# On some classes of Tchebychev distance based on intuitionistic fuzzy cardinality and intuitionistic fuzzy statistical description 

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#### Abstract

The Tchebychev distance on fuzzy sets (FSs) has been proposed to construct a measure of proximity between two modalities in a two-dimensional statistical description. The parameterized symmetric difference operations and cardinality for intuitionistic fuzzy sets (IFSs) has been proposed.

This paper extends to intuitionistic fuzzy set the Tchebychev distance and possibility measure on fuzzy sets. More precisely, we firstly use the parameterized symmetric difference operations and the cardinality on IFSs to propose a Tchebychev distance measure for IFSs. From these, we then deduce two examples of metrics. Secondly, we introduce an intuitionistic fuzzy mapping that preserves the properties of the fuzzy mapping. We use this mapping to propose a Tchebychev


possibility measure based on IF-cardinality. This leads to define a proximity measure between two modalities of a given character in a two-dimensional intuitionistic fuzzy statistical description.
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## 1 Introduction

In real life we are faced with situations where the informations are ambiguous and insufficient. To deal with these situations, Zadeh [29] introduced the theory of fuzzy set (FS). This theory allows for an element to belong to a group (set) with some degree of membership (partial membership). Continuing in the same direction as Zadeh [29], Atanassov [4] introduced the theory of intuitionistic fuzzy sets (IFSs). While the fuzzy set theory is concerned with membership and non-membership, the concept of IFS allows for the additional term called the degree of hesitancy. Since then, scholars in several fields use these theories for decision making [2,3,14,2123,27]. Possibility measures, distance measures and metrics are great decision support tools in these theories. The possibility theory $[12,30]$ is presented as an alternative framework to represent uncertain information. Distance measures and metrics was used in [ $8,10,15,17,20,28,31]$ for decision making in several environments such as medical diagnosis and pattern recognition for example.

In recent years, statistical theory is widely used in fuzziness and intuitionistic fuzziness. In descriptive statistics for example, when $X$ and $Y$ are crisp characters, $\left(X^{-1}(a), Y^{-1}(b)\right)$ is the set of individuals of $\Omega$ that is having both the modalities $a$ and $b$ on the respectively sets of observations $A$ and $B$. To summarize the information contained in the pair $(X, Y)$, we then construct a contingency table where at the crossroads of the row $a$ and the column $b$, we have the number of individuals of $\Omega$ that is having both the modalities $a$ and $b$, and denoted by $\operatorname{Card}(X=a, Y=b)$. This is to measure by the cardinal, the importance given to the pair $(a, b)$ by using quantitative evaluation of the data. However when dealing with vague and ambiguous information, we need a qualitative evaluation of the data. Then another measure like possibility measure to evaluate the set $(X=a, Y=b)$, comes into play. Gwet [18] proposed the distance which extends to the fuzzy case the Tchebychev distance on crisp sets; then he used this distance to construct a proximity measure between two character modalities in a two-dimensional fuzzy statistical description. This measure being based on the cardinal of the symmetric difference between two fuzzy sets. Several authors such as Antonov [1], Ejegwa [13], Huawen [19] and Taiwo et al. [24] introduced and studied the properties of some difference and symmetric difference operations for IFSs. Determined from the intuitionistic fuzzy R-implication operators, Taiwo et al. [24] introduced more general difference and symmetric difference operations for IFSs and present various characterizations of these operators. They further derive one property of cardinality of the symmetric difference between IFSs.

This work extends the result of Gwet [18] to the intuitionistic fuzzy case. To be more precise, we first use the symmetric difference proposed by Taiwo et al. [24] and the notion
of cardinality of an intuitionistic fuzzy set proposed by Tripathy et al. [25] to propose a more generalized Tchebychev distance measures based on intuitionistic fuzzy implication. We then deduce two examples of metrics. That will be the extensions of the Gwet's distance to the intuitionistic fuzzy case. Notice that Atanassova [7] proposed on "Remarks on the cardinality of the intuitionistic fuzzy sets", the first research on the cardinality of an IFS. Secondly, we give a definition of an intuitionistic fuzzy mapping and we show that, the properties of the fuzzy mapping defined by Gwet [18] are preserved by this mapping. At the end we use this mapping to propose a Tchebychev possibility measure based on IF-cardinality. This help to propose a proximity measure between two modalities of characters in a two-dimensional fuzzy statistical description.

The remaining part of the paper is organized as follows. Section 2 gives basic notions of IFSs. Section 3 deals with some classes of Tchebychev distance measures and metrics based on intuitionistic fuzzy implication. Section 4 examines the concept of intuitionistic fuzzy mapping. Section 5 presents the concept of intuitionistic fuzzy statistical description and Section 6 comprises the concluding remarks part.

## 2 Preliminaries

In this section, we recall some basic notions that form the background of our theoretical framework. Throughout the paper, $\Omega$ is a nonempty and finite universe, $\left(L^{*}, \leq_{L^{*}}, \vee, \wedge\right)$ is a complete lattice with unit elements $0_{L^{*}}=(0,1)$ and $1_{L^{*}}=(1,0)$ where $L^{*}=\left\{\left(\omega_{1}, \omega_{2}\right) \in[0,1] \times[0,1] \mid\right.$ $\left.\omega_{1}+\omega_{2} \leq 1\right\}, \leq_{L^{*}}$ be an order on $L^{*}$ defined by $\forall\left(\omega_{1}, \omega_{2}\right),\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}\right) \in L^{*}$,

$$
\left(\omega_{1}, \omega_{2}\right) \leq_{L^{*}}\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}\right) \Longleftrightarrow\left(\omega_{1} \leq \omega_{1}^{\prime} \text { and } \omega_{2} \geq \omega_{2}^{\prime}\right) .
$$

The meet operator $\wedge$ and join operator $\vee$ on $L^{*}$ are defined as follows.

$$
\left(\omega_{1}, \omega_{2}\right) \wedge\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}\right)=\left(\min \left(\omega_{1}, \omega_{1}^{\prime}\right), \max \left(\omega_{2}, \omega_{2}^{\prime}\right)\right)
$$

and

$$
\left(\omega_{1}, \omega_{2}\right) \vee\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}\right)=\left(\max \left(\omega_{1}, \omega_{1}^{\prime}\right), \min \left(\omega_{2}, \omega_{2}^{\prime}\right)\right)
$$

We recall now some operators on fuzzy sets and IFSs ( $[9,11,16,24]$ ).

### 2.1 On IFS and its operations

In the rest of this paper, $\top$ is a fuzzy t -norm and $S$ is a fuzzy t-conorm, $I_{\top}$ and $J_{S}$ the fuzzy implication and co-implication operators associated with $\top$ and $S$, respectively. $\mathcal{T}=(\top, S)$ is a t-representable intuitionistic fuzzy t-norm (t-representable IF-t-norm), that is, a binary operation $\mathcal{T}$ on $L^{*}$ defined by: $\forall \omega=\left(\omega_{1}, \omega_{2}\right), \mathbf{y}=\left(y_{1}, y_{2}\right) \in L^{*}, \mathcal{T}(\omega, \mathbf{y})=\left(\top\left(\omega_{1}, y_{1}\right), S\left(\omega_{2}, y_{2}\right)\right)$.

The IF-R-implication to the corresponding fuzzy co-implication $J_{S}$ associated with $S$ and fuzzy R-implication $I_{\top}$ associated with $\top$ is defined by: $\forall \omega=\left(\omega_{1}, \omega_{2}\right), \mathbf{y}=\left(y_{1}, y_{2}\right) \in L^{*}$,

$$
I_{\mathcal{T}}(\omega, \mathbf{y})=\left(\min \left(I_{\top}\left(\omega_{1}, y_{1}\right), 1-J_{S}\left(\omega_{2}, y_{2}\right)\right), J_{S}\left(\omega_{2}, y_{2}\right)\right) .
$$

In the following, we recall preliminaries on IFS and its operations (Antonov [1], Atanassov [4, 6], Taiwo et al. [24], Tzvetkov [26]).

An intuitionistic fuzzy set (IFS) $D$ on $\Omega$ is defined by
$D=\left\{\left\langle\omega, \mu_{D}(\omega), \nu_{D}(\omega)\right\rangle \mid \mu_{D}(\omega), \nu_{D}(\omega) \in[0,1], 0 \leq \mu_{D}(\omega)+\nu_{D}(\omega) \leq 1, \omega \in \Omega\right\}$,
where $\mu_{D}(\omega), \nu_{D}(\omega)$ are respectively the degrees of membership and non-membership of $\omega$ in $D . \pi_{D}(\omega)=1-\mu_{D}(\omega)-\nu_{D}(\omega)$ is called the intuitionistic fuzzy index or hesitancy degree of $\omega$ in $D$. If $\mu_{D}(\omega)+\nu_{D}(\omega)=1$ for all $\omega \in \Omega$, then $D$ is a fuzzy set on $\Omega$.

Throughout this paper, $F S(\Omega)$ will be the set of all fuzzy sets on $\Omega$ and $\operatorname{IFS}(\Omega)$ will be the set of all intuitionistic fuzzy sets on $\Omega$.

Let $A, B$ be two IFSs defined on $\Omega$. The intuitionistic fuzzy symmetric difference associated with $\mathcal{T}$ of $A$ and $B$ is the IFS on $\Omega$ denoted by $A \Delta \mathcal{T} B$ and defined by the membership and non-membership degrees as follows: For all $\omega \in \Omega$,

$$
\begin{align*}
\mu_{A \Delta \mathcal{T} B}(\omega)= & J_{S}\left(\nu_{A}(\omega) \wedge \nu_{B}(\omega), \nu_{A}(\omega) \vee \nu_{B}(\omega)\right), \\
\nu_{A \Delta \mathcal{T} B}(\omega)= & \min \left\{I_{\top}\left(\mu_{A}(\omega) \vee \mu_{B}(\omega), \mu_{A}(\omega) \wedge \mu_{B}(\omega)\right),\right.  \tag{1}\\
& \left.1-J_{S}\left(\nu_{A}(\omega) \wedge \nu_{B}(\omega), \nu_{A}(\omega) \vee \nu_{B}(\omega)\right)\right\} .
\end{align*}
$$

Taiwo et al. [24] showed that: when $A$ and $B$ are fuzzy sets on $\Omega$ and, $S$ and $\top$ dual, the intuitionistic fuzzy symmetric difference operator is an extension of the fuzzy symmetric difference operator proposed by Fono et al. [15]. More precisely, $A \Delta_{\mathcal{T}} B$ is a fuzzy set (i.e $\left.\nu_{A \Delta_{\mathcal{T}} B}(\omega)=1-\mu_{A \Delta_{\mathcal{T}} B}(\omega) \forall \omega \in \Omega\right)$ associated to $T$ and defined by $\mu_{A \Delta_{\top} B}(\omega)=\mu_{A \Delta_{\mathcal{T}} B}(\omega)$ where $A \triangle_{\top} B$ is a fuzzy symmetric difference between $A$ and $B$ proposed by Fono et al. [15].

We have the following classical properties of the intuitionistic fuzzy symmetric difference operation ( [24]).
Let $A, B$ and $C$ be IFSs on $\Omega . A-\mathcal{T} B$ is the intuitionistic fuzzy difference (IF-difference) operation defined by Taiwo et al. [24]. The following properties for intuitionistic fuzzy symmetric difference operations hold:
$A \Delta_{\mathcal{T}} B=\left(A-_{\mathcal{T}} B\right) \cup\left(B-_{\mathcal{T}} A\right) ; A \Delta_{\mathcal{T}} B=B \Delta_{\mathcal{T}} A$; If $A \subseteq B$, then $A \Delta_{\mathcal{T}} B=B-_{\mathcal{T}} A$ and $A \Delta_{\mathcal{T}} A=\varnothing$.

The following example gives the expressions of memberships and non-memberships degrees of IF symmetric difference operator associated with t-representable IF t-norms $\mathcal{T}_{L}$ of Łukasiewicz and $\mathcal{T}_{P}$ product, respectively (Taiwo et al. [24]).

Example 1. Let $A$ and $B$ be two intuitionistic fuzzy sets defined on $\Omega$. For all $\omega \in \Omega$,

1. The IF symmetric difference between $A$ and $B$ associated with $\mathcal{T}_{L}=\left(\top_{L}, S_{L}\right)$ is given by:

$$
\mu_{A \Delta \tau_{L} B}(\omega)= \begin{cases}0, & \text { if }\left(\mu_{A}(\omega), \nu_{A}(\omega)\right)=\left(\mu_{B}(\omega), \nu_{B}(\omega)\right)  \tag{2}\\ \left(\nu_{A}(\omega)-\nu_{B}(\omega)\right) \vee\left(\nu_{B}(\omega)-\nu_{A}(\omega)\right), & \text { if }\left(\mu_{A}(\omega), \nu_{A}(\omega)\right) \neq\left(\mu_{B}(\omega), \nu_{B}(\omega)\right),\end{cases}
$$

and

$$
\nu_{A \Delta \tau_{L} B}(\omega)= \begin{cases}1, & \text { if }\left(\mu_{A}(\omega), \nu_{A}(\omega)\right)  \tag{3}\\
\begin{array}{rl}
1-\left(\mu_{A}(\omega)-\mu_{B}(\omega)\right) \vee\left(\mu_{B}(\omega)-\mu_{A}(\omega)\right), & \\
\min \left\{1-\left(\nu_{A}(\omega)-\nu_{B}(\omega)\right) \vee\left(\nu_{B}(\omega)\right)\right.
\end{array} \\
\left.\left.\nu_{B}(\omega)-\nu_{A}(\omega)\right)\right\} & \text { else. }\end{cases}
$$

2. The IF symmetric difference between $A$ and $B$ associated with $\mathcal{T}_{P}=\left(\top_{P}, S_{P}\right)$ is given by:

$$
\mu_{A \Delta \tau_{P} B}(\omega)= \begin{cases}0, & \text { if } \nu_{A}(\omega)=\nu_{B}(\omega)  \tag{4}\\ \frac{\nu_{A}(\omega) \nu_{B}(\omega)-\nu_{A}(\omega) \wedge \nu_{B}(\omega)}{1-\nu_{A}(\omega) \wedge \nu_{B}(\omega)}=\frac{\left.\left(\nu_{A}(\omega)-\nu_{B}(\omega)\right)\right)\left(\nu_{B}(\omega)-\nu_{A}(\omega)\right)}{\left(1-\nu_{A}(\omega)\right) V\left(1-\nu_{B}(\omega)\right)}, & \text { else. }\end{cases}
$$

and

$$
\nu_{A \Delta \tau_{P}}(\omega)=\left\{\begin{array}{lr}
1, & \text { if }\left(\mu_{A}(\omega), \nu_{A}(\omega)\right)  \tag{5}\\
& =\left(\mu_{B}(\omega), \nu_{B}(\omega)\right) \\
\min \left\{\frac{\mu_{A}(\omega) \wedge \mu_{B}(\omega)}{\mu_{A}(\omega) \vee \mu_{B}(\omega)}, 1-\frac{\left(\nu_{A}(\omega)-\nu_{B}(\omega)\right) \vee\left(\nu_{B}(\omega)-\nu_{A}(\omega)\right)}{\left(1-\nu_{A}(\omega)\right) \vee \vee\left(1-\nu_{B}(\omega)\right)}\right\}, & \text { if }\left(\mu_{A}(\omega), \nu_{A}(\omega)\right) \\
& \neq\left(\mu_{B}(\omega), \nu_{B}(\omega)\right) .
\end{array}\right.
$$

In the following subsection, we recall distance propose by Gwet [18] in the case of fuzzy sets.

### 2.2 On Tchebychev distance measure and metrics for fuzzy sets

We recall here that, the Tchebychev metric $d_{1}$ between crisp sets $A$ and $B$ is defined as follow:

$$
d_{1}(A, B)=\bigvee_{\omega \in \Omega}\left|1_{A}(\omega)-1_{B}(\omega)\right|
$$

where $1_{A}$ and $1_{B}$ are the memberships indicators functions. To extend $d_{1}$ into fuzzy sets, Gwet [18] proposed the mapping $d_{\top}$ defined as follow: Let $A$ and $B$ be fuzzy sets on $\Omega$.

$$
d_{\top}(A, B)=\bigvee_{\omega \in \Omega} \mu_{A \triangle_{T} B}(\omega)
$$

where $\triangle_{T}$ is a fuzzy symmetric difference operator associated to the fuzzy $t$-norm $T$. Under the fuzzy t-norms $\top_{L}$ of Łukasiewicz and $\top_{P}$ of product, he proved that $d_{\top}$ is a metric. More precisely, he showed that,

$$
\begin{equation*}
d_{\top_{P}}(A, B)=\bigvee_{\omega \in \Omega} \frac{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|}{\mu_{A}(\omega) \vee \mu_{B}(\omega)} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{\top_{L}}(A, B)=\bigvee_{\omega \in \Omega}\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|, \tag{7}
\end{equation*}
$$

which is a natural extension of $d_{1}$. He called $d_{T}$ the generalized Tchebychev distance or the distance of the height of the symmetric difference. It is a formulation analogous to that of the
distance of the cardinality of the symmetric difference widely used in Data Analysis and recently defined by Fono et al. [15] as follow:

$$
d_{\mathrm{T}}^{\prime}(A, B)=\frac{1}{n} \operatorname{Card}\left(A \triangle_{\mathrm{T}} B\right)=\frac{1}{n} \sum_{i=1}^{n} \mu_{A \triangle_{\mathrm{T}} B}\left(\omega_{i}\right) .
$$

Notice that $d_{\top}^{\prime}$ is a distance measure for fuzzy set. Then it is easy to prove that $d_{\top}$ is also a distance measure for fuzzy set.

In the following section, we will propose some classes of Tchebychev distance measures and metrics for IFSs based on the preview symmetric difference between IFSs. We assume that $\Omega=\left\{\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right\}$ have $n$ elements.

## 3 Some classes of Tchebychev distance measures and metrics based on intuitionistic fuzzy implication

In the following subsection, we extend to the intuitionistic fuzzy case the distance measure $d_{\top}$.

### 3.1 Generalized Tchebychev distance measure and metrics for intuitionistic fuzzy sets

We first recall here that Tripathy et al. [25] defined the intuitionistic fuzzy cardinality (IFcardinality) of an IFS $A$ on $\Omega$ denoted by $\operatorname{Count}(A)$ as follow:

$$
\operatorname{Count}(A)=\left(\sum_{i=1}^{n} \mu_{A}\left(\omega_{i}\right), \sum_{i=1}^{n}\left(1-\nu_{A}\left(\omega_{i}\right)\right)\right)
$$

Let $A, B \in \operatorname{IFS}(\Omega)$. Using a linear combination of the components, of the cardinality, of the symmetric difference, we propose the following mapping:

$$
\begin{equation*}
d_{\mathcal{T}}^{0}(A, B)=\frac{1}{2 n} \sum_{i=1}^{n}\left(1+\mu_{A \Delta_{\mathcal{T}} B}(\omega)-\nu_{A \Delta_{\mathcal{T}} B}(\omega)\right) . \tag{8}
\end{equation*}
$$

We then deduce the following mapping:

$$
\begin{equation*}
d_{\mathcal{T}}(A, B)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{A \Delta_{\mathcal{T}} B}(\omega)-\nu_{A \Delta_{\mathcal{T}} B}(\omega)\right) \tag{9}
\end{equation*}
$$

In view of its construction and the definition of the symmetric difference on the basis of IFimplication, these mappings are in our opinion based on IF-cardinality of the symmetric difference and on the IF-implication.

The following Definition 3.1 gives definitions and useful properties of distance measures and metrics between IFSs.

Definition 3.1. $[8,10,28,31]$ Let $d: \operatorname{IFS}(\Omega) \times \operatorname{IFS}(\Omega) \rightarrow[0,1]$ be a mapping.
a) $d$ is a distance measure if for all $A, B, C \in \operatorname{IFS}(\Omega)$, the following properties hold: (i) $0 \leq d(A, B) \leq 1$; (ii) $d(A, B)=0$ if and only if $A=B$; (iii) $d(A, B)=d(B, A)$; (iv) If $A \subseteq B \subseteq C$, then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$.
b) $d$ is a metric (distance) if for all $A, B, C \in \operatorname{IFS}(\Omega), d$ satisfies the following properties: axiom (i) $d(A, B)=0$ if and only if $A=B$; axiom (ii) $d(A, B)=d(B, A)$ and axiom (iii) $d(A, C) \leq d(A, B)+d(B, C)$.

The following main result shows that, the mappings defined by Eqs. (8)-(9) are distance measures.

Theorem 1. $d_{\mathcal{T}}^{0}$ and $d_{\mathcal{T}}$ are distance measures associated with $\mathcal{T}$.
$d_{\mathcal{T}}$ is this distance measure which we call the generalize Tchebychev distance measure or the distance measure of the height of symmetric difference for intuitionistic fuzzy sets. $d_{\mathcal{T}}^{0}$ can be called the distance measure of the cardinality of the symmetric difference.

Let $A, B, C \in \operatorname{IFS}(\Omega)$. To prove Theorem 1, we first notice that it is easy to prove the following additional properties (Eq. (11) and Eq. (12)) for IF symmetric difference by using the definition, the properties of the symmetric difference and the following well-known properties (Eq. (10)) of $J_{S}$ and $I_{\mathrm{\top}}$, for all $a, b \in[0,1]$,

$$
\begin{gather*}
\left\{\begin{array}{cc}
(i) & J_{S}(a, b)=0 \text { if and only if } \quad a \geq b \\
(i i) & I_{\top}(a, b)=1 \text { if and only if } \\
a \leq b
\end{array}\right.  \tag{10}\\
A \Delta_{\mathcal{T}} B=\varnothing \text { if and only if } A=B . \tag{11}
\end{gather*}
$$

$$
\begin{equation*}
\text { If } A \subseteq B \subseteq C \text {, then } A \Delta_{\mathcal{T}} B \subseteq A \Delta_{\mathcal{T}} C \text { and } B \Delta_{\mathcal{T}} C \subseteq A \Delta_{\mathcal{T}} C \tag{12}
\end{equation*}
$$

Proof. $\quad$ (i) $0 \leq d_{\mathcal{T}}(A, B) \leq 1$ since $A \Delta_{\mathcal{T}} B \in \operatorname{IFS}(X) \Longrightarrow 0 \leq 1+\mu_{A \Delta_{\mathcal{T}} B}(\omega)-\nu_{A \Delta_{\mathcal{T}} B}(\omega) \leq$ $2 \forall \omega \in \Omega$.
(ii) We prove that $d_{\mathcal{T}}(A, B)=0 \Longleftrightarrow A=B$. We recall first that $\varnothing$ is an IFS defined by $\mu_{\varnothing}=0$ and $\nu_{\varnothing}=1$. Thereby $\left(d_{\mathcal{T}}(A, B)=0 \Longleftrightarrow \mu_{A \Delta_{\mathcal{T}} B}(\omega)=0\right.$ and $\nu_{A \Delta_{\mathcal{T}} B}(\omega)=1$ $\forall \omega \in \Omega) \Longleftrightarrow\left(A \Delta_{\mathcal{T}} B=\varnothing\right)$. From Eq. (11) the result follows.
(iii) We prove that $d_{\mathcal{T}}(A, B)=d_{\mathcal{T}}(B, A)$. This is obvious duce to the fact that, $A \Delta_{\mathcal{T}} B=$ $B \Delta_{\mathcal{T}} A$.
(iv) Assume that $A \subseteq B \subseteq C$. We prove that $d_{\mathcal{T}}(A, C) \geq d_{\mathcal{T}}(A, B)$ and $d_{\mathcal{T}}(A, C) \geq$ $d_{\mathcal{T}}(B, C)$. From Eq. (12) $\mu_{A \Delta_{\mathcal{T}} B}(\omega) \leq \mu_{A \Delta_{\mathcal{T}} C}(\omega), \mu_{B \Delta_{\mathcal{T}} C}(\omega) \leq \mu_{A \Delta_{\mathcal{T}} C}(\omega), 1-\nu_{A \Delta_{\mathcal{T}} B}(\omega) \leq$ $1-\nu_{A \Delta_{\mathcal{T}} C}(\omega), 1-\nu_{B \Delta_{\mathcal{T}} C}(\omega) \leq 1-\nu_{A \Delta_{\mathcal{T}} C}(\omega)$. The result follows immediately from Eq. (9).

The following example gives the expressions of $d_{\mathcal{T}_{L}}$ and $d_{\mathcal{T}_{P}}$.

Example 2. Let $A, B \in \operatorname{IFS}(\Omega)$.

$$
\begin{equation*}
d_{\mathcal{T}_{L}}(A, B)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(\max \left\{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|\right\}+\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|\right) \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{\mathcal{T}_{P}}(A, B)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(\max \left\{\frac{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|}{\mu_{A}(\omega) \vee \mu_{B}(\omega)}, \frac{\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|}{1-\nu_{A}(\omega) \wedge \nu_{B}(\omega)}\right\}+\frac{\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|}{1-\nu_{A}(\omega) \wedge \nu_{B}(\omega)}\right) \tag{14}
\end{equation*}
$$

Assume that $\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}, A=\left\{\left(\omega_{1}, 0.8,0.1\right),\left(\omega_{2}, 0.6,0.4\right),\left(\omega_{3}, 0.4,0.5\right)\right\}$ and $B=$ $\left\{\left(\omega_{1}, 0.4,0.4\right),\left(\omega_{2}, 0.6,0.3\right),\left(\omega_{3}, 0.7,0.3\right)\right\}$. Then $d_{\mathcal{T}_{L}}(A, B)=\frac{1}{2}(0.7 \vee 0.2 \vee 0.5)=0.35$ and $d_{\mathcal{T}_{P}}(A, B)=\frac{1}{2}\left(\frac{5}{6} \vee \frac{2}{7} \vee \frac{5}{7}\right)=\frac{5}{12}$.

The following result shows that $d_{\mathcal{T}_{L}}$ and $d_{\mathcal{T}_{P}}$ defined by Eq. (13) and Eq. (14) respectively, are metrics that extend into intuitionistic fuzzy sets the metrics $d_{\top_{L}}$ defined by Eq. (7) and $d_{\top_{P}}$ defined by Eq. (6), respectively.

## Proposition 3.1.

1) The mappings $d_{\mathcal{T}_{L}}$ and $d_{\mathcal{T}_{P}}$ are metrics.
2) If $A$ and $B$ are fuzzy set on $\Omega$, then $d_{\mathcal{T}_{L}}(A, B)=d_{\top_{L}}(A, B)$ and $d_{\mathcal{T}_{P}}(A, B)=d_{\top_{P}}(A, B)$.

Proof. Proof of 1) We prove that $d_{\mathcal{T}_{L}}$ and $d_{\mathcal{T}_{P}}$ are metrics. From Definition 3.1 and Theorem 1, it is sufficient to prove the triangular inequalities. Let $A, B, C \in \operatorname{IFS}(\Omega)$ and let $\omega \in \Omega$.
We prove that $d_{\mathcal{T}_{L}}(A, C) \leq d_{\mathcal{T}_{L}}(A, B)+d_{\mathcal{T}_{L}}(B, C)$. From

$$
\left\{\begin{array}{l}
\left|\mu_{A}(\omega)-\mu_{C}(\omega)\right| \leq\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|+\left|\mu_{B}(\omega)-\mu_{C}(\omega)\right|  \tag{15}\\
\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right| \leq\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|+\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|
\end{array}\right.
$$

Eq. (16) holds.

$$
\begin{gather*}
\max \left\{\left|\mu_{A}(\omega)-\mu_{C}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right|\right\} \\
\leq \max \left\{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|+\left|\mu_{B}(\omega)-\mu_{C}(\omega)\right|,\right.  \tag{16}\\
\left.\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|+\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|\right\}
\end{gather*}
$$

Then Eq. (17) holds since $\forall a, b, c, d \in \mathbb{R},(a+b) \vee(c+d) \leq(a \vee c)+(b \vee d)$.

$$
\begin{align*}
& \quad \max \left\{\left|\mu_{A}(\omega)-\mu_{C}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right|\right\} \\
& \leq \max \left\{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|\right\}  \tag{17}\\
& \quad+\max \left\{\left|\mu_{B}(\omega)-\mu_{C}(\omega)\right|,\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|\right\}
\end{align*}
$$

Thus

$$
\begin{align*}
& \quad \max \left\{\left|\mu_{A}(\omega)-\mu_{C}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right|\right\}+\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right| \\
& \leq \max \left\{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|,\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|\right\}+\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|  \tag{18}\\
& \quad+\max \left\{\left|\mu_{B}(\omega)-\mu_{C}(\omega)\right|,\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|+\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|\right\}
\end{align*}
$$

The result follows immediately since for all elements $a_{i}, b_{i}$ of the finite set,

$$
\bigvee_{i}\left(a_{i}+b_{i}\right) \leq \bigvee_{i}\left(a_{i}+\vee b_{i}\right) .
$$

We prove that $d_{\mathcal{T}_{P}}(A, C) \leq d_{\mathcal{T}_{P}}(A, B)+d_{\mathcal{T}_{P}}(B, C)$. Since $d_{T_{P}}$ defined by Eq. (6) is a metric, the following Eq. (19) holds.

$$
\begin{equation*}
\frac{\left|\mu_{A}(\omega)-\mu_{C}(\omega)\right|}{\mu_{A}(\omega) \vee \mu_{C}(\omega)} \leq \frac{\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|}{\mu_{A}(\omega) \vee \mu_{B}(\omega)}+\frac{\left|\mu_{B}(\omega)-\mu_{C}(\omega)\right|}{\mu_{B}(\omega) \vee \mu_{C}(\omega)} \forall \omega \in \Omega . \tag{19}
\end{equation*}
$$

We prove now that the following Eq. (20) holds.

$$
\begin{equation*}
\frac{\left|\nu_{A}(\omega)-\nu_{C}(\omega)\right|}{1-\nu_{A}(\omega) \wedge \nu_{C}(\omega)} \leq \frac{\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|}{1-\nu_{A}(\omega) \wedge \nu_{B}(\omega)}+\frac{\left|\nu_{B}(\omega)-\nu_{C}(\omega)\right|}{1-\nu_{B}(\omega) \wedge \nu_{C}(\omega)} \quad \forall \omega \in \Omega \tag{20}
\end{equation*}
$$

Let $a, b, c \in[0,1]$. Then

$$
\begin{aligned}
& \frac{|a-b|}{(1-a) \wedge(1-b)}=1-\left(\frac{1-a}{1-b}\right) \wedge\left(\frac{1-b}{1-a}\right) ; \\
& \frac{|a-c|}{(1-a) \wedge(1-c)}=1-\left(\frac{1-a}{1-c}\right) \wedge\left(\frac{1-c}{1-a}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{|b-c|}{(1-b) \wedge(1-c)}= & 1-\left(\frac{1-b}{1-c}\right) \wedge\left(\frac{1-c}{1-b}\right) \\
\left(\frac{1-a}{1-b}\right) \wedge\left(\frac{1-b}{1-a}\right)+\left(\frac{1-b}{1-c}\right) \wedge\left(\frac{1-c}{1-b}\right)= & \left(\frac{1-a}{1-b}+\frac{1-b}{1-c}\right) \wedge\left(\frac{1-a}{1-b}+\frac{1-c}{1-b}\right) \\
& \wedge\left(\frac{1-b}{1-a}+\frac{1-b}{1-c}\right) \wedge\left(\frac{1-b}{1-a}+\frac{1-c}{1-b}\right) \\
& \leq\left(\frac{1-a}{1-b}+\frac{1-c}{1-b}\right) \wedge\left(\frac{1-b}{1-a}+\frac{1-b}{1-c}\right) \\
& \leq\left(1+\frac{1-a}{1-c}\right) \wedge\left(1+\frac{1-c}{1-a}\right) \\
& \leq 1+\frac{1-a}{1-c} \wedge \frac{1-c}{1-a} .
\end{aligned}
$$

Passing to the complement of 1 , we deduce that:

$$
\begin{equation*}
1-\left(\frac{1-a}{1-c}\right) \wedge\left(\frac{1-c}{1-a}\right) \leq 1-\left(\frac{1-a}{1-b}\right) \wedge\left(\frac{1-b}{1-a}\right)+1-\left(\frac{1-b}{1-c}\right) \wedge\left(\frac{1-c}{1-b}\right) \tag{21}
\end{equation*}
$$

By setting $a=\nu_{A}(\omega), b=\nu_{B}(\omega)$ and $c=\nu_{C}(\omega), \forall \omega \in \Omega$, in Eq. (21), Eq. (20) holds.
By combining Eq. (19) and Eq. (20) like in Eq. (16), Eq. (17) and Eq. (18), the result follows.
 and $d_{\mathcal{T}_{P}}(A, B)=d_{\top_{P}}(A, B)$. Since $A$ and $B$ are fuzzy sets on $\Omega$, then $\nu_{A}(\omega)=1-\mu_{A}(\omega)$ and $\nu_{B}(\omega)=1-\mu_{B}(\omega) \forall \omega \in \Omega$. Therefore $\left|\mu_{A}(\omega)-\mu_{B}(\omega)\right|=\left|\nu_{A}(\omega)-\nu_{B}(\omega)\right|$ and $1-\nu_{A}(\omega) \wedge \nu_{B}(\omega)=\left(1-\nu_{A}(\omega)\right) \vee\left(1-\nu_{B}(\omega)\right)=\mu_{A}(\omega) \vee \mu_{B}(\omega)$. The results follow immediately from the expressions of $d_{\mathcal{T}_{L}}$ and $d_{\top_{L}}$ defined by Eq. (13) and Eq. (7) respectively, and from the expressions of $d_{\mathcal{T}_{P}}$ and $d_{T_{P}}$ defined by Eq. (14) and Eq. (6) respectively.

The following result shows that, the generalize Tchebychev distance measure $d_{\mathcal{T}}$ and the new distance measure $d_{\mathcal{T}}^{0}$ are the extensions of the distance measures $d_{\top}$ of Gwet [18] and $d_{\top}^{\prime}$ of Fono et al. [15] defined on fuzzy sets respectively, into intuitionistic fuzzy sets.

Proposition 3.2. Let $A$ and $B$ be fuzzy sets on $\Omega$. If $\top$ and $S$ are dual, then $d_{\mathcal{T}}^{0}(A, B)=d_{\top}^{\prime}(A, B)$ and $d_{\mathcal{T}}(A, B)=d_{\top}(A, B)$.

Proof. Assume that $\top$ and $S$ are dual. We prove that $d_{\mathcal{T}}^{0}(A, B)=d_{\top}^{\prime}(A, B)$ and $d_{\mathcal{T}}(A, B)=$ $d_{\top}(A, B)$. From Taiwo et al. [24], since $\top$ and $S$ are dual, and $A$ and $B$ are fuzzy sets on $\Omega, A \Delta_{\mathcal{T}} B$ is a fuzzy set (i.e $\nu_{A \Delta_{\mathcal{T}} B}(\omega)=1-\mu_{A \Delta_{\mathcal{T}} B}(\omega)$ ) associated to $\top$ and defined by $\mu_{A \Delta_{T} B}(\omega)=\mu_{A \Delta_{\mathcal{T}} B}(\omega)$. Using these results of Taiwo et al. [24] in Eqs. (8)-(9), the result follows.

We must use the cardinality to define the possibility measure on IFSs. This require the intuitionistic fuzzy mapping.
In the following section, we will propose a mapping on the intuitionistic fuzzy sets.

## 4 Mapping on the intuitionistic fuzzy sets

A modality $a$ of a character $X$ is observed by the individual $\omega$ of a population $\Omega$ if and only if $X(\omega)=a$. This reflects the fact that $a$ is the image of $\omega$ by mapping $X$. This assignment operation is symbolized by a crisp mapping $X$.

### 4.1 Fuzzy mapping

Gwet [18] extends crisp mapping to fuzzy mapping that we recall in this subsection. Let $E$ and $F$ be two sets. A fuzzy correspondence between $E$ and $F$ is a fuzzy relationship $R$ of $E$ towards $F$. Such relation takes the form of an array with two entries $E$ and $F$, where at the crossroads the row $x$ and the column $y$, we have the number $\mu_{R}(x, y) \in[0,1]$ which evaluate a membership degree with which $x$ and $y$ are related. Given an element $x$ of $E$, the cut follow $x$, is a fuzzy set $C(x)$ defined by:

$$
\mu_{C(x)}(y)=\mu_{R}(x, y) \quad \forall y \in F .
$$

The correspondence $R$ is a fuzzy mapping if

$$
\bigvee_{y \in F} \mu_{C(x)}(y)=1 \quad \forall x \in E .
$$

This condition shows that on each row of the fuzzy correspondence table, we find at least once the number 1. Or even that each element of the starting set has at least one crisp image in the target set.

The image of an element $x$ of $E$ by the mapping $X$ is a fuzzy set $X(x)$ defined by:

$$
X(x)=C(x)
$$

Let $X$ be a crisp mapping of $E$ towards $F$ and $A$ be a fuzzy part of $E$. The extension principle of Zadeh [29] makes it possible to define from $A$, a fuzzy part of $F$ denoted by $X(A)$ by

$$
\mu_{X(A)}(y)=\bigvee_{x \in E} \mu_{A}(x) \wedge 1_{X(x)=y}
$$

When $X$ is a fuzzy mapping, this principle can be extended in fuzzy case by:

$$
\mu_{X(A)}(y)=\bigvee_{x \in E} \mu_{A}(x) \wedge \mu_{X(x)}(y)
$$

$X(A)$ is said to be the image of the fuzzy set $A$ by the fuzzy mapping $X$.
Notice that when $\mu_{X(x)}(y)=1, y=X(x)$ is called crisp image of $x$.
Let $X$ be a fuzzy mapping and $B$ be a fuzzy part of $F$. The reciprocal image of $B$ by $X$ is defined by

$$
\mu_{X^{-1}(B)}(x)=\bigvee_{y \in F} \mu_{B}(y) \wedge \mu_{X(x)}(y)
$$

Then $\mu_{X^{-1}(y)}(x)=\mu_{X(x)}(y)$. We notice that a row $x$ of the fuzzy correspondence table represents the image of $x$ by $X$ and the column $y$ represented the reciprocal image of $y$ by $X$. Subsequently, we will write $X=y$ to represent $X^{-1}(\{y\})$,

### 4.2 Intuitionistic fuzzy mapping

Let us now see how we can extend all of these definitions to the intuitionistic fuzzy case. Let $E$ and $F$ be two referential sets. An intuitionistic fuzzy correspondence between $E$ and $F$ is an intuitionistic fuzzy relationship $R$ of $E$ towards $F$. Such a relation takes the form of an array with two entries $E$ and $F$ (Table 1) where at the crossroads, the row $x$ and the column $y$, we have the pair $\left\langle\mu_{R}(x, y), \nu_{R}(x, y)\right\rangle \in[0,1]^{2}$ such that $\mu_{R}(x, y)+\nu_{R}(x, y) \leq 1$, which evaluate a membership and nonmembership degrees with which $x$ and $y$ are related. Given an element $x$ of $E$, the cut follow $x$, is a intuitionistic fuzzy set $C(x)$ defined by:

$$
\left(\mu_{C(x)}(y), \nu_{C(x)}(y)\right)=\left(\mu_{R}(x, y), \nu_{R}(x, y)\right) \quad \forall y \in F .
$$

Table 1. Intuitionistic fuzzy correspondence $R$ between $E$ and $F$

| $E \backslash F$ | $\ldots$ | $y$ | $\ldots$ |
| :---: | :---: | :---: | :---: |
| $\ldots$ |  |  |  |
| $x$ | $\ldots$ | $\left(\mu_{R}(x, y), \nu_{R}(x, y)\right)$ | $\ldots$ |
| $\ldots$ |  |  |  |

The following is definition of intuitionistic fuzzy mapping.
Definition 4.1. Let $R$ be an intuitionistic fuzzy correspondence between $E$ and $F$. Then $R$ is an intuitionistic fuzzy mapping (IF-mapping) if

$$
\begin{equation*}
\left(\bigvee_{y \in F} \mu_{C(x)}(y), \bigwedge_{y \in F} \nu_{C(x)}(y)\right)=(1,0) \quad \forall x \in E . \tag{22}
\end{equation*}
$$

Definition 4.1 coincides with the truth degree of the evaluation function defined in [5]. Eq. (22) shows that on each row of the intuitionistic fuzzy correspondence table (Table 1), we find at least one pair of numbers $(1,0)$. Or even that each element of the starting set has at least one crisp image in the target set.

The image of an element $x$ of $E$ by the mapping $X$ is an intuitionistic fuzzy set $X(x)$ defined by:

$$
X(x)=C(x)
$$

Let $X$ be a fuzzy mapping of $E$ towards $F$ and $A$ be an intuitionistic fuzzy set of $E$. The extension principle of Atanassov [4] makes it possible to define from $A$, an intuitionistic fuzzy set of $F$ denoted by $X(A)$ by

$$
\left(\mu_{X(A)}(y), \nu_{X(A)}(y)\right)=\left(\bigvee_{x \in E} \mu_{A}(x) \wedge \mu_{X(x)}(y), \bigwedge_{x \in E} \nu_{A}(x) \vee\left(1-\mu_{X(x)}(y)\right)\right) .
$$

The following is definition of image of an intuitionistic fuzzy set by an intuitionistic fuzzy mapping.

Definition 4.2. Let $X$ be an intuitionistic fuzzy mapping of $E$ towards $F$ and $A$ be an intuitionistic fuzzy set of $E$. The image of $A$ by $X$ is an intuitionistic fuzzy set of $F$, denoted $X(A)$ and defined by:

$$
\left(\mu_{X(A)}(y), \nu_{X(A)}(y)\right)=\left(\bigvee_{x \in E} \mu_{A}(x) \wedge \mu_{X(x)}(y), \bigwedge_{x \in E} \nu_{A}(x) \vee \nu_{X(x)}(y)\right)
$$

Notice that when $\mu_{X(x)}(y)+\nu_{X(x)}(y)=1, y=X(x)$ is called fuzzy image of $x$. This extends in fuzzy case crisp image of $x$ defined by Gwet [18]

The following is definition of reciprocal image of an intuitionistic fuzzy set by an intuitionistic fuzzy mapping.

Definition 4.3. Let $X$ be an intuitionistic fuzzy mapping and $B$ be an intuitionistic fuzzy set of $F$. The reciprocal image of $B$ by $X$ is an intuitionistic fuzzy set of $E$ defined by

$$
\left(\mu_{X^{-1}(B)}(x), \nu_{X^{-1}(B)}(x)\right)=\left(\bigvee_{y \in F} \mu_{B}(y) \wedge \mu_{X(x)}(y), \bigwedge_{y \in F} \nu_{B}(y) \vee \nu_{X(x)}(y)\right) \forall x \in E
$$

We notice that $\mu_{X^{-1}(y)}(x)=\mu_{X(x)}(y)$ and $\nu_{X^{-1}(y)}(x)=\nu_{X(x)}(y)$. Then a row of $x$ of the intuitionistic fuzzy correspondence table (Table 1) represents the image of $x$ by $X$ and the column of $y$ represents the reciprocal image of $y$ by $X$.

The following is definition of the composite of two IF-mappings.
Definition 4.4. Let $X: E \longrightarrow F$ and $Y: F \longrightarrow G$ be two IF-mappings. The composite mapping $Y \circ X: E \longrightarrow G$ is the intuitionistic fuzzy set defined by

$$
\left(\mu_{Y \circ X(x)}(z), \nu_{Y \circ X(x)}(z)\right)=\left(\bigvee_{y \in F} \mu_{X(x)}(y) \wedge \mu_{Y(y)}(z), \bigwedge_{y \in F} \nu_{X(x)}(y) \vee \nu_{Y(y)}(z)\right) \forall x \in E, \forall z \in G
$$

From the definition of the intuitionistic fuzzy image, we obtain the associativity of the max-min composition by $Y \circ X(x)=Y(X(x))$.

The following proprieties of fuzzy mappings proposed by Gwet [18] hold for intuitionistic fuzzy mappings.

Proposition 4.1. Let $X: E \longrightarrow F$ be an intuitionistic fuzzy mapping, $A_{1}, A_{2}$ be two intuitionistic fuzzy sets of $E$ and $B_{1}, B_{2}$ be two intuitionistic fuzzy sets of $F$ then:

1. $X^{-1}(F)=E$;
2. $\bigcup_{y \in F}(X=y)=E$;
3. $A_{1} \subset A_{2} \Longleftrightarrow X\left(A_{1}\right) \subset X\left(A_{2}\right) ;$ and $B_{1} \subset B_{2} \Longleftrightarrow X^{-1}\left(B_{1}\right) \subset X^{-1}\left(B_{2}\right)$;
4. $X\left(A_{1} \cup A_{2}\right)=X\left(A_{1}\right) \cup X\left(A_{2}\right)$ and $X^{-1}\left(B_{1} \cup B_{2}\right)=X^{-1}\left(B_{1}\right) \cup X^{-1}\left(B_{2}\right)$
5. $X\left(A_{1} \cap A_{2}\right) \subset X\left(A_{1}\right) \cap X\left(A_{2}\right)$ and $X^{-1}\left(B_{1} \cap B_{2}\right) \subset X^{-1}\left(B_{1}\right) \cap X^{-1}\left(B_{2}\right)$

Proof. 1. To prove that $X^{-1}(F)=E$, it is sufficient to prove that, for all $x \in E, \mu_{X^{-1}(F)}(x)=$ 1 and $\nu_{X^{-1}(F)}(x)=0$. Let $x \in E$.

$$
\begin{aligned}
\left(\mu_{X^{-1}(F)}(x), \nu_{X^{-1}(F)}(x)\right) & =\left(\bigvee_{y \in F} \mu_{F}(y) \wedge \mu_{X(x)}(y), \bigwedge_{y \in F} \nu_{F}(y) \vee \nu_{X(x)}(y)\right) \\
& =\left(\bigvee_{y \in F} \mu_{X(x)}(y), \bigwedge_{y \in F} \nu_{X(x)}(y)\right) \\
& =(1,0) \text { since } X \text { is an IF-mapping. }
\end{aligned}
$$

2. We prove that $\bigcup_{y \in F}(X=y)=E$. It is sufficient to prove that, for all $x \in E, \mu_{y \in F}^{\bigcup}(X=y)(x)=1$ and $\nu_{y \in F}(X=y)(x)=0$. Let $x \in E$.

$$
\begin{aligned}
&\left(\mu_{y \in F}(X=y)\right. \\
&(x), \nu \bigcup_{y \in F}(X=y) \\
&(x))=\left(\bigvee_{y \in F} \mu_{X(x)}(y), \bigwedge_{y \in F} \nu_{X(x)}(y)\right) \\
&=(1,0) \text { since } X \text { is an IF-mapping. }
\end{aligned}
$$

3. We prove firstly that $A_{1} \subset A_{2} \Longleftrightarrow X\left(A_{1}\right) \subset X\left(A_{2}\right)$.

$$
\begin{aligned}
A_{1} \subset A_{2} & \Longleftrightarrow \mu_{A_{1}}(x) \leq \mu_{A_{2}}(x) \text { and } \nu_{A_{1}}(x) \geq \nu_{A_{2}}(x) \forall x \in E \\
& \Longleftrightarrow \mu_{A_{1}}(x) \wedge \mu_{X(x)}(y) \leq \mu_{A_{2}}(x) \wedge \mu_{X(x)}(y) \\
& \text { and } \nu_{A_{1}}(x) \vee \nu_{X(x)}(y) \geq \nu_{A_{2}}(x) \vee \nu_{X(x)}(y) \forall x \in E \forall y \in F \\
& \Longleftrightarrow \bigvee_{x \in E} \mu_{A_{1}}(x) \wedge \mu_{X(x)}(y) \leq \bigvee_{x \in E} \mu_{A_{2}}(x) \wedge \mu_{X(x)}(y) \\
& \text { and } \bigwedge_{x \in E} \nu_{A_{1}}(x) \vee \nu_{X(x)}(y) \geq \bigwedge_{x \in E} \nu_{A_{2}}(x) \vee \nu_{X(x)}(y) \forall y \in F \\
& \Longleftrightarrow \mu_{X\left(A_{1}\right)}(y) \leq \mu_{X\left(A_{2}\right)}(y) \\
& \text { and } \nu_{X\left(A_{1}\right)}(y) \geq \nu_{X\left(A_{2}\right)}(y) \forall y \in F .
\end{aligned}
$$

We prove secondly that $B_{1} \subset B_{2} \Longleftrightarrow X^{-1}\left(B_{1}\right) \subset X^{-1}\left(B_{2}\right)$.

$$
\begin{aligned}
B_{1} \subset B_{2} & \Longleftrightarrow \mu_{B_{1}}(y) \leq \mu_{B_{2}}(y) \text { and } \nu_{B_{1}}(y) \geq \nu_{B_{2}}(y) \forall y \in F \\
& \Longleftrightarrow \mu_{B_{1}}(y) \wedge \mu_{X=y}(x) \leq \mu_{B_{2}}(y) \wedge \mu_{X=y}(x) \\
& \text { and } \nu_{B_{1}}(y) \vee \nu_{X=y}(x) \geq \nu_{B_{2}}(y) \vee \nu_{X=y}(x) \forall x \in E \forall y \in F \\
& \Longleftrightarrow \bigvee_{y \in F} \mu_{B_{1}}(y) \wedge \mu_{X=y}(x) \leq \bigvee_{y \in F} \mu_{B_{2}}(y) \wedge \mu_{X=y}(x) \\
& \text { and } \bigwedge_{y \in F} \nu_{B_{1}}(y) \vee \nu_{X=y}(x) \geq \bigwedge_{y \in F} \nu_{B_{2}}(y) \vee \nu_{X=y}(x) \forall x \in E \\
& \Longleftrightarrow \mu_{X^{-1}\left(B_{1}\right)}(x) \leq \mu_{X^{-1}\left(B_{2}\right)}(x) \\
& \text { and } \nu_{X^{-1}\left(B_{1}\right)}(x) \geq \nu_{X^{-1}\left(B_{2}\right)}(x) \forall x \in E .
\end{aligned}
$$

4. We prove that $X\left(A_{1} \cup A_{2}\right)=X\left(A_{1}\right) \cup X\left(A_{2}\right)$.

$$
\begin{aligned}
\mu_{X\left(A_{1} \cup A_{2}\right)}(y) & =\bigvee_{x \in E} \mu_{A_{1} \cup A_{2}}(x) \wedge \mu_{X(x)}(y) \forall y \in F \\
& =\left(\bigvee_{x \in E} \mu_{A_{1}}(x) \wedge \mu_{X(x)}(y)\right) \vee\left(\bigvee_{x \in E} \mu_{A_{2}}(x) \wedge \mu_{X(x)}(y)\right) \forall y \in F \\
& =\mu_{X\left(A_{1}\right)}(y) \vee \mu_{X\left(A_{2}\right)}(y) \forall y \in F
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{X\left(A_{1} \cup A_{2}\right)}(y) & =\bigwedge_{x \in E} \nu_{A_{1} \cup A_{2}}(x) \vee \nu_{X(x)}(y) \forall y \in F \\
& =\left(\bigwedge_{x \in E} \nu_{A_{1}}(x) \vee \mu_{X(x)}(y)\right) \wedge\left(\bigwedge_{x \in E} \nu_{A_{2}}(x) \vee \nu_{X(x)}(y)\right) \forall y \in F \\
& =\nu_{X\left(A_{1}\right)}(y) \wedge \nu_{X\left(A_{2}\right)}(y) \forall y \in F .
\end{aligned}
$$

We prove that $X^{-1}\left(B_{1} \cup B_{2}\right)=X^{-1}\left(B_{1}\right) \cup X^{-1}\left(B_{2}\right)$.

$$
\begin{aligned}
\mu_{X^{-1}\left(B_{1} \cup B_{2}\right)}(x) & =\bigvee_{y \in F} \mu_{B_{1} \cup B_{2}}(y) \wedge \mu_{X=y}(x) \forall x \in E \\
& =\left(\bigvee_{y \in F} \mu_{B_{1}}(y) \wedge \mu_{X=y}(x)\right) \vee\left(\bigvee_{y \in F} \mu_{B_{2}}(y) \wedge \mu_{X=y}(x)\right) \forall x \in E \\
& =\mu_{X^{-1}\left(B_{1}\right)}(x) \vee \mu_{X^{-1}\left(B_{2}\right)}(x) \forall x \in E .
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{X^{-1}\left(B_{1} \cup B_{2}\right)}(x) & =\bigwedge_{y \in F} \nu_{B_{1} \cup B_{2}}(y) \vee \nu_{X=y}(x) \forall x \in E \\
& =\left(\bigwedge_{y \in F} \nu_{B_{1}}(y) \vee \mu_{X=y}(x)\right) \wedge\left(\bigwedge_{y \in F} \nu_{B_{2}}(y) \vee \nu_{X=y}(x)\right) \forall x \in E \\
& =\nu_{X^{-1}\left(B_{1}\right)}(x) \wedge \nu_{X^{-1}\left(B_{2}\right)}(x) \forall x \in E
\end{aligned}
$$

5. To prove the last properties for $X$ and $X^{-1}$, we recall that $A_{1} \cap A_{2} \subset A_{i}$ and $B_{1} \cap B_{2} \subset B_{i}$ according to $i \in\{1,2\}$. The third property of Proposition 4.1 gives the result.

## 5 Intuitionistic fuzzy statistical description

In this Section, unless otherwise stated, $\Omega$ will be a population. An individual of $\Omega$ will be designated by the symbol $\omega$. $X$ is the intuitionistic fuzzy character which describes the population $\Omega$ if and only if $X$ is an IF-mapping to $\Omega$ in the set $A$ of observations called set of modalities.

### 5.1 Descriptor and signification

The following are definitions of descriptor of the individual $\omega$ and signification of the observation $a$.
Definition 5.1. The descriptor of $\omega$ is the intuitionsitic fuzzy set $X(w)$. The signification of the modality $a$ of $A$ is the IFS $X^{-1}(a)$.

The intuitionsitic fuzzy description of $\Omega$ by the character $X$ is presented in the form of an array with two entries (Table 2).

Table 2. Intuitionistic fuzzy description of $\Omega$

| $\Omega \backslash A$ | $\ldots$ | $a$ | $\ldots$ |
| :---: | :---: | :---: | :---: |
| $\ldots$ |  |  |  |
| $w$ | $\ldots$ | $\left(\mu_{X(\omega)}(a), \nu_{X(\omega)}(a)\right)$ | $\ldots$ |
| $\ldots$ |  |  |  |

The rank row $\omega$ of Table 2 represents the descriptor of $\omega$, and the rank column $a$ represents the signification of the modality $a$. At the crossroads of the row $\omega$ and column $a$, we have the pair of numbers $\langle k(\omega, a), l(\omega, a)\rangle=\left\langle\mu_{X(\omega)}(a), \nu_{X(\omega)}(a)\right\rangle$ which represents the membership and nonmembership degrees of association between the individual $\omega$ and the modality $a$. A modality $a_{j}$ from $A$ is said to be observed (significant) if and only if

$$
\left(\bigvee_{\omega \in \Omega} \mu_{X(\omega)}\left(a_{j}\right), \bigwedge_{\omega \in \Omega} \nu_{X(\omega)}\left(a_{j}\right)\right) \neq(0,1) .
$$

This means that there is at least one individual in the population that is associated, even weakly, in the modality $a_{j}$.

A modality $a_{j}$ from $A$ is said to be clearly observed (or fully significant) if and only if

$$
\left(\bigvee_{\omega \in \Omega} \mu_{X(\omega)}\left(a_{j}\right), \bigwedge_{\omega \in \Omega} \nu_{X(\omega)}\left(a_{j}\right)\right)=(1,0) .
$$

This means that there is at least one individual in the population that is clearly associated, in the modality $a_{j}$.

In this study, unless otherwise specified, a modality will always be assumed to be observed.

### 5.2 Possibilistic correspondence

In this subsection, we consider two intuitionistic fuzzy characters $X$ and $Y$ which describe the population $\Omega$ with values in the respective sets of observations $A$ and $B . X$ and $Y$ are also called intuitionistic fuzzy variables. Let $(X, Y): \Omega \longrightarrow[0,1]^{A} \times[0,1]^{B}$ be the joint variable defined by

$$
(X, Y)(\omega)=(X(\omega), Y(\omega))
$$

The join signification is defined by

$$
(X, Y)^{-1}(a, b)=(X=a, Y=b)=(X=a) \cap(Y=b) .
$$

The aim of this Subsection is to measure the joint signification ( $X=a, Y=b$ ) of the pair of observations $(a, b) \in A \times B$.

Notice that when $X$ and $Y$ are crisp characters, the set $(X=a, Y=b)$ is the set of individuals of $\Omega$ that having both the modalities $a$ and $b$. To summarize the information contained in the pair $(X, Y)$, we then construct a contingency table where at the crossroads of the row $a$ and the column $b$, we have the number of individuals of $\Omega$ that having both the modalities $a$ and $b$, and denoted by $\operatorname{Card}(X=a, Y=b)$. This is to measure by the cardinal, the importance given to the pair $(a, b)$ by using quantitative evaluation of the data. However when dealing with vague and ambiguous information, we need a qualitative evaluation of the data. That is the reason why we use another measure like possibility measure to evaluate the set ( $X=a, Y=b$ ), just like Gwet [18] does in fuzzy case. Then this work extends to the intuitionistic fuzzy case that of Gwet [18] in the fuzzy case.

In the next of this paper, we assume that, $X$ and $Y$ are IF-mappings; that means $(X=a, Y=b)$ is an IFS on $\Omega$. We then obtain a table (Table 3) of possibilistic correspondence $(\pi(a, b))_{a \in A, b \in B}$ with $\pi(a, b)=\Pi(X=a, Y=b)$, where $\Pi$ is a possibility measure on $\Omega$.

Table 3. Possibilistic correspondence

| $A \backslash B$ | $\ldots$ | $b$ | $\ldots$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $\ldots$ |  |  |  |  |
| $a$ | $\ldots$ | $\pi(a, b)$ | $\ldots$ | $\Pi(X=a)$ |
| $\ldots$ |  | $\ldots$ |  |  |
| $\ldots$ |  | $\Pi(Y=b)$ | $\ldots$ |  |

The aim of the following section is to propose a possibility measure on $\Omega$ which will help to construct a possibilistic correspondance.

### 5.3 Possibility measure

The possibility theory $[12,30]$ is presented as an alternative framework to represent uncertain information. It is closely linked to the theory of fuzzy sets and intuitionistic fuzzy sets. Before coming back to the population, $\Omega$ will be the finite referential set.

The following is the definition of the possibility measure.
Definition 5.2. The mapping $\Pi: \operatorname{IFS}(\Omega) \longrightarrow[0,1]$ is a possibility measure if the following properties hold: (i) $\Pi(\varnothing)=0$, (ii) $\Pi(A \cup B)=\Pi(A) \vee \Pi(B) \forall A, B \in \operatorname{IFS}(\Omega)$ Moreover, if $\Pi(\Omega)=1$, then the possibility measure is said to be normal.

This definition is close with the definition of the possibility measure on fuzzy set. The number $\Pi(A)$ quantifies to what extent the event $A \subset \Omega$ is possible. Note that the possibility measures satisfy the relation: $\Pi(A) \vee \Pi\left(A^{c}\right)=1$. Where $A^{c}$ is the complement of $A$ in $\Omega$.

In reality, it is improbable, even impossible for an alternative $\omega$ to be in the set $A$ and in the set $B$ with the memberships and the nonmemberships degrees which verify $\mu_{A}(\omega)<\mu_{B}(\omega)$ and $\nu_{A}(\omega)<\nu_{B}(\omega)$. A such alternative will be assumed to be non-existent in this work.

We consider the following mapping:

$$
\begin{equation*}
\Pi(A)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{A}(\omega)-\nu_{A}(\omega)\right) \tag{23}
\end{equation*}
$$

In view of its construction and the expression of $\operatorname{Count}(A)$, this mapping is in our opinion based on IF-cardinality of A.

The second main result of this paper gives an example of possibility measure for IFSs.
Theorem 2. The mapping $\Pi$ defined by Eq. (23) is a possibility measure.
Proof. $\Pi(\varnothing)=0$ and $\Pi(\Omega)=1$ are obvious.
Let $A, B \in \operatorname{IFS}(\Omega)$. We prove that $\Pi(A \cup B)=\Pi(A) \vee \Pi(B)$. Let $\omega_{0}, \omega_{1}$ and $\omega_{2}$ such that: $\Pi(A \cup B)=\frac{1}{2}\left(1+\mu_{A \cup B}\left(\omega_{0}\right)-\nu_{A \cup B}\left(\omega_{0}\right)\right), \Pi(A)=\frac{1}{2}\left(1+\mu_{A}\left(\omega_{1}\right)-\nu_{A}\left(\omega_{1}\right)\right)$ and $\Pi(B)=\frac{1}{2}\left(1+\mu_{B}\left(\omega_{2}\right)-\nu_{B}\left(\omega_{2}\right)\right)$.

If $\mu_{A}\left(\omega_{0}\right) \leq \mu_{B}\left(\omega_{0}\right)$, then $\nu_{A}\left(\omega_{0}\right) \geq \nu_{B}\left(\omega_{0}\right)$, and

$$
\begin{aligned}
\Pi(A \cup B) & =\frac{1}{2}\left(1+\mu_{B}\left(\omega_{0}\right)-\nu_{B}\left(\omega_{0}\right)\right) \leq \Pi(B) \\
& \leq \frac{1}{2}\left(1+\mu_{A \cup B}\left(\omega_{2}\right)-\nu_{A \cup B}\left(\omega_{2}\right)\right) \leq \Pi(A \cup B)
\end{aligned}
$$

If $\mu_{B}\left(\omega_{0}\right) \leq \mu_{A}\left(\omega_{0}\right)$, then $\nu_{B}\left(\omega_{0}\right) \geq \nu_{A}\left(\omega_{0}\right)$, and $\Pi(A \cup B)=\Pi(A)$ by using the same process. Thus $\Pi(A \cup B)=\Pi(A) \vee \Pi(B)$.

This result shows also that $\Pi(A)(\omega) \vee \Pi\left(A^{c}\right)(\omega)=1$ for all $\omega \in \Omega$ since $A \cup A^{c}=\Omega$.
Remark 1. When $A$ is a fuzzy set, the possibility measure $\Pi$ become

$$
\Pi(A)=\bigvee_{\omega \in \Omega} \mu_{A}(\omega)
$$

which is the well known possibility measure in fuzzy set ( [18]).
We deduce the possibility measure of $A \cap_{\mathcal{T}} B$ defined in [24] by

$$
A \cap_{\mathcal{T}} B=\left\{\left\langle\omega, \top\left(\mu_{A}(\omega), \mu_{B}(\omega)\right), S\left(\nu_{A}(\omega), \nu_{B}(\omega)\right)\right\rangle, \omega \in \Omega\right\}
$$

as follows:

$$
\Pi\left(A \cap_{\mathcal{T}} B\right)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mathrm{T}\left(\mu_{A}(\omega), \mu_{B}(\omega)\right)-S\left(\nu_{A}(\omega), \nu_{B}(\omega)\right)\right)
$$

and when $\mathcal{T}=(\min , \max )$, we have:

$$
\Pi\left(A \cap_{\mathcal{T}_{M}} B\right)=\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{A}(\omega) \wedge \mu_{B}(\omega)-\nu_{A}(\omega) \vee \nu_{B}(\omega)\right)
$$

We consider now the population $\Omega$ and the characters $X$ and $Y$ of the sets of the observations $A$ and $B$ respectively. We define the proximity in the sense of Tchebychev between the modalities $a$ and $a^{\prime}$ of $A$ as a possibility measure of the IFSs $(X, X)^{-1}\left(a, a^{\prime}\right)=\left(X=a, X=a^{\prime}\right)$ as follow:

$$
\begin{aligned}
\pi\left(a, a^{\prime}\right) & =\Pi\left(X=a, X=a^{\prime}\right)=\Pi\left(X=a \cap_{\mathcal{T}_{M}} X=a^{\prime}\right) \\
& =\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{X=a}(\omega) \wedge \mu_{X=a^{\prime}}(\omega)-\nu_{X=a}(\omega) \vee \nu_{X=a^{\prime}}(\omega)\right)
\end{aligned}
$$

Then the possibilistic correspondence between the modalities $a$ of $A$ and $b$ of $B$ is define as follow:

$$
\begin{aligned}
\pi(a, b) & =\Pi\left(X=a \cap_{\mathcal{T}_{M}} Y=b\right) \\
& =\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{X=a}(\omega) \wedge \mu_{Y=b}(\omega)-\nu_{X=a}(\omega) \vee \nu_{Y=b}(\omega)\right) \\
& =\frac{1}{2} \bigvee_{\omega \in \Omega}\left(1+\mu_{X=a}(\omega) \wedge \mu_{Y(\omega)}(b)-\nu_{X=a}(\omega) \vee \nu_{Y(\omega)}(b)\right)
\end{aligned}
$$

Remark 2. In practice, it can happen that we directly obtain a proximity between $a$ and $a^{\prime}$ without going through a population $\Omega$. In this case for $a, a^{\prime} \in A$,

$$
\begin{aligned}
\pi\left(a, a^{\prime}\right) & =\Pi\left(Y(X=a), Y\left(X=a^{\prime}\right)\right) \\
& =\frac{1}{2} \bigvee_{b \in B}\left(1+\mu_{Y(X=a)}(b) \wedge \mu_{Y\left(X=a^{\prime}\right)}(b)-\nu_{Y(X=a)}(b) \vee \nu_{Y\left(X=a^{\prime}\right)}(b)\right),
\end{aligned}
$$

and the possibilistic measure:

$$
\Pi(X=a)=\Pi(Y(X=a))=\frac{1}{2} \bigvee_{b \in B}\left(1+\mu_{Y(X=a)}(b)-\nu_{Y(X=a)}(b)\right)
$$

## 6 Conclusion

In this paper, we propose some classes of distance measures based on the cardinality of the symmetric difference, which extend to the intuitionistic fuzzy set the Tchebychev distance proposed by Gwet [18] in the fuzzy case. We then prove that if the chosen parameter is the IF-t-norm of Łukasiewicz or Product, these classes become metrics. To propose a proximity measure between two modalities of intuitionistic fuzzy characters in a two-dimensional intuitionistic fuzzy statistical description, we introduce an intuitionistic fuzzy mapping. We show that this mapping preserves the properties of fuzzy mappings. We finally propose a possibility measure based on IF-cardinality and we use it to define a proximity measure.

In this study, we have proposed possibility measure for IFS. One possible direction of interest is to investigate adaptability of our formulations to the case of intervals values intuitionistic fuzzy sets. These directions are left for future research.

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