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Intuitionistic fuzzy fractional boundary value problem

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Abstract: In this paper we investigate the existence and uniquness of intuitionistic fuzzy solution for three-point boundary value problem for fractional differential equation:

$$\begin{cases} D^{\alpha}X(t) = F(t, X_t, D^{\beta}X(t)) & t \in J := [0,1] \\ X(t) = \phi(t) & t \in [-r, 0] \ , \\ X(1) = X(\xi) \end{cases} \tag{0.1}$$

where D^{α}, D^{β} are the standard Riemann–Liouville fractional derivatives $(\alpha - \beta > 0)$ and $(1 < \alpha < 2)$, $(\xi \in [0,1[)$, $\quad F : J \times C_0 \times \mathbf{IF}^1 \longrightarrow \mathbf{IF}^1$ is an intuitionistic fuzzy function, $\phi \in C_0$, $\phi(0) = 0_{IF}$ and $C_0 = C\left([-r,0], IF^1\right)$. We denote by X_t the element of C_0 defined by $X_t(\theta) = X(t+\theta)$, $\theta \in [-r, 0]$.

Keywords: Intuitionistic fuzzy sets, Distance between intuitionistic fuzzy sets, Intuitionistic fractional derivative.
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1 Introduction

In a letter dated September 30th, 1695, L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the n^{th} derivative of the linear function $f(x) = x$, $\frac{d^n}{dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn". In these words, fractional calculus was born.

Following L'Hopital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences.

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative.

A lot of work in this field within the scope of the existence and uniqueness of the solution of a differential equation with fractional fractional: derivative.

$$(\mathcal{F}DE) \quad \begin{cases} D^{\alpha}x(t) = f(t, x(t)) & t \in [0, T] \\ x(0) = x_0 \end{cases}, \tag{1.1}$$

but we cannot usually be sure that the model is perfect. For example, the initial value in (1.1) may not be known precisely. It may take any value in the form of "less than x_0", "about x_0" or "more than x_0" .

Classical mathematics, however, fails to cope with this situation. Therefore, it is necessary to have other theories in order to handle this issue. Various theories exist for describing this situation and the most popular one is intuitionistic the fuzzy set theory.

2 Preliminaries

We denote by

$$\mathbf{IF}(\mathbb{R}) = \{\langle u, v\rangle : \mathbb{R} \longrightarrow [0,1]^2 \quad , \quad 0 \leq u(x) + v(x) \leq 1\}.$$

Definition 1. [2] *An element $\langle u, v\rangle \in \mathbf{IF}(\mathbb{R})$ is called an intuitionistic fuzzy number if it satisfy the following conditions:*

1. *$\langle u, v\rangle$ is normal, i.e., there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ et $u(x_1) = 1$.*
2. *u is fuzzy convex an v is fuzzy concave.*
3. *u is upper semi-continuous et v is lower semi-continuous.*
4. *$supp\langle u, v\rangle = \overline{\{x \in \mathbb{R} : v(x) < 1\}}$ is bounded.*

We denote by $\mathbf{IF}^1$ the collection of all intuitionistic fuzzy numbers. First, we define $0_{IF} \in \mathbf{IF}^1$ by

$$0_{IF}(t) = \begin{cases} (1,0) & if \quad t = 0 \\ (0,1) & if \quad t \neq 0 \end{cases}.$$

Definition 2. *Let $\langle u_1, v_1\rangle$, $\langle u_2, v_2\rangle \in \mathbf{IF}^1$, $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$, then*

1. $(\langle u_1, v_1\rangle \oplus \langle u_2, v_2\rangle)(z) = (\sup\limits_{z=x+y} min(u_1(x), u_2(y)), \inf\limits_{z=x+y} max(u_1(x), u_2(y)))$
2. $\lambda\langle u_1, v_1\rangle = \langle \lambda u_1, \lambda v_1\rangle \quad if \quad \lambda \neq 0$
3. $\lambda\langle u_1, v_1\rangle = 0_{IF} \quad if \quad \lambda = 0$

4. $[\langle u_1, v_1\rangle \oplus \langle u_2, v_2\rangle]^\alpha = [\langle u_1, v_1\rangle]^\alpha + [\langle u_2, v_2\rangle]^\alpha$

5. $[\langle u_1, v_1\rangle \oplus \langle u_2, v_2\rangle]_\alpha = [\langle u_1, v_1\rangle]_\alpha + [\langle u_2, v_2\rangle]_\alpha$

6. $[\lambda\langle u_1, v_1\rangle]^\alpha = \lambda[\langle u_1, v_1\rangle]^\alpha$

7. $[\lambda\langle u_1, v_1\rangle]_\alpha = \lambda[\langle u_1, v_1\rangle]_\alpha$

Let $\langle u, v\rangle \in \mathbf{IF}^1$ and $\alpha \in [0,1]$, then we define the following sets:

$$[\langle u, v\rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R} : u(x) \geq \alpha\},$$

$$[\langle u, v\rangle]_r^+(\alpha) = \sup\{x \in \mathbb{R} : u(x) \geq \alpha\},$$

$$[\langle u, v\rangle]_l^-(\alpha) = \inf\{x \in \mathbb{R} : v(x) \leq 1-\alpha\},$$

$$[\langle u, v\rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R} : v(x) \leq 1-\alpha\}.$$

Remark 1. *Let $\langle u, v\rangle \in \mathbf{IF}^1$ and $\alpha \in [0,1]$, then we have:*

$$[\langle u, v\rangle]^\alpha = [[\langle u, v\rangle]_l^-(\alpha), [\langle u, v\rangle]_r^-(\alpha)],$$

$$[\langle u, v\rangle]_\alpha = [[\langle u, v\rangle]_l^+(\alpha), [\langle u, v\rangle]_r^+(\alpha)].$$

Proposition 1. [3]
Let $\alpha, \beta \in [0,1]$ and $\langle u, v\rangle \in \mathbf{IF}^1$, then

1. $[\langle u, v\rangle]_\alpha \subset [\langle u, v\rangle]^\alpha$

2. $[\langle u, v\rangle]_\alpha et [\langle u, v\rangle]^\alpha$ *are nonempty compact convex sets.*

3. *if $\alpha \leq \beta$ then $[\langle u, v\rangle]^\beta \subset [\langle u, v\rangle]^\alpha$ and $[\langle u, v\rangle]_\beta \subset [\langle u, v\rangle]_\alpha$*

4. *if $\alpha_n \nearrow \alpha$ then $[\langle u, v\rangle]^\alpha = \cap_n [\langle u, v\rangle]^{\alpha_n}$ and $[\langle u, v\rangle]_\alpha = \cap_n [\langle u, v\rangle]_{\alpha_n}$*

Let $\alpha \in [0,1]$. We put

$$M_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$$

and

$$M^\alpha = \{x \in \mathbb{R} : v(x) \leq 1-\alpha\}$$

Lemma 1. [3] *Let $\{M^\alpha : \alpha \in [0,1]\}$ and $\{M_\alpha : \alpha \in [0,1]\}$ be two subset of $\mathbb{R}$ verify $(1)-(4)$ of Proposition 1, if u and v are defined by*

$$u(x) = \begin{cases} 0 & if \quad x \notin M_0 \\ sup\{\alpha \in [0,1] : x \in M_\alpha\} & if \quad x \in M_0 \end{cases},$$

$$v(x) = \begin{cases} 1 & if \quad x \notin M^0 \\ 1 - sup\{\alpha \in [0,1] : x \in M^\alpha\} & if \quad x \in M^0 \end{cases},$$

then $\langle u, v\rangle \in \mathbf{IF}^1$.

Lemma 2. [3] *Let I be a dense subset in $[0,1]$. If $[\langle u,v\rangle]_\alpha = [\langle w,z\rangle]_\alpha$ and $[\langle u,v\rangle]^\alpha = [\langle w,z\rangle]^\alpha$, $\forall \alpha \in I$ then $\langle u,v\rangle = \langle w,z\rangle$.*

Definition 3. [1] *Let $\langle u_1,v_1\rangle$, $\langle u_2,v_2\rangle \in \mathbf{IF}^1$, if there exists $\langle w,z\rangle \in \mathbf{IF}^1$ such that, $\langle u_1,v_1\rangle = \langle u_2,v_2\rangle \oplus \langle w,z\rangle$ then $\langle w,z\rangle$ is called the Generalized Hukuhara difference of $\langle u_1,v_1\rangle$ and $\langle u_2,v_2\rangle$ denoted by $\langle u_1,v_1\rangle \ominus^G \langle u_2,v_2\rangle$.*

Definition 4. [1] *Let $f : [a,b] \to \mathbf{IF}^1$ and $t_0 \in [a,b]$. We say that f is generalized Hukuhara differentiable at t_0 if there exists $f'(t_0) \in \mathbf{IF}^1$ such that:*

$$f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0+h) \ominus^G f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0) \ominus^G f(t_0-h)}{h}.$$

Definition 5. [3] *$F : [a,b] \longrightarrow \mathbf{IF}^1$ is strongly measurable if $\forall \alpha \in [0,1]$, the set-valued mappings $F_\alpha : [a,b] \longrightarrow \mathcal{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ and $F^\alpha : [a,b] \longrightarrow \mathcal{P}_K(\mathbb{R})$ defined by $F^\alpha(t) = [F(t)]_\alpha$ are Lebesgue measurable.*

Definition 6. [3] *Let $F : [a,b] \longrightarrow \mathbf{IF}^1$. We say that F is integrable on $[a,b]$ if there exists $\langle u,v\rangle \in \mathbf{IF}^1$ such that for each $\alpha \in [0,1]$*

$$\left[\int_a^b F(t)dt\right]^\alpha = \left\{\int_a^b f(t)dt \;\mid\; f : [a,b] \longrightarrow \mathbb{R} \textit{ is a measurable selection for } F_\alpha\right\}$$

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$$[\langle u,v\rangle]^\alpha = \left[\int_a^b F(t)dt\right]^\alpha$$

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and we write $\int_a^b F(t)dt = \langle u,v\rangle$.

Let $d_\infty : \mathbf{IF}^1 \times \mathbf{IF}^1 \longrightarrow [0,+\infty]$ be a mapping defined by:

$$\begin{aligned} d_\infty(\langle u,v\rangle,\langle w,z\rangle) = (&\frac{1}{4}\sup_{0\le\alpha\le 1} |[\langle u,v\rangle]_r^+(\alpha) - [\langle w,z\rangle]_r^+(\alpha)|^p d\alpha \\ &+\frac{1}{4}\sup_{0\le\alpha\le 1} |[\langle u,v\rangle]_l^+(\alpha) - [\langle w,z\rangle]_l^+(\alpha)|^p d\alpha \\ &+\frac{1}{4}\sup_{0\le\alpha\le 1} |[\langle u,v\rangle]_r^-(\alpha) - [\langle w,z\rangle]_r^-(\alpha)|^p d\alpha \\ &+\frac{1}{4}\sup_{0\le\alpha\le 1} |[\langle u,v\rangle]_l^-(\alpha) - [\langle w,z\rangle]_l^-(\alpha)|^p d\alpha)^{\frac{1}{p}}. \end{aligned}$$

Then we have the following result.

Proposition 2. [3] *$(\mathbf{IF}^1, d_\infty)$ is a metric space.*

3 Intuitionistic fuzzy fractional differential equation

For this purpose, we start by giving definitions of intuitionistic fractional integral and intuitionistic fractional derivative.

Definition 7. *Let $F(t) := \langle u, v\rangle(t) \in L\left([0,1], IF^1\right)$. The intuitionistic fuzzy fractional integral of order α of F denoted by*

$$I^\alpha F(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds$$

is defined by

$$[I^\alpha F(t)]^\theta = \left[I^\alpha F_l^-(t;\theta), I^\alpha F_r^-(t;\theta)\right], 0 < \theta < 1,$$
$$[I^\alpha F(t)]_\theta = \left[I^\alpha F_l^+(t;\theta), I^\alpha F_r^+(t;\theta)\right], 0 < \theta < 1,$$

where

$$F_l^+(t;\theta) = [\langle u,v\rangle]_l^+(t;\theta),$$
$$F_r^+(t;\theta) = [\langle u,v\rangle]_r^+(t;\theta),$$
$$F_l^-(t;\theta) = [\langle u,v\rangle]_l^-(t;\theta),$$
$$F_r^-(t;\theta) = [\langle u,v\rangle]_r^-(t;\theta).$$

Proposition 3. *Let $F, G \in L([0,1], \mathbf{IF}^1)$ and $a \in \mathbf{IF}^1$. Then, we have*

1. $I^\alpha(aF)(t) = aI^\alpha F(t)$.
2. $I^\alpha(F \oplus G)(t) = I^\alpha F(t) \oplus I^\alpha G(t)$.
3. $I^\alpha I^\beta F(t) = I^{\alpha+\beta}$.

Definition 8. *Let $F \in L([0,1], \mathbf{IF}^1)$, $\varphi(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (t-s)^{\alpha-1} F(s) ds$. The function F is called intuitionistic fuzzy Riemann–Liouville fractional differentiable of order $0 < \alpha < 1$ at t_0 if there exists an element $D^\alpha F(t_0) \in \mathbf{IF}^1$ such that*

$$D^\alpha F(t_0) = \lim_{h\to 0^+} \frac{\varphi(t_0+h) \ominus \varphi(t_0)}{h} = \lim_{h\to 0^-} \frac{\varphi(t_0) \ominus \varphi(t_0-h)}{h}.$$

Definition 9. *Let $F \in C^1([0,1], \mathbf{IF}^1) \cap L([0,1], \mathbf{IF}^1)$. The function F is called intuitionistic fuzzy Caputo fractional differentiable of order $0 < \alpha < 1$ at x if there exists an element ${}^cD^\alpha F(x) \in \mathbf{IF}^1$ such that*

$${}^cD^\alpha F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-s)^{\alpha-1} F'(s) ds.$$

Now, let us find a solution of the problem (0.1).

Let $\theta \in [0,1]$ then we have

$$\begin{cases} [D^\alpha X(t)]^\theta = \left[F(t, X_t, D^\beta X(t))\right]^\theta \\ [D^\alpha X(t)]_\theta = \left[F(t, X_t, D^\beta X(t))\right]_\theta \\ [X(t)]^\theta = [\phi(t)]^\theta \\ [X(t)]_\theta = [\phi(t)]_\theta \\ [X(1)]^\theta = [X(\xi)]^\theta \\ [X(1)]_\theta = [X(\xi)]_\theta \end{cases}$$

Since,

$$\left[F(t, X_t, D^\beta X(t))\right]^\theta = \left[F_l^-(t, X_t, D^\beta X(t);\theta), F_r^-(t, X_t, D^\beta X(t);\theta)\right]$$

$$\left[F(t, X_t, D^\beta X(t))\right]_\theta = \left[F_l^+(t, X_t, D^\beta X(t))\theta), F_r^+(t, X_t, D^\beta X(t);\theta)\right]$$

and

$$[D^\alpha X(t)]^\theta = \left[D^\alpha X_l^-(t;\theta), D^\alpha X_r^-(t;\theta)\right], 0 < \theta < 1$$

$$[D^\alpha X(t)]_\theta = \left[D^\alpha X_l^+(t;\theta), D^\alpha X_r^+(t;\theta)\right], 0 < \theta < 1,$$

then

$$\begin{cases} \left[D^\alpha X_l^-(t;\theta), D^\alpha X_r^-(t;\theta)\right] = \left[F_l^-(t, X_t, D^\beta X(t);\theta), F_r^-(t, X_t, D^\beta X(t);\theta)\right] \\ \left[D^\alpha X_l^+(t;\theta), D^\alpha X_r^+(t;\theta)\right] = \left[F_l^+(t, X_t, D^\beta X(t);\theta), F_r^+(t, X_t, D^\beta X(t);\theta)\right] \\ \left[X_l^-(t;\theta), X_r^-(t;\theta)\right] = \left[\phi_l^-(t;\theta), \phi_r^-(t;\theta)\right] \\ \left[X_l^+(t;\theta), X_r^+(t;\theta)\right] = \left[\phi_l^+(t;\theta), \phi_r^+(t;\theta)\right] \\ \left[X_l^-(1;\theta), X_r^-(1;\theta)\right] = \left[X_l^-(\xi;\theta), X_r^-(\xi;\theta)\right] \\ \left[X_l^+(1;\theta), X_r^+(1;\theta)\right] = \left[X_l^+(\xi;\theta), X_r^+(\xi;\theta)\right] \end{cases}$$

Thus, we get the following system

$$\begin{cases} D^\alpha X_l^-(t;\theta) = F_l^-(t, (X_l^-)_t, D^\beta X_l^-(t);\theta) & , t \in J := [0,1] \\ X_l^-(t;\theta) = \phi_l^-(t;\theta), \quad t \in [-r, 0] \\ X_l^-(1;\theta) = X_l^-(\xi;\theta) \\ D^\alpha X_r^-(t;\theta) = F_r^-(t, (X_r^-)_t, D^\beta X_r^-(t);\theta) & , t \in J := [0,1] \\ X_r^-(t;\theta) = \phi_r^-(t;\theta), \quad t \in [-r, 0] \\ X_r^-(1;\theta) = X_r^-(\xi;\theta) \\ D^\alpha X_l^+(t;\theta) = F_l^+(t, (X_l^+)_t, D^\beta X_l^+(t);\theta) & , t \in J := [0,1] \\ X_l^+(t;\theta) = \phi_l^+(t;\theta), \quad t \in [-r, 0] \\ X_l^+(1;\theta) = X_l^+(\xi;\theta) \\ D^\alpha X_r^+(t;\theta) = F_r^+(t, (X_r^+)_t, D^\beta X_r^+(t);\theta) & , t \in J := [0,1] \\ X_r^+(t;\theta) = \phi_r^+(t;\theta), \quad t \in [-r, 0] \\ X_r^+(1;\theta) = X_r^+(\xi;\theta) \end{cases}$$

We start by finding a solution of the following problem.

$$\begin{cases} D^\alpha X_l^-(t;\theta) = F_l^-(t, (X_l^-)_t, D^\beta X_l^-(t);\theta) & , t \in J := [0,1] \\ X_l^-(t;\theta) = \phi_l^-(t;\theta), \quad t \in [-r, 0] \\ X_l^-(1;\theta) = X_l^-(\xi;\theta) \end{cases}$$

By applying the operator I^α on the first equation of this problem we have in general form [2]:

$$I^\alpha D^\alpha X_l^-(t;\theta) = I^\alpha F_l^-(t,(X_l^-)_t, D^\beta X_l^-(t);\theta)$$

$$X_l^-(t;\theta) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} + I^\alpha F_l^-(t,(X_l^-)_t, D^\beta X_l^-(t);\theta).$$

For some $c_i \in \mathbb{R}, i = 0,1,\ldots,n-1, n = [\alpha]+1$.
Since $1 < \alpha < 2$ and $X_l^-(0;\theta) = \phi_l^-(0;\theta) = 0$, then

$$X_l^-(t;\theta) = c_1 t^{\alpha-1} + I^\alpha F_l^-(t,(X_l^-)_t, D^\beta X_l^-(t);\theta)$$

$$X_l^-(t;\theta) = c_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta).$$

Since

$$X_l^-(1;\theta) = c_1 + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds$$

$$X_l^-(\xi;\theta) = c_1 \xi^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)$$

and

$$X_l^-(1;\theta) = X_l^-(\xi;\theta),$$

then

$$\begin{aligned}(1-\xi^{\alpha-1})c_1 = {} & \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds.\end{aligned}$$

Thus

$$\begin{aligned}c_1 = {} & \frac{1}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds \\ & - \frac{1}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds.\end{aligned}$$

Finally,

$$\begin{aligned}(1)\quad X_l^-(t;\theta) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta) \\ & + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds \\ & - \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1} F_l^-(s,(X_l^-)_s, D^\beta X_l^-(s);\theta)ds\end{aligned}$$

In the same way, we obtain the others solutions:

$$(2)\quad X_r^-(t;\theta) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_r^-(s,(X_r^-)_s, D^\beta X_r^-(s);\theta)$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}F_r^-(s,(X_r^-)_s, D^\beta X_r^-(s);\theta)ds$$
$$-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}F_r^-(s,(X_r^-)_s, D^\beta X_r^-(s);\theta)ds$$

$$(3)\quad X_l^+(t;\theta) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_l^+(s,(X_l^+)_s, D^\beta X_l^+(s);\theta)$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}F_l^+(s,(X_l^-)_s, D^\beta X_l^+(s);\theta)ds$$
$$-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}F_l^+(s,(X_l^+)_s, D^\beta X_l^+(s);\theta)ds$$

$$(4)\quad X_r^+(t;\theta) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_r^+(s,(X_r^+)_s, D^\beta X_r^+(s);\theta)$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}F_r^+(s,(X_r^+)_s, D^\beta X_r^+(s);\theta)ds$$
$$-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}F_r^+(s,(X_r^+)_s, D^\beta X_r^+(s);\theta)ds$$

Finally, by Lemma (1) the solution of the Problem (3.1) is given by following formula

$$X(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,X_s,D^\beta X(s))$$
$$\oplus \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}F(s,X_s,D^\beta X(s))ds$$
$$\ominus^G \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}F(s,X_s,D^\beta X(s))ds.$$

Now we study the uniqness of this solution by using Banach fixed point theorem.
Let $H=\{X|X\in C([-r,1],\mathbf{IF}^1), D^\beta X\in C(J,\mathbf{IF}^1), X(t)=\phi(t), \forall t\in[-r,0]\}$.

Proposition 4. *Let* $u,v\in H$ *such that*

$$D(u(t),v(t)) = K\max_{t\in[-r,0]} d_\infty(u(t),v(t)) + L\max_{t\in[0,1]} d_\infty(D^\beta u(t), D^\beta v(t)).$$

Then (H,D) *is a complete metric space.*

Proof. Firstly, let us show that D is a metric on H.

1) $D(u(t),u(t)) = K\max_{t\in[-r,0]} d_\infty(u(t),u(t)) + L\max t\in[0,1] d_\infty(D^\beta u(t), D^\beta u(t)) = 0.$

2) $D(u(t),v(t)) < K\max_{t\in[-r,0]}\{d_\infty(u(t),z(t)) + d_\infty(z(t),v(t))\}$
$\quad + L\max_{t\in[0,1]}\{d_\infty(D^\beta u(t),D^\beta z(t)) + d_\infty(D^\beta z(t),D^\beta v(t))\}.$

3) $D(u(t),v(t)) < K\max_{\in[-r,1]} d_\infty(u(t),z(t)) + L\max_{t\in[0,1]} d_\infty(D^\beta u(t),D^\beta z(t))$
$\quad + K\max_{t\in[-r,1]} d_\infty(z(t),v(t)) + L\max_{t\in[0,1]} d_\infty(D^\beta z(t),D^\beta v(t)).$

Thus,

$$D(u(t), v(t)) < D(u(t), z(t)) + D(z(t), v(t)).$$

Now, (H, D) is a complete metric space indeed.

Let $(X_n)_n$ be a Cauchy sequence in H, then $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n, m > N_0$

$$D(X_n(t), X_m(t)) < \varepsilon.$$

Since

$$D(X_n(t), X_m(t)) = K \max_{\in[-r,1]} d_\infty(X_n(t), X_m(t)) + L \max_{t\in[0,1]} d_\infty(D^\beta X_n(t), D^\beta X_m(t)) < \varepsilon,$$

then

$$d_\infty(X_n(t), X_m(t)) < \varepsilon,$$
$$d_\infty(D^\beta X_n(t), D^\beta X_m(t)) < \varepsilon.$$

Since $(\mathbf{IF}^1, d_\infty)$ is a complete metric space, then $(X_n(t))_n$ converges to $X(t)$ in $\mathbf{IF}^1$.

Thus for $\varepsilon_1 = \frac{\varepsilon}{K+L} > 0$, $\exists N_1 \in \mathbb{N}$ such that

$$d_\infty(X_n(t), X(t)) < \varepsilon_1$$

then

$$K \max_{t\in[-r,1]} d_\infty(X_n(t), X(t)) < K\varepsilon_1$$

and

$$L \max_{t\in[0,1]} d_\infty(D^\beta X_n(t), D^\beta X(t)) < L\varepsilon_1.$$

Thus

$$D(X_n(t), X(t)) < \varepsilon.$$

Finally, (H, D) is a complete metric space. □

Theorem 1. *Assume that $F : J \times C_0 \times IF^1 \longrightarrow IF^1$, and there exist positive constants K, L such that*

$$d_\infty(F(t, u_t, D^\beta u(t)), F(t, v_t, D^\beta v(t))) < K d_\infty(u, v) + L d_\infty(D^\beta u(t), D^\beta v(t))$$

for all $u, v \in C_0$. Then

$$\left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}}\right) < 1$$

implies that the Problem (3.1) has a unique intuitionistic fuzzy solution on $[-r, 1]$.

Proof. Let $T : H \longrightarrow H$ such that

$$TX(t) = \begin{cases} \phi(t), \quad t \in [-r, 0] \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} F(s, X_s, D^\beta X(s)) ds \\ \oplus \dfrac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \displaystyle\int_0^\xi (\xi-s)^{\alpha-1} F(s, X_s, D^\beta X(s)) ds \\ \ominus^G \dfrac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \displaystyle\int_0^1 (1-s)^{\alpha-1} F(s, X_s, D^\beta X(s)) ds \quad t \in J := [0, 1] \end{cases}$$

For $u, v \in H$ we have $d_\infty(u(t), v(t)) = 0, \forall t \in [-r, 0]$, and for $t \in J$ we have

$$d_\infty(Tu(t), Tv(t)) < \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$
$$+ \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \int_0^\xi (\xi-s)^{\alpha-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$
$$+ \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \int_0^1 (1-s)^{\alpha-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$

$$d_\infty(Tu(t), Tv(t)) < \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [K d_\infty(u_s(\rho), v_s(\rho)) + L d_\infty(D^\beta u(s), D^\beta v(s))]ds$$
$$+ \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \int_0^\xi (\xi-s)^{\alpha-1} [K d_\infty(u_s(\rho), v_s(\rho)) + L d_\infty(D^\beta v(s), D^\beta v(s))]ds$$
$$+ \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \int_0^1 (1-s)^{\alpha-1} [K d_\infty(u_s(\rho), v_s(\rho) + L d_\infty(D^\beta u(s), D^\beta v(s))]ds$$

$$d_\infty(Tu(t), Tv(t)) < \frac{1}{\Gamma(\alpha+1)} \sup_{t \in J} (t^\alpha + \frac{t^{\alpha-1}\xi^\alpha}{1-\xi^{\alpha-1}} + \frac{t^{\alpha-1}}{1-\xi^{\alpha-1}}) D(u, v)$$

$$d_\infty(Tu(t), Tv(t)) < \frac{1}{\Gamma(\alpha+1)} (1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}}) D(u, v)$$

On the other hand,

$$d_\infty(D^\beta Tu(t), D^\beta Tv(t)) < \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$
$$+ \frac{t^{\alpha-\beta-1}}{(1-\xi^{\alpha-1})} \int_0^\xi (\xi-s)^{\alpha-\beta-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$
$$+ \frac{t^{\alpha-\beta-1}}{(1-\xi^{\alpha-1})} \int_0^1 (1-s)^{\alpha-1} d_\infty(F(s, u_s, D^\beta u(s)), F(s, v_s, D^\beta v(s)))ds$$

$$d_\infty(D^\beta Tu(t), D^\beta Tv(t)) < \frac{1}{\Gamma(\alpha-\beta+1)} \sup_{t \in J} \left(t^{\alpha-\beta} + \frac{t^{\alpha-\beta-1}\xi^\alpha}{1-\xi^{\alpha-1}} + \frac{t^{\alpha-\beta-1}}{1-\xi^{\alpha-1}} \right) D(u, v)$$

$$d_\infty(D^\beta Tu(t), D^\beta Tv(t)) < \frac{1}{\Gamma(\alpha-\beta+1)} \left(1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}} \right) D(u, v).$$

Therefore, for each $t \in J$, we have:

$$D(Tu(t), Tv(t)) < \left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)} \right) \left(1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}} \right) D(u, v).$$

So, T is a contraction and thus T has a unique fixed point X on H, then $X(t)$ is the unique solution to Problem (3.1) on $[-r, 1]$. □

References

[1] Bede, B., & Gal, S. G. (2005). Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, 151, 581–599.

[2] Kilbas, A. A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Sifferential Equations*, North-Holland Mathematical studies 204, Ed van Mill, Amsterdam.

[3] Melliani, S., Elomari, M. Chadli, L. S., & Ettoussi, R. (2015). Intuitionistic fuzzy metric space, *Notes on Intuitionistic Fuzzy Sets*, 21(1), 43–53.

[4] Salahshour, S., Allahviranloo, T., Abbasbandy, S., & Baleanu, D. (2012). Existence and uniqueness results for fractional differential equations with uncertainty. *Adv. Diff. Equ.*, 112, doi:10.1186/1687-1847-2012-112.