhere convergence of sequence of intuitionistic fuzzy observables $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ given by

$$g_n(x_1, \ldots, x_n) = h_n \circ g_n^{-1},$$

where $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint intuitionistic fuzzy observable of intuitionistic fuzzy observables $x_1, \ldots, x_n$ and $g_n : R^n \to R$ is a Borel measurable function. We show the connection between $\mathbf{m}$-almost everywhere convergence of this sequence of intuitionistic fuzzy observables and P-almost everywhere convergence of random variables in classical probability space induced by Kolmogorov construction.

Remark. Note that in a whole text we use a notation "IF" in short as the phrase "intuitionistic fuzzy."

2 IF-events, IF-states and IF-observables

First we start with definitions of basic notions.

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ will be called the membership function, $\nu_A : \Omega \longrightarrow [0,1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Łukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),$$
$$\mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))$$

and the partial ordering is then given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In paper we use max-min connectives defined by

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$

$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the de Morgan rules

$$(a \vee b)^* = a^* \wedge b^*,$$

$$(a \wedge b)^* = a^* \vee b^*,$$

where $a^* = 1 - a$.

Example 2.3. *Fuzzy set $f : \Omega \longrightarrow [0,1]$ can be regarded as IF-set, if we put*

$$\mathbf{A} = (f, 1_\Omega - f).$$

If $f = \chi_A$, then the corresponding IF-set has the form

$$\mathbf{A} = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the intersection of sets and $\leq$ to the set inclusion.

In the IF-probability theory [7, 9] instead of the notion of probability, we use the notion of state.

Definition 2.4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$*;*

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$*, then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$*;*

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$*,* $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Probably the most useful result in the IF-state theory is the following representation theorem ([7]):

Theorem 2.5. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = (1-\alpha) \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega \nu_A dP\right)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1_\Omega - x^\sharp(A)\big)$ *for each* $A \in \mathcal{B}(R)$, *then* $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ *are observables, where* $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - \text{measurable}\}$.

Remark 2.7. *Sometimes we need to work with n-dimensional IF-observable $x : \mathcal{B}(R^n) \to \mathcal{F}$ defined as a mapping with the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$ *we simply say that* x *is an IF-observable.*

Similarly to the classical case, the following theorem can be proved ([9]).

Theorem 2.8. *Let* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be an IF-observable,* $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ *be an IF-state. Define the mapping* $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *is a probability measure.*

3 Product operation, joint IF-observable and function of several IF-observables

In [5] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ and showed an example of this operation.

Definition 3.1. *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfying the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$;

(ii) *the operation $\cdot$ is commutative and associative;*

(iii) *if* $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$*, then*

$$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$$

and

$$\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$$

for each $(c_1, c_2) \in \mathcal{F}$*;*

(iv) *if* $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$*,* $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$*, then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$.

The following theorem provides an example of product operation for IF-events.

Theorem 3.2 ([5, Theorem 1])**.** *The operation* $\cdot$ *defined by*

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ *is product operation on* $\mathcal{F}$*.*

In [8] B. Riečan defined the notion of a joint IF-observable and proved its existence.

Definition 3.3. *Let* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be two IF-observables. The joint IF-observable of the IF-observables* x, y *is a mapping* $h : \mathcal{B}(R^2) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$*,* $h(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$*, then*

$$h(A \cup B) = h(A) \oplus h(B) \text{ and } h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$*, then* $h(A_n) \nearrow h(A)$*;*

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$*.*

Theorem 3.4 ([8, Theorem 3.3])**.** *For each two IF-observables* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *there exists their joint IF-observable.*

Remark 3.5. *The joint IF-observable of IF-observables* x, y *from Definition 3.3 is a two-dimensional IF-observable.*

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. Regarding this we provide the following definition.

Definition 3.6. *Let* $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ *be IF-observables,* h_n *their joint IF-observable and* $g_n : R^n \to R$ *a Borel measurable function. Then we define the IF-observable* $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ *by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$*.*

4 Lower and upper limits, $\mathbf{m}$-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables and showed the connection between two kinds of $\mathbf{m}$-almost everywhere convergence.

Definition 4.1. *We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup\limits_{n\to\infty}$, if there exists an IF-observable $\overline{x} : \mathcal{B}(R) \to \mathcal{F}$ such that*

$$\overline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for every $t \in R$. We write $\overline{x} = \limsup\limits_{n\to\infty} x_n$.

Note that if another IF-observable y satisfies the above condition, then $\mathbf{m} \circ y = \mathbf{m} \circ \overline{x}$.

Definition 4.2. *A sequence $(x_n)_n$ of IF-observables has $\liminf\limits_{n\to\infty}$, if there exists an IF-observable $\underline{x}$ such that*

$$\underline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for all $t \in R$. Notation: $\underline{x} = \liminf\limits_{n\to\infty} x_n$.

Theorem 4.3 ([4, Theorem 3.3]). *The IF-observables $\overline{x}$, $\underline{x}$ from Definition 4.1 and Definition 4.2 can be expressed in the following form*

$$\overline{x}(A) = \Big(\overline{x^\flat}(A), 1_\Omega - \overline{x^\sharp}(A)\Big),$$

$$\underline{x}(A) = \Big(\underline{x^\flat}(A), 1_\Omega - \underline{x^\sharp}(A)\Big),$$

for each $A \in \mathcal{B}(R)$. Here $\overline{x^\flat}$, $\underline{x^\flat}$ are upper and lower limits of sequence $(x_n^\flat)_1^\infty$ of observables in tribe $\mathcal{T}$ and $\overline{x^\sharp}$, $\underline{x^\sharp}$ are upper and lower limits of sequence $(x_n^\sharp)_1^\infty$ of observables in tribe $\mathcal{T}$ (see [6]).

Proposition 4.1 ([4, Proposition 3.1]). *If a sequence of IF-observables $(x_n)_n$ has $\overline{x} = \limsup\limits_{n\to\infty} x_n$ and $\underline{x} = \liminf\limits_{n\to\infty} x_n$, then*

$$\overline{x}((-\infty, t)) \leq \underline{x}((-\infty, t)),$$

for every $t \in R$.

Proposition 4.2 ([4, Proposition 4.1]). *A sequence $(x_n)_n$ of an IF-observables converges $\mathbf{m}$-almost everywhere to 0 if and only if*

$$\begin{aligned}\mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) &= \mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) = \\ &= \mathbf{m}\big(0_{\mathcal{F}}((-\infty, t))\big),\end{aligned}$$

for every $t \in R$.

In accordance to Proposition 4.2 we can extend the notion of $\mathbf{m}$-almost everywhere convergence in the following way.

Definition 4.4. *A sequence $(x_n)_n$ of an IF-observables converges $\mathbf{m}$-almost everywhere to an IF-observable x, if*

$$\begin{aligned}\mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) &= \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \\ &= \mathbf{m}\big(x((-\infty, t))\big),\end{aligned}$$

for every $t \in R$.

5 P-almost everywhere convergence and m-almost everywhere convergence

The main result of this section is given in Theorem 5.1. The main step is presented in the following proposition.

Recall, that the corresponding probability space is $(R^N, \sigma(\mathcal{C}), P)$, where $\mathcal{C}$ is the family of all sets of the form

$$\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\},$$

and P is the probability measure determined by the equality

$$P\big(\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\}\big) = \mathbf{m}\big(x_1(A_1) \cdot \ldots \cdot x_n(A_n)\big).$$

The corresponding projections $\xi_n : R^N \to R$ are defined by the equality

$$\xi_n\big((t_i)_{i=1}^{\infty}\big) = t_n.$$

Proposition 5.1. *Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ the sequence of corresponding projections, $g_n : R^n \to R$ be a Borel measurable functions $(n = 1, 2, \ldots)$. Then*

$$\begin{aligned}P\Big(\{u \in R^N \;:\; &\limsup_{n\to\infty} g_n\big(\xi_1(u), \ldots, \xi_n(u)\big) < t\}\Big) \le \\ &\le \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big), \\ P\Big(\{u \in R^N \;:\; &\liminf_{n\to\infty} g_n\big(\xi_1(u), \ldots, \xi_n(u)\big) < t\}\Big) \ge \\ &\ge \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big).\end{aligned}$$

Proof. We have

$$
\begin{aligned}
P\Big(\{u \in R^N \quad : \quad & \limsup_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\}\Big) = \\
&= P\Big(\bigcup_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\Big\{u \in R^N : g_n(u_1,\dots,u_n) < t - \frac{1}{p}\Big\}\Big) = \\
&= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\bigcap_{n=k}^{k+i}\Big(\pi_n^{-1}\Big(g_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big)\Big)\Big) = \\
&= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\pi_{k+i}^{-1}\Big(\bigcap_{n=k}^{k+i} g_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big)\Big) = \\
&= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} \mathbf{m}\Big(h_{k+i}\Big(\bigcap_{n=k}^{k+i} g_n^{-1}\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big)\Big) \le \\
&\le \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} \mathbf{m}\Big(\bigwedge_{n=k}^{k+i}(h_{k+i}\circ g_n^{-1})\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \\
&= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} \mathbf{m}\Big(\bigwedge_{n=k}^{k+i} g_n(x_1,\dots,x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \\
&= \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} g_n(x_1,\dots,x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big).
\end{aligned}
$$

The second inequality can be proved similarly. □

Theorem 5.1. *Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ be the sequence of corresponding projections, $(g_n)_n$ be a sequence of Borel measurable functions $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1,\dots,\xi_n)\big)_n$ converges P-almost everywhere, then the sequence $\big(g_n(x_1,\dots,x_n)\big)_n$ converges $\mathbf{m}$-almost everywhere and*

$$
\mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\dots,x_n)\big((-\infty,t)\big)\Big) = \mathbf{m}\Big(\liminf_{n\to\infty} g_n(x_1,\dots,x_n)\big((-\infty,t)\big)\Big)
$$

for each $t \in R$. Moreover

$$
P\Big(\{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\}\Big) = \mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\dots,x_n)\big((-\infty,t)\big)\Big)
$$

for each $t \in R$.

Proof. Let the sequence $\big(g_n(\xi_1,\dots,\xi_n)\big)_n$ converges P-almost everywhere, then

$$
\begin{aligned}
P\Big(\{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\}\Big) &= \\
= P\Big(\{u \in R^N : \liminf_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\}\Big).&
\end{aligned} \tag{1}
$$

Put

$$
\varphi(t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} g_n(x_1,\dots,x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),
$$

$$
\psi(t) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} g_n(x_1,\dots,x_n)\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),
$$

and

$$\eta_n(u) = g_n(\xi_1(u), \dots, \xi_n(u)).$$

Since $\varphi(t) \le \psi(t)$, then

$$\mathbf{m}(\varphi(t)) \le \mathbf{m}(\psi(t)).$$

By Proposition 5.1 and (1) we obtain

$$\begin{aligned}\mathbf{m}(\psi(t)) \le P\Big(\{u \in R^N : \liminf_{n\to\infty} \eta_n(u) < t\}\Big) &= P\Big(\{u \in R^N : \limsup_{n\to\infty} \eta_n(u) < t\}\Big) \\ &\le \mathbf{m}(\varphi(t)).\end{aligned}$$

Hence

$$\mathbf{m}(\varphi(t)) = \mathbf{m}(\psi(t)),$$

and moreover

$$P\Big(\{u \in R^N : \limsup_{n\to\infty} \eta_n(u) < t\}\Big) = \mathbf{m}(\varphi(t)) = \mathbf{m}(\psi(t)).$$

Denote the common value by

$$F(t) = P\Big(\{u \in R^N : \limsup_{n\to\infty} \eta_n(u) < t\}\Big) = \mathbf{m}(\varphi(t)) = \mathbf{m}(\psi(t)).$$

Since $\limsup_{n\to\infty} \eta_n$ is a random variable, then $F : R \to [0,1]$ is a distribution function. Evidently

$$\begin{aligned}\mathbf{m}\Big(\bigvee_{n=1}^{\infty} \varphi(n)\Big) &= \lim_{n\to\infty} \mathbf{m}\big(\varphi(n)\big) = \lim_{n\to\infty} F(n) = 1, \\ \mathbf{m}\Big(\bigwedge_{n=1}^{\infty} \varphi(-n)\Big) &= \lim_{n\to\infty} \mathbf{m}\big(\varphi(-n)\big) = \lim_{n\to\infty} F(-n) = 0.\end{aligned}$$

Since $\mathbf{m}$ is faithful, we obtain

$$\bigvee_{n=1}^{\infty} \varphi(n) = (1_\Omega, 0_\Omega), \ \bigwedge_{n=1}^{\infty} \varphi(-n) = (0_\Omega, 1_\Omega). \tag{2}$$

Let $t_n \nearrow t$. Evidently $\varphi(t_n) \le \varphi(t)$, hence

$$\bigvee_{n=1}^{\infty} \varphi(n) \le \varphi(t).$$

On the other hand to each $p \in N$ there exist $j, q \in N$ such that $\big(-\infty, t - \frac{1}{p}\big) \subset \big(-\infty, t_j - \frac{1}{q}\big)$, hence

$$g_n(x_1, \dots, x_n)\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big) \le g_n(x_1, \dots, x_n)\Big(\Big(-\infty, t_j - \frac{1}{q}\Big)\Big)$$

and therefore

$$\bigwedge_{n=k}^{\infty} g_n(x_1, \dots, x_n)\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big) \le \bigwedge_{n=k}^{\infty} g_n(x_1, \dots, x_n)\Big(\Big(-\infty, t_j - \frac{1}{q}\Big)\Big) \le \varphi(t_j) \le \bigvee_{j=1}^{\infty} \varphi(t_j).$$

Since the relation holds for every k and p, we obtain

$$\varphi(t) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big) \leq \bigvee_{j=1}^{\infty} \varphi(t_j).$$

Hence

$$\varphi(t) = \bigvee_{j=1}^{\infty} \varphi(t_j). \tag{3}$$

Let us summarize: φ is non-decreasing, $\bigvee_{n=1}^{\infty} \varphi(n) = (1_\Omega, 0_\Omega)$, $\bigwedge_{n=1}^{\infty} \varphi(-n) = (0_\Omega, 1_\Omega)$, $t_n \nearrow t$ implies $\varphi(t_n) \nearrow \varphi(t)$.

Put $y_n = g_n(x_1, \ldots, x_n)$. Using max-min connectives $\vee$, $\wedge$, the De Morgan rules and equality $y_n(A) = \big(y_n^\flat(A), 1_\Omega - y_n^\sharp(A)\big)$ we obtain

$$\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n(A) = \Big(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\flat(A), 1_\Omega - \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\sharp(A)\Big)$$

for each $A \in \mathcal{B}(R)$, where $y_n^\flat, y_n^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables. Put

$$\begin{aligned} \varphi^\flat(t) &= \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\flat\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big), \\ \varphi^\sharp(t) &= \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\sharp\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big). \end{aligned}$$

Then

$$\varphi(t) = \big(\varphi^\flat(t), 1_\Omega - \varphi^\sharp(t)\big).$$

Since φ is non-decreasing, therefore $\varphi^\flat, \varphi^\sharp$ are non-decreasing. More by (2) we have

$$\begin{aligned} (1_\Omega, 0_\Omega) &= \bigvee_{n=1}^{\infty} \varphi(n) = \bigvee_{n=1}^{\infty} \big(\varphi^\flat(n), 1_\Omega - \varphi^\sharp(n)\big) = \Big(\bigvee_{n=1}^{\infty} \varphi^\flat(n), 1_\Omega - \bigvee_{n=1}^{\infty} \varphi^\sharp(n)\Big), \\ (0_\Omega, 1_\Omega) &= \bigwedge_{n=1}^{\infty} \varphi(-n) = \bigwedge_{n=1}^{\infty} \big(\varphi^\flat(-n), 1_\Omega - \varphi^\sharp(-n)\big) = \Big(\bigwedge_{n=1}^{\infty} \varphi^\flat(-n), 1_\Omega - \bigwedge_{n=1}^{\infty} \varphi^\sharp(-n)\Big). \end{aligned}$$

Hence

$$\bigvee_{n=1}^{\infty} \varphi^\flat(n) = 1_\Omega, \ \bigwedge_{n=1}^{\infty} \varphi^\flat(-n) = 0_\Omega, \tag{4}$$

$$\bigvee_{n=1}^{\infty} \varphi^\sharp(n) = 1_\Omega, \ \bigwedge_{n=1}^{\infty} \varphi^\sharp(-n) = 0_\Omega. \tag{5}$$

Since $t_n \nearrow t$ implies

$$\big(\varphi^\flat(t_n), 1_\Omega - \varphi^\sharp(t_n)\big) = \varphi(t_n) \nearrow \varphi(t) = \big(\varphi^\flat(t), 1_\Omega - \varphi^\sharp(t)\big),$$

then

$$\varphi^\flat(t_n) \nearrow \varphi^\flat(t), \tag{6}$$

$$1_\Omega - \varphi^\sharp(t_n) \searrow 1_\Omega - \varphi^\sharp(t) \Longleftrightarrow \varphi^\sharp(t_n) \nearrow \varphi^\sharp(t). \tag{7}$$

For fixed $\omega \in \Omega$ and arbitrary $t \in R$ put

$$F^\flat_\omega(t) = \varphi^\flat(t)(\omega),\ F^\sharp_\omega(t) = \varphi^\sharp(t)(\omega).$$

Evidently $F^\flat_\omega, F^\sharp_\omega : R \to [0,1]$ are the non-decreasing functions and by (4), (6) and by (5), (7) we obtain that $F^\flat_\omega, F^\sharp_\omega$ are the distribution functions. Denote by $\lambda^\flat_\omega$, $\lambda^\sharp_\omega$ the corresponding Stieltjes probability measures and define $\overline{y}^\flat, \overline{y}^\sharp : \mathcal{B}(R) \to \mathcal{T}$ by the equalities

$$\overline{y}^\flat(A)(\omega) = \lambda^\flat_\omega(A),\ \overline{y}^\sharp(A)(\omega) = \lambda^\sharp_\omega(A).$$

Then $\overline{y}^\flat, \overline{y}^\sharp$ are the observables and

$$\overline{y}^\flat\big((-\infty,t)\big)(\omega) = \lambda^\flat_\omega\big((-\infty,t)\big) = F^\flat_\omega(t) = \varphi^\flat(t)(\omega), \tag{8}$$

$$\overline{y}^\sharp\big((-\infty,t)\big)(\omega) = \lambda^\sharp_\omega\big((-\infty,t)\big) = F^\sharp_\omega(t) = \varphi^\sharp(t)(\omega), \tag{9}$$

for each $\omega \in \Omega$. Using Theorem 4.3 and (8), (9) we have that there exists IF-observable $\overline{y} = \limsup_{n\to} y_n$ given by

$$\overline{y}\big((-\infty,t)\big) = \Big(\overline{y}^\flat\big((-\infty,t)\big), 1_\Omega - \overline{y}^\sharp\big((-\infty,t)\big)\Big) = \big(\varphi^\flat(t), 1_\Omega - \varphi^\sharp(t)\big) = \varphi(t).$$

The existence of IF-observable $\underline{y} = \liminf_{n\to} y_n = \psi$ can be proved similarly. Since $\mathbf{m}(\varphi(t)) = \mathbf{m}(\psi(t))$, then

$$\mathbf{m}\Big(\overline{y}\big((-\infty,t)\big)\Big) = \mathbf{m}\Big(\underline{y}\big((-\infty,t)\big)\Big)$$

for each $t \in R$ and by Definition 4.4 we have that $(y_n)_n = \big(g_n(x_1,\dots,x_n)\big)_n$ converges $\mathbf{m}$-almost everywhere. Moreover

$$P\Big(\{u \in R^N : \limsup_{n\to\infty} \eta_n(u) < t\}\Big) = \mathbf{m}\Big(\overline{y}\big((-\infty,t)\big)\Big)$$

for each $t \in R$. □

6 Conclusion

The Theorem 5.1 is important for the proof of the Individual ergodic theorem in intuitionistic fuzzy case, where we work with the sequence of several IF-observables induced by the Borel function.

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