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# On two formulations of the IF state representation theorem

**Beloslav Riečan**

Department of Mathematics
Faculty of Natural Sciences, Matej Bel University
Tajovského 40, 974 01 Banská Bystrica, Slovakia

**Abstract:** There are two IF-state representation theorems, [2,4]. They represent IF-state by some classical Kolmogorovian probabilities. Of course, they must be equivalent, but the formulations correspond with the constructions of the probabilities.
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## 1 Butnariu–Klement formulation

Recall the definition of IF-sets and some operations with them.

**Definition 1.** Let $\Omega$ be a nonempty set. An IF-subsets of $\Omega$ is a pair $A$ of mappings $A = (\mu_A, \nu_A), \mu_A : \Omega \to [0,1], \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$. If $A, B$ are IF-sets then we define

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

and

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_A, \nu_A \geq \nu_B.$$

The following terminology is probably inspired by the quantum theory. Therefore we speak about states instead of probabilities.

**Definition 2.** Start with a measurable space $(\Omega, \mathcal{S})$, hence $\mathcal{S}$ is a $\sigma$-algebra of subsets of $\Omega$. An IF-event is such an IF-set $A = (\mu_A, \nu_A)$ that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable. Let $\mathcal{F}$ be the family of all IF–events in $\Omega$. A mapping $m : \mathcal{F} \to [0, 1]$ is called an IF-state, if the following conditions are satisfied:

(i) $m((1_\Omega, 0_\Omega)) = 1, m((0_\Omega, 1_\Omega)) = 0;$

(ii) $A \odot B = (0_\Omega, 1_\Omega)) \Longrightarrow m(A \oplus B) = m(A) + m(B);$

(iii) $A_n \nearrow A(i.e. \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A) \Longrightarrow m(A_n) \nearrow m(A).$

Now we shall present the first solution of a problem of Radko Mesiar: find all IF-states on $\Omega$.

**Theorem 1.** To each state $m : \mathcal{F} \to [0, 1]$ there exist exactly one probability measure $P : \mathcal{S} \to [0, 1]$ and exactly one $\alpha \in [0, 1]$ such that

$$m(A) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$$

for any $A \in \mathcal{F}$.

*Proof.* In [5] it was based on the Butnariu–Klement theorem ([1]): $m$ can be found in the form $m(A) = f(\int_\Omega \mu_A dP, \int_\Omega \nu_A dP)$. If $\beta, Q$ is another pair, then

$$(1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP) = (1 - \beta) \int_\Omega \mu_A dQ + \beta(1 - \int_\Omega \nu_A dQ).$$

Put $\mu_A = \nu_A = 0_\Omega$. Then we obtain $\alpha = \beta$. If we put $\mu_A = \chi_A, \nu_A = 0_\Omega$, then we obtain

$$(1 - \alpha) \int_\Omega \mu_A dP + \alpha = (1 - \alpha) \int_\Omega \mu_A dQ + \alpha,$$

hence if $\alpha \neq 1$ we obtain $P(A) = Q(A)$ for any $A \in \mathcal{S}$, hence $P = Q$. Let $\alpha = 1$. Then again

$$1 - P(A) = 1 - \int \chi_A dP = 1 - \int \chi_A dQ = 1 - Q(A),$$

for any $A \in \mathcal{S}$, hence $P = Q$. □

## 2 Cignoli formulation

The proof without the Butnariu–Klement theorem has been presented in [2].

**Theorem 2.** To each state $m : \mathcal{F} \to [0, 1]$ there are probability measures $P, Q : \mathcal{S} \to [0, 1]$ and $\alpha \in [0, 1]$ such that

$$m(A) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ)$$

for any $A = (\mu_A . \nu_A) \in \mathcal{F}$.

**Theorem 3.** Theorem 2 follows from Theorem 1.

*Proof.* Put $Q = P$. Then

$$m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dP.$$

Hence the proof. □

Now we show that also Theorem 2 implies Theorem 1.

**Theorem 4.** Let $P, Q, R : \mathcal{S} \to [0,1]$ be probability measures. and $\alpha, \beta \in [0,1]$ such that

$$(1-\beta)\int_\Omega \mu_A dR + \beta(1 - \int_\Omega \nu_A dR) =$$

$$= \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ.$$

for any $A \in \mathcal{S}$. Then $\alpha = \beta$ and $P = Q = R$.

*Proof.* First put $\mu_A = 0_\Omega, \nu_A = 0_\Omega$. Then $\beta = \alpha$. Put now $\mu_A = 0, \nu_A = \chi_A$. Then

$$\alpha(1 - R(A)) = \alpha(1 - Q(A)).$$

for any $A \in \mathcal{S}$. Therefore, if $\alpha \neq 0$, then $R = Q$. If $\alpha = 0$ then $R(A) = P(A)$ for any $A \in \mathcal{S}$, hence $R = P$.

Now put $\mu_A = \chi_A, \nu_A = 0_\Omega$. Then $(1-\alpha)R(A) + \alpha = P(A) + \alpha - \alpha Q(A)$.

If $\alpha \neq 0$ then $(1-\alpha)R(A) = P(A) - \alpha R(A)$, hence $R = P = Q$).

If $\alpha = 0$, then $R = P$, and $Q$ can be arbitrary. □

# 3 Grzegorzewski formulation

Consider a probability space $(\Omega, \mathcal{S}, P)$. Then in [3] the probability $P(A)$ of an event $A = (\mu_A, \nu_A)$ has been defined as a compact interval by the equality

$$m(A) = [\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP].$$

Let $\mathcal{J}$ be the family of all compact intervals. Then the mapping $m : \mathcal{F} \to \mathcal{J}$ can be defined axiomatically similarly as in [4].

**Definition 3.** A mapping $m : \mathcal{F} \to \mathcal{J}$ is called an IF-probability, if the following conditions hold:

(i) $m((1_\Omega, 0_\Omega)) = [1,1], m((0_\Omega, 1_\Omega)) = [0,0]$;

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B) = m(A) + m(B)$;

(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$.

(Recall that $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ means that $\alpha_n \nearrow \alpha, \beta_n \nearrow \beta$, but $A_n = (\mu_{A_n}, \nu_{A_n}) \nearrow A = (\mu_A, \nu_A)$ means $\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$.)

Very well known is the following assertion.

**Theorem 5.** Let $m : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $m(A) = [m^b(A), m^\sharp(A)], A \in \mathcal{F}$. Then $m^b : \mathcal{F} \to [0,1], m^\sharp : \mathcal{F} \to [0,1]$ are IF-states.

Of course, we know the general form of IF-states. Therefore the following theorem is evident.

**Theorem 6.** Let $m : \mathcal{F} \to \mathcal{J}$ be an IF-probability. Then there exist Kolmogorov probabilities $P^b : \mathcal{S} \to [0,1], P^\sharp : \mathcal{S} \to [0,1]$ and $\alpha^b, \alpha^\sharp \in [0,1]$ such that

$$m^b(A) = \int_\Omega \mu_A dP^b + \alpha^b(1 - \int_\Omega \nu_A dP^b),$$

$$m^\sharp(A) = \int_\Omega \mu_A dP^\sharp + \alpha^\sharp(1 - \int_\Omega \nu_A dP^\sharp),$$

where $\alpha^b \leq \alpha^\sharp$ and $\alpha^b(1 - P^b(A)) \leq \alpha^\sharp(1 - P^\sharp(A)), A \in \mathcal{F}$.

*Proof.* We have

$$m^b(A) = \int_\Omega \mu_A dP^b + \alpha^b(1 - \int_\Omega \nu_A dP^b) \leq$$

$$\leq m^\sharp(A) = \int_\Omega \mu_A dP^\sharp + \alpha^\sharp(1 - \int_\Omega \nu_A dP^\sharp).$$

Put $\mu_A = \nu_A = 0_\Omega$. Then we directly obtain $\alpha^b \leq \alpha^\sharp$. Moreover $\mu_A = o_\Omega, \nu_A = \chi_A$ implies

$$\alpha^b(1 - P^b(A)) \leq \alpha^\sharp(1 - P^\sharp(A))$$

for any $A \in \mathcal{S}$. □

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