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# Semi linear equation with fuzzy parameters

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**Abstract**

In this paper we studie the solution concept for a semi linear equation with fuzzy parameters. The extension principle described by L. A. Zadeh [11] provides a natural way for obtaining the notion of fuzzy solution. The fuzzy extension of the solution operator is shown to provide the unique solution in the formar case.

## 1 Introduction

Fuzzy set theory is a powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models. While its main directions of development have been information theory, data analysis, artificial intelligence, decision theory, control, and image processing (see e.g. [3, 12, 13]), fuzzy set theory is increasingly used as a means for modelling and evaluating the influence of imprecisely known parameters in mathematical, technical, physical models. The purpose of this paper is to work out this approch when the models are constitued by partial differential equations.

Based on the fuzzy description of parameters and mathematical objects, we shall be concerned here with partial differential equation in the scalar case of the form

$$\begin{aligned} u_t + \lambda u_x &= au \\ u(x,0) &= u_0(x) \end{aligned}$$

Here the parameters $a$ and $\lambda$ will be fuzzy numbers. The solution $u(x,t)$ at any fixed point $(x,t)$ will be a fuzzy number as well.

## 2 Partial differential equations

Consider a semi-linear equation :

$$u_t + \lambda u_x = au \tag{1}$$

for a function $u = u(x,t)$, where $\lambda$ and $a$ are two real constants. Along a line of the family

$$x - \lambda t = \xi = \text{constante}$$

("characteristic line "in the $xt$-plane) we have for a solution $u$ of

$$\frac{du}{dt} = \frac{d}{dt}u(\lambda t + \xi, t) = \lambda u_x + u_t = au \tag{2}$$

the general solution of (2) has the form

$$u(x,t) = u_0(\xi)\exp(at) = u_0(x - \lambda t)\exp(at) \tag{3}$$

Formula (3) represents the general solution $u$ uniquely in terms of its initial values

$$u(x,0) = u_0(x)$$

Conversely, every $u$ of the form (3) is a solution of (1) with initial values $u_0$ provided $u$ is a class $C^1(\mathbb{R})$. We notice that the value of $u$ at any point $(x,t)$ depend only on the initial value $u_0$ at the single argument $\xi = x - \lambda t$, the abscissa of the point of intersection of the characteristic line through $(x,t)$ with the initial line, the $x$-axis.
We consider $\Omega = \mathbb{R} \times \mathbb{R}_+$
introducing the equation operator

$$E : C^1(\Omega) \to C(\Omega) : v \to [(x,t) \to v_t + \lambda v_x - av],$$

the restriction operator $\forall (x,t) \in \Omega$

$$R_{x,t} : C^1(\Omega) \to \mathbb{R} : \mathrm{v} \to \mathrm{v(x,t)} = \mathrm{u_0(x - \lambda t)e^{at}},$$

and the solution operator

$$L : \mathbb{R}^2 \to \mathrm{C}^1(\Omega) : (\lambda, \mathrm{a}) \to \mathrm{L}(\lambda, \mathrm{a}).$$

# 3 Fuzzy sets and fuzzy numbers

Geven a set $X$, a fuzzy set $A$ over $X$ is a map

$$m_A : X \longrightarrow [0,1]$$

called the membership of $A$ (it is convenient to distinguish between $A$ and its membership functions $m_A$ to be able to employ the usual language of set theory). Thus given $x \in A$, $m_A(x)$ is considered the degree to which, respectively the possibility that, $x$ belong to $A$ (In calssical sets theory, $m_A$ would correspond to the characteristic function of $A$). This concept allows to model uncertainty in situations where more information than just upper and lower bounds is available (in contrast to interval analysis), but no probability distribution are available. This situation often arises e.g. in engineering practice, when parameters are estimated partially in subjective way.

We denote the family of fuzzy sets over $X$ by $\mathcal{F}(\mathcal{X})$. The $\alpha$-level sets are the classical sets

$$A^{\alpha \geq} = \{x \in X : m_A(x) \geq \alpha\}$$

A fuzzy real number is an element $A \in \mathcal{F}(\mathbb{R})$ such that all level sets $A^{\alpha \geq}$ are compact intervals $(0 < \alpha \leq 1)$ and $A^{1 \geq}$ is not empty. The graph of $m_A$ has a monotonically increasing left branch, a central point or plateau of membership degree one, and a monotonically decreasing right branch. Similary, one can define fuzzy vectors, fuzzy functions etc.The *extension principle* introduced by [11] allows the evaluation of functions on e.g. fuzzy numbers according to the following definition : Let

$$f : \mathbb{R}^n \to \mathbb{R}$$

be a function, define the extension [3], [6]

$$f : (\mathcal{F}(\mathbb{R}))^n \to \mathcal{F}(\mathbb{R})$$

by

$$m_{f(a_1,\dots,a_n)}(y) = \sup_{y=f(x_1,\dots,x_n)} \inf\left(m_{a_1}(x_1),\dots,m_{a_n}(x_n)\right)$$

It can be shown that in case $f$ is continuous, $f(a_1,\dots,a_n)$ is a fuzzy number as well, and

$$f\left(a_1,\dots,a_n\right)^{\alpha\geq} = f\left(a_1^{\alpha\geq},\dots,a_n^{\alpha\geq}\right)$$

the set theoretic image of the level sets. Thus the upper and lower endpoints of the interval $f(a_1,\dots,a_n)^{\alpha\geq}$ can be obtained by minimizing / maximizing $f$ over $a_1^{\alpha\geq} \times \dots \times a_n^{\alpha\geq}$.
we denote by 0 the crisp zero function in $\mathcal{F}(\mathbb{R}^n)$, that is,

$$m_0(f) = \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1** *Let $A$ a fuzzy set, we define*

- *$A$ is normalized if he exist an element $x$ in $A$ such that $m_A(x) = 1$.*
- *The $\alpha$-level sets $A^{\alpha\geq}$ for $(0 < \alpha < 1)$ of a fuzzy set $A$ are the classical (crisp) sets.*
- *$A$ is convex if and only if its $\alpha$-level are convexs.*
- *A fuzzy number is a convex, normalized fuzzy subset of the domain $A$*

The concept of fuzzy number is an extension of the notion of real number : its encodes approximate but non probabilistic quantitative knowledge [4].

For instance, let us consider a semi-linear equation where it is not easy or convenient to measure a certain variable (e.g. propagation speed); futhermore suppose that we have a qualitative, imprecise knowledge that in certain operating conditions the variable is around 2 $m/s$. This does not imply a probabilistic distribution of the variable's value but rather a possibilistic distribution, which may be represented by a fuzzy number (see Figure.1)

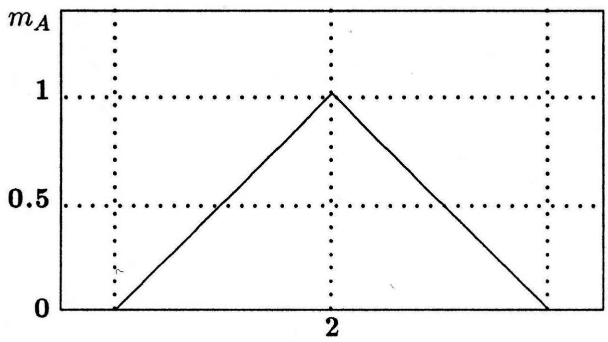


Figure 1: Triangular fuzzy number

## 4 Fuzzy semi-linear equation

Let us consider a fuzzy semi-linear equation

$$\begin{cases} \tilde{u}_t + \tilde{\lambda} u_x = \tilde{a}\tilde{u} \\ \tilde{u}(x,0) = u_0(x) \end{cases} \tag{4}$$

with $\tilde{\lambda}$ and $\tilde{a}$ are two fuzzy numbers, the initial condition $u_0$ is a classic function in $C(\mathbb{R})$. by the extension principle

$$\begin{array}{lll} E & : \mathcal{F}\left(\mathrm{C}^1(\Omega)\right) & \to \mathcal{F}\left(\mathrm{C}(\Omega)\right) \\ R_{x,t} & : \mathcal{F}\left(\mathrm{C}^1(\Omega)\right) & \to \mathcal{F}(\mathbb{R}) \\ L & : \mathcal{F}\left(\mathbb{R}^2\right) & \to \mathcal{F}\left(\mathrm{C}^1(\Omega)\right) \end{array}$$

$\forall\, (x,t) \in \mathbb{R} \times \mathbb{R}_+$

$$\begin{array}{llll} L_{x,t}: & \mathcal{F}\left(\mathbb{R}^2\right) & \to & \mathcal{F}(\mathbb{R}) \\ & \left(\tilde{\lambda}, \tilde{a}\right) & \mapsto & \tilde{u}(x,t) \quad \text{solution of (4)} \end{array}$$

The fuzzy value $L_{x,t}\left(\tilde{\lambda}, \tilde{a}\right)$ may be computed by the extension principle in this way

$$m_{L_{x,t}(\tilde{\lambda},\tilde{a})}(y) = \sup\left\{\inf\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) : y = L_{x,t}(\lambda, a)\right\}$$

**Lemma 1** *We have*

$$R_{x,t} \circ L = L_{x,t} \quad \textit{in } \mathcal{F}\left(\mathbb{R}^2\right)$$

**proof :**

$$m_{L_{x,t}(\tilde{\lambda},\tilde{a})}(y) = \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ y = L_{x,t}(\lambda,a)}} \min\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right)$$

$$m_{L(\tilde{\lambda},\tilde{a})}(f) = \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ f(x,t) = L_{x,t}(\lambda,a)}} \min\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right)$$

$$\begin{aligned} m_{R_{x,t} \circ L(\tilde{\lambda},\tilde{a})}(y) &= \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ y = R_{x,t} \circ L(\lambda,a)}} \min\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) \\ &= \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ y = u_0(x - \lambda t)e^{at}}} \min\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) \\ &= \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ y = L_{x,t}(\lambda,a)}} \min\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) \\ &= m_{L(\tilde{\lambda},\tilde{a})}(y) \end{aligned}$$

we have

$$m_L(f) = \sup\left\{\inf\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) : f = L(\lambda, a)\right\}$$

and

$$\begin{aligned} m_{L_{x,t}(\tilde{\lambda},\tilde{a})}(y) &= \sup\sup\left\{m_L(f) : f \in \mathrm{C}^1(\Omega) \text{ with } \mathrm{y} = \mathrm{f(x,t)}\right\} \\ &= \sup_{\substack{(\lambda, a) \in \mathbb{R}^2 \\ y = L_{x,t}(\lambda,a)}} \min(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)) \end{aligned}$$

**Definition 2** *An element $\tilde{u} \in \mathcal{F}\left(\mathrm{C}^1(\Omega)\right)$ is called a fuzzy solution with the initial data $u_0 \in \mathrm{C}(\Omega)$, if $E(\tilde{u}) = 0$ in $\mathcal{F}\left(\mathrm{C}(\Omega)\right)$, $R_{x,t}(\tilde{u}) = u_0\left(x - \tilde{\lambda} t\right) e^{\tilde{a}t}$ in $\mathcal{F}(\mathbb{R})$ ,*

**Proposition 1** *Given $\tilde{\lambda}$, $\tilde{a} \in \mathcal{F}(\mathrm{I\!R})$, $\tilde{u} = L\left(\tilde{\lambda}, \tilde{a}\right)$ is a fuzzy solution to problem (4)*

**proof :**

To show that $\tilde{u} = L\left(\tilde{\lambda}, \tilde{a}\right)$ solves the fuzzy partial differential equation, we compute :

$$\begin{aligned} m_{E(\tilde{u})=L(\tilde{\lambda},\tilde{a})}(w) &= \sup\left\{m_{\tilde{u}=L(\tilde{\lambda},\tilde{a})}(v) : w = E(v)\right\} \\ &= \sup\left\{\sup\left\{\inf\left(m_{\tilde{\lambda}}(\lambda), m_{\tilde{a}}(a)\right) : v = L(\lambda, a)\right\} : w = E(v)\right\} \end{aligned}$$

if $w \neq 0$ and $w = E(v)$; then $\left\{(\lambda, a) \in \mathrm{I\!R}^2 : \mathrm{v} = \mathrm{L}(\lambda, \mathrm{a})\right\} = \emptyset$, so the inner supremum is zero and $m_{EL(\tilde{\lambda},\tilde{a})}(w) = 0$.

if $w = 0$, we may take $(\lambda, a) \in \mathrm{I\!R}^2$ with $m_{\tilde{\lambda}}(\lambda) = m_{\tilde{a}}(a) = 1$ and $v = L(\lambda, a)$. Then $E(w) = 0$, and so the supremum equals 1.

Let $S = \left\{u \in \mathrm{C}^1(\Omega) : \mathrm{E(u)} = 0\right\}$. We can view $\mathcal{F}(\mathcal{S})$ as a subset of $\mathcal{F}\left(\mathrm{C}^1(\Omega)\right)$, setting the membership degree of any $u \in \mathrm{C}^1(\Omega)\backslash\mathrm{S}$ to some $\tilde{u} \in F(S)$ equal to zero.

**Lemma 2** *If $\tilde{u} \in \mathcal{F}\left(\mathrm{C}^1(\Omega)\right)$ is a solution to (1) , then $\tilde{u}$ belongs to $\mathcal{F}(\mathcal{S})$*

**proof :**

we have that

$$m_{E(\tilde{u})}(v) = \sup\left\{m_{\tilde{u}}(w) : v = E(w)\right\}$$

suppose there exist $v \notin S$ , such that $m_{(\tilde{u})}(v) > 0$. Putting $w = E(v)$ we have $m_{E(\tilde{u})}(w) \geq m_{\tilde{u}}(v) > 0$, contradicting the hypothesis that $E(\tilde{u}) = 0$.

**Proposition 2** *The fuzzy solution $\tilde{u} \in \mathcal{F}\left(\mathrm{C}^1(\Omega)\right)$ to (1) is unique.*

**proof :**

Since $L : \mathrm{I\!R}^2 \to \mathrm{S}$ is bijective, the same is true of the extension $L : \mathcal{F}\left(\mathrm{I\!R}^2\right) \to \mathcal{F}(\mathcal{S})$. If $u \in \mathcal{F}\left(\mathrm{C}^1(\Omega)\right)$ is a solution, it belongs to $\mathcal{F}(\mathcal{S})$ by the lemma and hence is uniquely determined by the initial data.

It would be interesting to consider a further fuzzification of the selected class of semilinear differential equations, this time basing on intuitionistic fuzzy sets.

Intuitionistic fuzzy sets (IFS) (see [1]) were defined in 1983. In addition to the degree of membership $\mu$ known from fuzzy sets, here one more degree is added - that of non-membership, denoted by $\nu$. They do not need to sum to 1. The value $1 - \mu - \nu$ is called a degree of uncertainty and is denoted by $\pi$.

Intuitionistic fuzzy real numbers were initially proposed by Burillo, Bustince and Mohedano [2]. Another approach to this concept is developed by M. Stoeva [10].

Basing on this notion, one can define suitably intuitionistic fuzzy vectors, intuitionistic fuzzy functions etc.

In a next research we shall introduce new formulations (using IFS) of the theorems and definitions from the paper.

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