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Algebraic structures of circular intuitionistic fuzzy triples: Monoids, multi-unitary groups, and lattice properties

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Abstract: This paper investigates the algebraic structures induced by binary operations on Circular Intuitionistic Fuzzy Triples (C-IFTs). For a fixed upper bound on the circular radius, the neutral and absorbing elements of the standard union, intersection, algebraic addition, and algebraic multiplication are determined, considering both minimum and maximum aggregation of the radius components. It is proved that the corresponding algebraic structures are commutative monoids and commutative multi-unitary groups (μ-groups), although they do not form ordinary

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groups on the entire domain of C-IFTs. The sets of μ-inverse elements are described explicitly, and the dual μ-groups are determined.

Furthermore, it is established that the standard union with maximum radius aggregation and intersection with minimum radius aggregation form a bounded distributive lattice. The relationships between the operations are examined under the standard radius-preserving negation. It is shown that the duality generated by the negation preserves the radius operation, whereas the duality of μ-groups requires the opposite one.

The results characterize the algebraic properties of the selected C-IFT operations and provide a basis for further study of aggregation procedures and circular intuitionistic fuzzy index-matrix constructions.

Keywords: Circular intuitionistic fuzzy triples, Circular intuitionistic fuzzy sets, Binary operations, Commutative monoids, Multi-unitary groups, Distributive lattices, Radius aggregation.
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1 Introduction

Intuitionistic Fuzzy Sets (IFSs) [1, 2] and their evaluations, the Intuitionistic Fuzzy Pairs (IFPs) [8], are equipped with a large family of operations. The algebraic properties of these operations are important both theoretically and in applications, since aggregation procedures, index-matrix operations, and optimization algorithms may depend on associativity, commutativity, and the existence of neutral elements. For IFPs, the operations "$+$" and "$\cdot$" were investigated from this perspective in [11].

In [4, Section 1.7], Atanassov introduced the concept of a multi-unitary group (μ-group), extending the classical group framework by considering an associative operation with a neutral element and a distinguished element playing the role of a generalized inverse target. This concept includes structures such as $(\mathcal{P}(X), \cup, \varnothing, X)$, which are not groups in the ordinary sense.

Circular Intuitionistic Fuzzy Sets (C-IFSs) [5] extend IFSs by incorporating a radius that describes a neighborhood of an intuitionistic fuzzy point. Circular Intuitionistic Fuzzy Triples (C-IFTs) provide the basic numerical representation of these evaluations and are relevant to circular intuitionistic fuzzy index matrices [3, 6] and decision-making and optimization models [9, 10].

Previous studies have investigated algebraic properties of intuitionistic fuzzy pairs and index matrices. In particular, algebraic structures associated with modified intuitionistic fuzzy index matrices were examined in [6]. For C-IFT operations involving variable radii, however, the neutral and absorbing elements must be specified together with their radius components. This motivates a systematic investigation of the corresponding algebraic structures.

The aim of the present paper is to investigate the algebraic properties of C-IFTs under four binary operations, each combined with either $\min$ or $\max$ on the radius components. For these eight operations, we establish commutative monoid and μ-group structures, determine neutral and absorbing elements, characterize ordinary invertible and μ-inverse elements, and investigate the duality of the corresponding μ-groups.

We also establish a bounded distributive lattice structure under the componentwise partial order and distinguish the duality induced by negation from the duality of μ-groups. The results are developed within the framework introduced in [4], with particular attention to the role of the radius operation.

The obtained results provide an algebraic foundation for subsequent investigations of multidimensional circular intuitionistic fuzzy index matrices.

The remainder of the paper is organized as follows. Section 2 recalls the basic definitions, operations, relations, and the concept of a μ-group. Section 3 establishes the commutative monoid and μ-group structures, characterizes invertible and μ-inverse elements. Section 4 investigates the bounded distributive lattice structure and the two forms of duality. Finally, Section 5 presents the discussion and conclusions.

2 Preliminaries and algebraic conventions

This section recalls the fundamental concepts of Circular Intuitionistic Fuzzy Sets (C-IFSs) and Circular Intuitionistic Fuzzy Triples (C-IFTs), together with their basic operations and relations. These concepts provide the mathematical foundation for the investigation of the corresponding algebraic structures.

2.1 Circular intuitionistic fuzzy sets and circular intuitionistic fuzzy triples

Intuitionistic Fuzzy Sets (IFSs), introduced in [1, 2], are characterized by membership and non-membership degrees satisfying the intuitionistic fuzzy admissibility condition. Circular Intuitionistic Fuzzy Sets (C-IFSs), introduced in [5], extend this representation by associating a radius with the corresponding intuitionistic fuzzy point.

Following [7], let $R = \sqrt{2}$ denote the upper bound of the radius component.

Definition 1 (Circular Intuitionistic Fuzzy Set)**.** *Let X be a non-empty universe. A Circular Intuitionistic Fuzzy Set (C-IFS) A on X is represented as*

$$A = \{\langle x, \mu_A(x), \nu_A(x); r_A(x)\rangle : x \in X\}, \tag{1}$$

where $\mu_A(x), \nu_A(x) \in [0,1]$, $\mu_A(x) + \nu_A(x) \leq 1$, and $r_A(x) \in [0, R]$. Here, $\mu_A(x)$ and $\nu_A(x)$ denote the membership and non-membership degrees, respectively, while $r_A(x)$ represents the radius associated with the corresponding intuitionistic fuzzy point.

Following the circular representation in [5, 6], the corresponding numerical element can be expressed as a circular intuitionistic fuzzy triple.

Definition 2. *A Circular Intuitionistic Fuzzy Triple (C-IFT) is an ordered triple*

$$x = \langle a, b; r\rangle, \tag{2}$$

where $a, b \in [0,1],\ a + b \leq 1,\ r \in [0, R]$. The set of all admissible C-IFTs is denoted by

$$\mathcal{C}_R = \{\langle a, b; r\rangle : a, b \in [0,1],\ a + b \leq 1,\ r \in [0, R]\}. \tag{3}$$

The first two components of a C-IFT form an admissible intuitionistic fuzzy pair, while the third component specifies the associated circular radius.

In the algebraic framework considered here, the radius is treated as an independent stored component within the prescribed interval. No additional requirement that the corresponding closed disk be entirely contained in the intuitionistic fuzzy triangle is imposed.

Thus, as a set, $\mathcal{C}_R = \Delta \times [0, R]$, where $\Delta = \{(a, b) \in [0, 1]^2 + b \leq 1\}$ is the domain of intuitionistic fuzzy pairs. Each operation considered below combines an operation on Δ with either minimum or maximum on the radius interval. The resulting algebraic structures therefore have a direct-product form.

2.2 Operations on circular intuitionistic fuzzy triples

The basic operations on intuitionistic fuzzy pairs have been studied in [4, 11]. Their extension to circular intuitionistic fuzzy structures is considered in [5, 6].

Let

$$x = \langle a, b; r_1 \rangle, \qquad y = \langle c, d; r_2 \rangle$$

be two elements of $\mathcal{C}_R$.

A general binary operation on C-IFTs can be represented by

$$x \circ^{\circ_r}_{(\circ_1, \circ_2)} y = \langle \circ_1(a, c), \circ_2(b, d); \circ_r(r_1, r_2) \rangle \,, \tag{4}$$

where $\circ_1$ and $\circ_2$ act on the membership and non-membership components, respectively, while

$$\circ_r \in \{\min, \max\}$$

acts on the radius components.

For a fixed radius operation $\circ_r$, the basic intuitionistic fuzzy operations are defined as follows in standard and abbreviated forms:

$$x \vee^{\circ_r}_{(\max,\min)} y = x \vee^{\circ_r}_1 y = \langle \max(a, c), \min(b, d); \circ_r(r_1, r_2) \rangle \,, \tag{5}$$

$$x \wedge^{\circ_r}_{(\min,\max)} y = x \wedge^{\circ_r}_1 y = \langle \min(a, c), \max(b, d); \circ_r(r_1, r_2) \rangle \,, \tag{6}$$

$$x \wedge^{\circ_r}_2 y = x \wedge^{\circ_r}_{(+,\cdot)} y = \langle a + c - ac, bd; \circ_r(r_1, r_2) \rangle \,, \tag{7}$$

$$x \vee^{\circ_r}_2 y = x \vee^{\circ_r}_{(\cdot,+)} y = \langle ac, b + d - bd; \circ_r(r_1, r_2) \rangle \,, \tag{8}$$

The operations $\vee^{\circ_r}_1$ and $\wedge^{\circ_r}_1$ are the standard union (maximum on the membership degrees) and intersection (minimum on the membership degrees). The operations $\wedge^{\circ_r}_2$ and $\vee^{\circ_r}_2$ correspond to algebraic addition and algebraic multiplication, respectively.

The classical negation is defined by

$$\neg x = \langle b, a; r_1 \rangle. \tag{9}$$

Thus, negation exchanges the membership and non-membership degrees while preserving the radius component.

The algebraic properties of the above binary operations depend on the corresponding componentwise operations and on the selected radius aggregation rule $\circ_r$.

2.3 Relations on circular intuitionistic fuzzy triples

The componentwise ordering of intuitionistic fuzzy pairs is discussed in [4, 6]. For C-IFTs, the radius component is additionally taken into account.

Let $x = \langle a, b; r_1 \rangle$, $y = \langle c, d; r_2 \rangle$.

The equality relation is defined by

$$x = y \iff a = c, \quad b = d, \quad r_1 = r_2. \tag{10}$$

The componentwise partial order is defined by

$$x \preceq y \iff a \leq c, \quad b \geq d, \quad r_1 \leq r_2. \tag{11}$$

Equivalently,

$$x \succeq y \iff a \geq c, \quad b \leq d, \quad r_1 \geq r_2. \tag{12}$$

The corresponding strict partial order is given by

$$x \prec y \iff x \preceq y \text{ and } x \neq y. \tag{13}$$

Similarly,

$$x \succ y \iff x \succeq y \text{ and } x \neq y. \tag{14}$$

The relation $\preceq$ is reflexive, antisymmetric, and transitive. Consequently, it defines a partial order on $\mathcal{C}_R$.

Not every pair of C-IFTs is comparable under this relation.

In the subsequent algebraic analysis, the componentwise partial order (11) is used to investigate the corresponding lattice structures of C-IFTs.

The relation $\preceq$ is an algebraic order on the stored components and does not, in general, express inclusion of the corresponding circular regions.

2.4 Multi-unitary groups

The concept of a multi-unitary group (μ-group) was introduced in [4] as a generalization of the classical group structure.

Definition 3 (Multi-unitary group)**.** *Following [4, Section 1.7] (formulas (1.7.11)–(1.7.15) there), a multi-unitary group (μ-group) is a quadruple*

$$(M, *, e_*, e_\circ),$$

where M is a non-empty set, $* : M \times M \to M$ is an associative binary operation, $e_* \in M$ is a two-sided neutral element satisfying $x * e_* = e_* * x = x$ for every $x \in M$, and $e_\circ \in M$ satisfies

$$\forall x \in M \; \exists y \in M : \quad x * y = y * x = e_\circ. \tag{15}$$

The structure is called a commutative μ-group if the operation $*$ is commutative.

The element $e_\circ$ is not required to be a second ordinary neutral element. If $e_* = e_\circ$, the definition coincides with that of a classical group.

Following [4], two μ-groups are called dual *if and only if they have the forms*

$$(M, *, e_*, e_\circ) \quad \textit{and} \quad (M, \circ, e_\circ, e_*)$$

for some binary operations $*$ and $\circ$ over M.

For example, $(\mathcal{P}(X), \cup, \varnothing, X)$ and $(\mathcal{P}(X), \cap, X, \varnothing)$ are dual commutative μ-groups [4].

For IFPs, the commutativity and associativity of the union, intersection, "$+$" and "$\cdot$", as well as the roles of $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ as their neutral and absorbing elements, are known [4, 8, 11]. The new element in the case of C-IFTs is the radius component, whose neutral and absorbing values depend on the selected radius operation.

Throughout the paper, for $\circ_r \in \{\min, \max\}$, we denote by $\overline{\circ_r}$ the other radius operation, i.e., $\overline{\min} = \max$ and $\overline{\max} = \min$, and by

$$r^e_{\circ_r} = \begin{cases} 0, & \circ_r = \max, \\ R, & \circ_r = \min, \end{cases} \qquad r^z_{\circ_r} = \begin{cases} R, & \circ_r = \max, \\ 0, & \circ_r = \min, \end{cases}$$

the neutral and the absorbing element of $\circ_r$ on $[0, R]$.

3 Monoid and multi-unitary group structures

This section investigates the algebraic properties of the operations defined in Section 2. In particular, closure, associativity, commutativity, and the existence of neutral and absorbing elements are established.

Lemma 1 (Closure)**.** *For every fixed $\circ_r \in \{\min, \max\}$, the operations $\vee_1^{\circ_r}$, $\wedge_1^{\circ_r}$, $\wedge_2^{\circ_r}$, and $\vee_2^{\circ_r}$ are closed on $\mathcal{C}_R$.*

Proof. Let $x = \langle a, b; r_1\rangle$, $y = \langle c, d; r_2\rangle$ be arbitrary elements of $\mathcal{C}_R$.

Since $r_1, r_2 \in [0, R]$, both $\min(r_1, r_2)$ and $\max(r_1, r_2)$ belong to $[0, R]$.

For $\vee_1^{\circ_r}$, if $a \geq c$, then

$$\max(a, c) + \min(b, d) \leq a + b \leq 1.$$

If $c \geq a$, the inequality follows analogously from $c + d \leq 1$.

For $\wedge_1^{\circ_r}$ the check is similar.

For $\wedge_2^{\circ_r}$, the admissibility conditions $b \leq 1 - a$ and $d \leq 1 - c$ imply

$$bd \leq (1 - a)(1 - c) = 1 - (a + c - ac).$$

Consequently,

$$(a + c - ac) + bd \leq 1.$$

For $\vee_2^{\circ_r}$ the check is similar.

All resulting membership and non-membership degrees belong to $[0, 1]$. Hence all four operations are closed on $\mathcal{C}_R$. □

Theorem 1 (Commutative monoid structures)**.** *For every fixed radius operation $\circ_r \in \{\min, \max\}$, each of the algebraic structures*

$$(\mathcal{C}_R, \vee_1^{\circ_r}), \quad (\mathcal{C}_R, \wedge_1^{\circ_r}),$$
$$(\mathcal{C}_R, \wedge_2^{\circ_r}), \quad (\mathcal{C}_R, \vee_2^{\circ_r})$$

is a commutative monoid.

The corresponding neutral elements e_* and absorbing elements $e_\circ$ are given in Table 1.

Table 1. Neutral and absorbing elements of the C-IFT operations.

Operation	**Neutral element e_***	**Absorbing element $e_\circ$**
$\vee_1^{\min}$, $\wedge_2^{\min}$	$\langle 0, 1; R\rangle$	$\langle 1, 0; 0\rangle$
$\vee_1^{\max}$, $\wedge_2^{\max}$	$\langle 0, 1; 0\rangle$	$\langle 1, 0; R\rangle$
$\wedge_1^{\min}$, $\vee_2^{\min}$	$\langle 1, 0; R\rangle$	$\langle 0, 1; 0\rangle$
$\wedge_1^{\max}$, $\vee_2^{\max}$	$\langle 1, 0; 0\rangle$	$\langle 0, 1; R\rangle$

Proof. Closure has been established in Lemma 1.

The operations $\min$, $\max$, $u + v - uv$ and uv are associative and commutative.

Since the four operations on C-IFTs are defined componentwise using these scalar operations, they are also associative and commutative.

For $\vee_1^{\circ_r}$ and $\wedge_2^{\circ_r}$, the intuitionistic fuzzy pair $\langle 0, 1\rangle$ is neutral, whereas $\langle 1, 0\rangle$ is absorbing.

For $\wedge_1^{\circ_r}$ and $\vee_2^{\circ_r}$, the roles of these two pairs are reversed.

On the interval $[0, R]$, the neutral element of $\min$ is R, and its absorbing element is 0. For $\max$, the neutral element is 0, and its absorbing element is R.

Combining the corresponding components yields exactly the elements listed in Table 1.

Therefore, all eight structures are commutative monoids. □

Corollary 1 (Multi-unitary group structures)**.** *Each of the eight commutative monoids established in Theorem 1, together with its corresponding elements e_* and $e_\circ$ from Table 1, forms a commutative multi-unitary group (μ-group) in the sense of Definition 3.*

Proof. By Theorem 1, each considered operation is closed, associative, and commutative, and possesses a two-sided neutral element e_*.

Moreover, the corresponding element $e_\circ$ is absorbing. Therefore, for every $x \in \mathcal{C}_R$,

$$x * e_\circ = e_\circ * x = e_\circ.$$

Choosing $y = e_\circ$ in condition (15) establishes the required μ-group property. □

Remark 1. *The established μ-group structures should not be identified with ordinary groups. The element $e_\circ$ is absorbing rather than a second neutral element.*

Furthermore, the proof of Corollary 1 uses the absorbing element as a common μ-partner for all elements of $\mathcal{C}_R$. Consequently, the μ-group classification follows directly from the algebraic properties of the corresponding monoids.

Theorem 2 (Absence of ordinary group structures)**.** *None of the eight commutative monoids established in Theorem 1 is an ordinary group.*

Proof. The operations $\vee_1^{\circ_r}$ and $\wedge_1^{\circ_r}$ are idempotent, since

$$x \vee_1^{\circ_r} x = x, \qquad x \wedge_1^{\circ_r} x = x$$

for every $x \in \mathcal{C}_R$.

In an ordinary group, the equality $x * x = x$ implies $x = e_*$ by cancellation. Since $\mathcal{C}_R$ contains more than one element, neither of these operations can define a group structure.

For the operations $\wedge_2^{\circ_r}$ and $\vee_2^{\circ_r}$, Table 1 identifies distinct neutral and absorbing elements.

Suppose that one of these structures were a group with absorbing element $e_\circ$. Then, for every $x \in \mathcal{C}_R$,

$$e_\circ * x = e_\circ = e_\circ * e_*.$$

By the cancellation law, $x = e_*$ for every $x \in \mathcal{C}_R$.

This contradicts the nontriviality of $\mathcal{C}_R$. Therefore, none of the eight monoids is an ordinary group. □

Proposition 1 (Invertible elements)**.** *For each of the eight commutative monoids of Theorem 1, the neutral element e_* is the only element possessing an inverse with respect to the corresponding monoid operation.*

Proof. Let $*$ denote one of the considered binary operations.

For the idempotent operations $\vee_1^{\circ_r}$ and $\wedge_1^{\circ_r}$, suppose that $x * y = e_*$.

Then, by associativity and idempotency,

$$x = x * e_* = x * (x * y) = (x * x) * y = x * y = e_*.$$

For the algebraic addition $\wedge_2^{\circ_r}$, suppose that $x \wedge_2^{\circ_r} y = e_*$.

Since the membership and non-membership components of e_* are 0 and 1, respectively, we obtain

$$a + c - ac = 0, \qquad bd = 1.$$

As $a, b, c, d \in [0, 1]$, these equalities imply

$$a = c = 0, \qquad b = d = 1.$$

The radius component must satisfy $\circ_r(r_1, r_2) = r_*$, where r_* denotes the neutral element of the chosen radius operation.

If $\circ_r = \min$, then $r_* = R$ and

$$\min(r_1, r_2) = R$$

implies $r_1 = r_2 = R$.

If $\circ_r = \max$, then $r_* = 0$ and

$$\max(r_1, r_2) = 0$$

implies $r_1 = r_2 = 0$.

Thus, $x = y = e_*$.

For the algebraic multiplication $\vee_2^{\circ_r}$, the equality $x \vee_2^{\circ_r} y = e_*$ requires

$$ac = 1, \qquad b + d - bd = 0.$$

Consequently,

$$a = c = 1, \qquad b = d = 0.$$

The radius condition again implies $r_1 = r_2 = r_*$.

Hence, for each of the eight monoids, the only invertible element is its neutral element. □

The proof of Corollary 1 uses only one μ-partner, namely $e_\circ$. The following theorem describes all μ-partners.

Theorem 3 (μ-inverse elements)**.** *Let $*$ be one of the eight operations of Theorem 1 with radius operation $\circ_r$, and let $e_\circ$ be its absorbing element. For $x = \langle a, b; r_1\rangle \in \mathcal{C}_R$, let*

$$\mathrm{Inv}(x) = \{y \in \mathcal{C}_R : \ x * y = e_\circ\}.$$

Then

$$\mathrm{Inv}(x) = \big\{\langle c, d; r_2\rangle \in \mathcal{C}_R : (a = 1 \textit{ or } c = 1) \textit{ and } (r_1 = r^z_{\circ_r} \textit{ or } r_2 = r^z_{\circ_r})\big\}$$

for $* \in \{\vee_1^{\circ_r}, \wedge_2^{\circ_r}\}$, and

$$\mathrm{Inv}(x) = \big\{\langle c, d; r_2\rangle \in \mathcal{C}_R : (b = 1 \textit{ or } d = 1) \textit{ and } (r_1 = r^z_{\circ_r} \textit{ or } r_2 = r^z_{\circ_r})\big\}$$

for $* \in \{\wedge_1^{\circ_r}, \vee_2^{\circ_r}\}$. In particular, $e_\circ \in \mathrm{Inv}(x)$ for every x, and $\mathrm{Inv}(x) = \{e_\circ\}$ if and only if $a < 1$ (respectively, $b < 1$) and $r_1 \neq r^z_{\circ_r}$.

Proof. Let $y = \langle c, d; r_2\rangle$ and $* = \wedge_2^{\circ_r}$. Then $e_\circ = \langle 1, 0; r^z_{\circ_r}\rangle$, and $a + c - ac = 1$ if and only if $(1 - a)(1 - c) = 0$, i.e., $a = 1$ or $c = 1$. In both cases $bd = 0$, because $a = 1$ implies $b = 0$ and $c = 1$ implies $d = 0$. Since $r^z_{\circ_r}$ is an end point of $[0, R]$, the equality $\circ_r(r_1, r_2) = r^z_{\circ_r}$ holds if and only if $r_1 = r^z_{\circ_r}$ or $r_2 = r^z_{\circ_r}$. For $* = \vee_1^{\circ_r}$, we have $\max(a, c) = 1$ if and only if $a = 1$ or $c = 1$, and then $\min(b, d) = 0$. The cases $\wedge_1^{\circ_r}$ and $\vee_2^{\circ_r}$ are obtained by interchanging the roles of the membership and non-membership degrees.

If $a < 1$ and $r_1 \neq r^z_{\circ_r}$, every $y \in \mathrm{Inv}(x)$ satisfies $c = 1$, hence $d = 0$, and $r_2 = r^z_{\circ_r}$; thus $y = e_\circ$. If $a = 1$, then every $\langle c, d; r^z_{\circ_r}\rangle$ belongs to $\mathrm{Inv}(x)$, and if $r_1 = r^z_{\circ_r}$, then every $\langle 1, 0; r_2\rangle$ belongs to $\mathrm{Inv}(x)$. The argument for $\wedge_1^{\circ_r}$ and $\vee_2^{\circ_r}$ is the same. □

Remark 2. *The absorbing element is the unique μ-partner precisely when the radius differs from its absorbing value and the membership degree, for union and algebraic addition, or the non-membership degree, for intersection and algebraic multiplication, is less than* 1.

4 Distributive lattice and duality properties

This section investigates the lattice structure induced by the standard operations on Circular Intuitionistic Fuzzy Triples and establishes relationships between the operations under the standard negation.

The componentwise partial order $\preceq$ was introduced in Equation (11).

Theorem 4 (Bounded distributive lattice)**.** *The algebraic structure*

$$\left(\mathcal{C}_R, \vee_1^{\max}, \wedge_1^{\min}, \bot, \top\right),$$

where $\bot = \langle 0, 1; 0\rangle$, $\top = \langle 1, 0; R\rangle$, *is a bounded distributive lattice with respect to the partial order* $\preceq$*. In particular, the absorption laws*

$$x \wedge_1^{\min} (x \vee_1^{\max} y) = x, \tag{16}$$

$$x \vee_1^{\max} (x \wedge_1^{\min} y) = x \tag{17}$$

hold for all $x, y \in \mathcal{C}_R$*.*

Moreover, both distributive laws are satisfied.

Proof. Let $x = \langle a, b; r_1\rangle$, $y = \langle c, d; r_2\rangle$. By definition,

$$x \vee_1^{\max} y = \langle \max(a, c), \min(b, d); \max(r_1, r_2)\rangle$$

and

$$x \wedge_1^{\min} y = \langle \min(a, c), \max(b, d); \min(r_1, r_2)\rangle.$$

Under the partial order $\preceq$, these operations give the least upper bound and greatest lower bound, respectively.

Closure follows from Lemma 1.

The absorption laws follow componentwise from the identities

$$\min(u, \max(u, v)) = u$$

and

$$\max(u, \min(u, v)) = u.$$

Furthermore, on a totally ordered set, $\min$ and $\max$ satisfy the distributive identities

$$\min(u, \max(v, w)) = \max(\min(u, v), \min(u, w))$$

and

$$\max(u, \min(v, w)) = \min(\max(u, v), \max(u, w)).$$

Applying these identities to the three components establishes both distributive laws.

Finally, for every $x \in \mathcal{C}_R$,

$$\bot \preceq x \preceq \top.$$

Hence, the structure is a bounded distributive lattice. □

Proposition 2 (Duality of operations under standard negation)**.** *For every fixed* $\circ_r \in \{\min, \max\}$ *and all* $x, y \in \mathcal{C}_R$*, the following identities hold:*

$$\neg(x \vee_1^{\circ_r} y) = (\neg x) \wedge_1^{\circ_r} (\neg y), \tag{18}$$

$$\neg(x \wedge_1^{\circ_r} y) = (\neg x) \vee_1^{\circ_r} (\neg y), \tag{19}$$

$$\neg(x \wedge_2^{\circ_r} y) = (\neg x) \vee_2^{\circ_r} (\neg y), \tag{20}$$

$$\neg(x \vee_2^{\circ_r} y) = (\neg x) \wedge_2^{\circ_r} (\neg y). \tag{21}$$

Furthermore,

$$\neg(\neg x) = x. \tag{22}$$

Proof. Let

$$x = \langle a, b; r_1 \rangle, \qquad y = \langle c, d; r_2 \rangle.$$

Using the standard negation defined in (9), we obtain

$$\neg(x \vee_1^{\circ_r} y) = \langle \min(b, d), \max(a, c); \circ_r(r_1, r_2) \rangle \, .$$

On the other hand,

$$(\neg x) \wedge_1^{\circ_r} (\neg y) = \langle \min(b, d), \max(a, c); \circ_r(r_1, r_2) \rangle \, .$$

Thus, identity (18) is established.

Similarly,

$$\neg(x \wedge_1^{\circ_r} y) = \langle \max(b, d), \min(a, c); \circ_r(r_1, r_2) \rangle \, ,$$

which coincides with $(\neg x) \vee_1^{\circ_r} (\neg y)$.

For algebraic addition,

$$\neg(x \wedge_2^{\circ_r} y) = \langle bd, a + c - ac; \circ_r(r_1, r_2) \rangle \, .$$

By the definition of $\vee_2^{\circ_r}$,

$$(\neg x) \vee_2^{\circ_r} (\neg y) = \langle bd, a + c - ac; \circ_r(r_1, r_2) \rangle \, .$$

Therefore, identity (20) holds.

The fourth identity follows analogously.

Finally,

$$\neg(\neg x) = \neg\langle b, a; r_1 \rangle = \langle a, b; r_1 \rangle = x.$$

This completes the proof. □

Remark 3. *The standard negation preserves the radius component.*

Consequently, although $\vee_1^{\max}$ and $\wedge_1^{\min}$ form a bounded distributive lattice, the standard negation does not generally reverse the partial order in the radius component.

Therefore, the identities established in Proposition 2 express the correspondence between pairs of operations with the same radius rule, but do not imply that the bounded distributive lattice is a De Morgan lattice under the standard negation.

Theorem 5 (Dual μ-groups)**.** *Let $\circ_r \in \{\min, \max\}$, $* \in \{\vee_1^{\circ_r}, \wedge_2^{\circ_r}\}$ and $\circ \in \{\wedge_1^{\overline{\circ_r}}, \vee_2^{\overline{\circ_r}}\}$. Then the commutative μ-groups*

$$\big(\mathcal{C}_R, *, \langle 0, 1; r^e_{\circ_r} \rangle, \langle 1, 0; r^z_{\circ_r} \rangle\big) \quad \textit{and} \quad \big(\mathcal{C}_R, \circ, \langle 1, 0; r^z_{\circ_r} \rangle, \langle 0, 1; r^e_{\circ_r} \rangle\big)$$

are dual. In particular, $(\mathcal{C}_R, \vee_1^{\max}, \bot, \top)$ and $(\mathcal{C}_R, \wedge_1^{\min}, \top, \bot)$ are dual commutative μ-groups, and so are $(\mathcal{C}_R, \wedge_2^{\max}, \bot, \top)$ and $(\mathcal{C}_R, \vee_2^{\min}, \top, \bot)$.

Proof. By Theorem 1 and Corollary 1, the first object is a commutative μ-group. Since $r^e_{\overline{\circ_r}} = r^z_{\circ_r}$ and $r^z_{\overline{\circ_r}} = r^e_{\circ_r}$, Table 1 shows that the neutral element of $\circ$ is $\langle 1, 0; r^z_{\circ_r} \rangle$ and its absorbing element is $\langle 0, 1; r^e_{\circ_r} \rangle$. Hence, the second object is a commutative μ-group, and the two objects have the forms required in Definition 3. □

Remark 4. *For C-IFTs, the two notions of duality considered in this section differ. Standard negation relates union to intersection and algebraic addition to algebraic multiplication while preserving the radius rule (Proposition 2). In contrast, duality of the μ-groups in Theorem 5 requires opposite radius rules, because the neutral and absorbing elements exchange roles. For IFPs, where no radius component is present, the pairs associated by standard negation are also dual μ-groups. However, Definition 3 permits additional pairings: union and algebraic multiplication are also dual in this sense, as are algebraic addition and intersection.*

5 Discussion and conclusions

This paper has investigated the algebraic structures induced by selected binary operations on C-IFTs.

For the standard union, intersection, algebraic addition, and algebraic multiplication, with radius aggregation determined by either $\min$ or $\max$, the corresponding neutral and absorbing elements have been identified. It has been established that all eight resulting algebraic structures are commutative monoids.

Moreover, when equipped with their corresponding absorbing elements, these monoids satisfy the multi-unitary group (μ-group) condition.

However, none of these structures forms an ordinary group on the entire domain of C-IFTs. In fact, the neutral element is the only invertible element in each of the considered monoids.

The operations $\vee_1^{\max}$ and $\wedge_1^{\min}$ have been shown to determine a bounded distributive lattice with respect to the componentwise partial order.

The standard radius-preserving negation establishes corresponding identities between union and intersection and between algebraic addition and multiplication for each fixed radius rule. Nevertheless, this negation does not generally reverse the radius order and therefore does not turn the resulting bounded distributive lattice into a De Morgan lattice.

The sets of μ-inverse elements have been described explicitly. The absorbing element is the unique μ-partner precisely when the radius differs from its absorbing value and the membership degree, for union and algebraic addition, or the non-membership degree, for intersection and algebraic multiplication, is less than (1).

Moreover, the duality of μ-groups in the sense of [4] pairs operations with opposite radius rules, for example $\vee_1^{\max}$ with $\wedge_1^{\min}$, whereas the negation pairs operations with the same radius rule. This distinction does not arise for intuitionistic fuzzy pairs.

The results have been established for the explicitly defined domain $\mathcal{C}_R$ and the radius aggregation operations $\circ_r \in \{\min, \max\}$. They should not be extended without additional proofs to more restrictive geometric admissibility conditions or to other radius aggregation rules.

Future research may investigate subalgebras and congruence relations of C-IFT structures, alternative radius aggregation operations, and the extension of the obtained algebraic results to Circular Intuitionistic Fuzzy Index Matrices. Particular attention may be given to multidimensional index-matrix constructions and uncertainty-aware decision-making models.

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