Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2023, Volume 29, Number 2, Pages 99–113
DOI: 10.7546/nifs.2023.29.2.99-113

Intuitionistic L-fuzzy classical prime and intuitionistic L-fuzzy 2-absorbing submodules

P. K. Sharma

Post-Graduate Department of Mathematics, D.A.V. College
Jalandhar, Punjab, India
e-mail: pksharma@davjalandhar.com

Received: 26 April 2023 **Revised:** 18 May 2023
Accepted: 20 May 2023 **Online First:** 3 July 2023

Abstract: Let L be a complete lattice. We introduce and characterise intuitionistic L-fuzzy classical prime submodule and intuitionistic L-fuzzy 2-absorbing submodules of a unitary module M over a commutative ring R with identity. We compare both of these submodules with intuitionistic L-fuzzy prime submodules. It is proven that in the case of the multiplication module M, the two notions of intuitionistic L-fuzzy classical prime submodules and intuitionistic L-fuzzy prime submodules coincide. Many other related results concerning these notions are obtained.
Keywords: Intuitionistic L-fuzzy submodule, Intuitionistic L-fuzzy classical prime submodule, Intuitionistic L-fuzzy 2-absorbing submodule, Intuitionistic L-fuzzy prime (2-absorbing) ideal.
2020 Mathematics Subject Classification: 03F55, 08A72, 13C05, 13C13, 13C99.

1 Introduction

Throughout this paper R is a commutative ring with a nonzero identity, and M is a unitary R-module and L stand for a complete lattice with least element 0 and greatest element 1. For every submodule N of M, we denote the annihilator of M/N by $(N :_R M)$, i.e., $(N :_R M) = \{r \in R | rM \subseteq N\}$. In his paper [5], Badawi introduced the notion of 2-absorbing ideals of a commutative ring, where a proper ideal I of R is said to be 2-absorbing provided that whenever

$a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. In [10], this concept was generalised to weakly 2-absorbing submodules of M by Darani and Soheilnia. Let N be a proper submodule of M. Then, N is said to be a 2-absorbing submodule of M provided that whenever $a, b \in R$ and $m \in M$, with $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$.

Behboodi and Koohy introduced the notion of weakly prime submodules in [7], where a proper submodule N of M is said to be weakly prime if whenever $a, b \in R$, and $m \in M$ with $abm \in N$, then either $am \in N$ or $bm \in N$. Atani and Farzalipour gave a different definition for weakly prime submodules in [3]. According to their definition, a proper submodule N of M is called weakly prime provided that for every $a \in R$ and $m \in M$ with $0 \neq am \in N$, then either $m \in N$ or $a \in (N :_R M)$. To avoid ambiguity, Behboodi et al. renamed weakly prime submodules to classical prime submodules [8].

We recall that a proper submodule N of M is called a prime submodule of M if, for every $a \in R$ and $m \in M$, $am \in N$ implies that either $m \in N$ or $a \in (N :_R M)$. The notion of prime submodules was first introduced and studied in [11] and recently it has received a good deal of attention from several authors [7, 14]. Clearly every prime submodule is classical prime submodule and every classical prime submodule is 2-absorbing submodule. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$(see [1, 16]).

Atanassov and Stoeva [2] generalised the notion of L-fuzzy subsets given by Goguen [13] to an intuitionistic L-fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Wang and He [24] and Deschrijver and Kerre [12] studied the relationship between intuitionistic fuzzy sets and L-fuzzy sets and some extensions of fuzzy set theory. Palaniappan and others [18] studied intuitionistic L-fuzzy subgroups. Meena and Thomas [17] discussed the notion of intuitionistic L-fuzzy subrings. Sharma et al. [15,21,22] discussed intuitionistic L-fuzzy submodules, intuitionistic L-fuzzy prime and primary submodules of a module. The author in [23] studied, the set of all intuitionistic L-fuzzy prime submodules of M, and topologized it in a similar way to that of $Spec(M)$.

In this paper, we introduce and examine two generalised notions of prime submodules, namely classical prime submodules and 2-absorbing submodules, in the intuitionistic fuzzy environment over the lattice range. We will investigate the relationship between these notions. Apart from these, some algebraic properties of these notions will be established, which will be of great help for the further study of these algebraic and topological structures.

2 Preliminaries

Throughout the paper, R is a commutative ring, M is a unitary R-module, and L stands for a complete lattice with least element 0 and greatest element 1. θ denotes the zero element of M.

Definition 2.1. [2,15] Let $(L, \leq)$ be a complete lattice with an evaluative order reversing operation $N : L \to L$. Let X is a non-empty set. An intuitionistic L-fuzzy set A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to L$ and $\nu_A : X \to L$ define respectively the degree of membership and the degree of non membership for every $x \in X$

satisfying $\mu_A(x) \le N(\nu_A(x))$. When $\mu_A(x) = N(\nu_A(x))$, for all $x \in X$, then A is called L-fuzzy set. A complete order reversing involution is a map $N : L \to L$ such that

(i) $N(0) = 1$ and $N(1) = 0$;

(ii) If $\alpha \le \beta$, then $N(\beta) \le N(\alpha)$;

(iii) $N(N(\alpha)) = \alpha$;

(iv) $N(\vee_{i=1}^n \alpha_i) = \wedge_{i=1}^n N(\alpha_i)$ and $N(\wedge_{i=1}^n \alpha_i) = \vee_{i=1}^n N(\alpha_i)$.

We also denote an intuitionistic L-fuzzy set by simply $ILFS$ and the set of all $ILFSs$ on X by $ILFS(X)$. For $A, B \in ILFS(X)$ we say $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \ne B$. Let $A \in ILFS(X)$ and $p, q \in L$. Then the set $A_{(p,q)} = \{x \in X : \mu_A(x) \ge p \text{ and } \nu_A(x) \le q\}$ is called the (p, q)-cut subset of X with respect to A. In particular the set $A_{(1,0)} = \{x \in X : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$ is denoted by A_*. By an intuitionistic L-fuzzy point $(ILFP)$ $x_{(p,q)}$ of $X, x \in X$ and $p, q \in L\backslash\{0\}$ such that $p \vee q \le 1$, we mean $x_{(p,q)} \in ILFS(X)$ defined by

$$x_{(p,q)}(y) = \begin{cases} (p, q), & \text{if } y = x \\ (0, 1), & \text{if otherwise.} \end{cases}$$

If $x_{(p,q)}$ is an intuitionistic L-fuzzy point of X and $x_{(p,q)} \subseteq A$, we write $x_{(p,q)} \in A$. Thus $x_{(p,q)} \in A \Leftrightarrow \mu_A(x) \ge p, \nu_A(x) \le q$. Let A be an ILFS of X and $Y \subseteq X$. Then the intuitionistic L-fuzzy characteristic function χ_Y w.r.t. Y is defined as:

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases} ; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.2. [17] Let $A \in ILFS(R)$. Then A is called an intuitionistic L-fuzzy ideal $(ILFI)$ of R if for all $x, y \in R$, the followings are satisfied:

(i) $\mu_A(x - y) \ge \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(xy) \ge \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x - y) \le \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(xy) \le \nu_A(x) \wedge \nu_A(y)$.

Definition 2.3. [15] Let $A \in ILFS(M)$. Then A is called an intuitionistic L-fuzzy submodule $(ILFSM)$ of M if for all $x, y \in M, r \in R$, the following are satisfied:

(i) $\mu_A(x - y) \ge \mu_A(x) \wedge \mu_A(y)$;

(ii) $\nu_A(x - y) \le \nu_A(x) \vee \nu_A(y)$;

(iii) $\mu_A(rx) \ge \mu_A(x)$;

(iv) $\nu_A(rx) \leq \nu_A(x)$;

(v) $\mu_A(\theta) = 1$;

(vi) $\nu_A(\theta) = 0$.

Let $ILFSM(M)$ denote the set of all intuitionistic L-fuzzy submodules of M and $ILFI(R)$ denote the set of all intuitionistic L-fuzzy ideals of R. We note that when $R = M$, then $A \in ILFSM(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in ILFI(R)$.

Definition 2.4. [21] Let $C \in ILFS(R)$ and $B \in ILFS(M)$. Define the composition $C \circ B$, and product CB respectively as follows: For all $w \in M$,

$$\mu_{C\circ B}(w) = \begin{cases} \sup[\mu_C(r) \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases},$$

$$\nu_{C\circ B}(w) = \begin{cases} \inf[\nu_C(r) \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\mu_{CB}(w) = \begin{cases} \sup[\inf_{i=1}^{n}\{\mu_C(r_i) \wedge \mu_B(x_i)\}] & \text{if } w = \Sigma_{i=1}^{n} r_i x_i, r_i \in R, x_i \in M, n \in N \\ 0, & \text{if } w \text{ is not expressible as } w = \Sigma_{i=1}^{n} r_i x_i \end{cases},$$

$$\nu_{CB}(w) = \begin{cases} \inf[Sup_{i=1}^{n}\{\nu_C(r_i) \vee \nu_B(x_i)\}] & \text{if } w = \Sigma_{i=1}^{n} r_i x_i, r_i \in R, x_i \in M, n \in N \\ 1, & \text{if } w \text{ is not expressible as } w = \Sigma_{i=1}^{n} r_i x_i \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1, respectively. Clearly, $C \circ B \subseteq CB$. Further, for any $a_{(p,q)} \in ILFP(R)$, $x_{(u,v)} \in ILFP(M)$, we have $\langle a_{(p,q)}\rangle\langle x_{(u,v)}\rangle = \langle a_{(p,q)} x_{(u,v)}\rangle$.

Definition 2.5. [19] For $A, B \in ILFS(M)$ and $C \in ILFS(R)$, define the residual quotient $(A\!:\!B)$ and $(A\!:\!C)$ as follows:

$(A\!:\!B) = \bigcup\{D : D \in ILFS(R)$ such that $D \cdot B \subseteq A\}$ and

$(A\!:\!C) = \bigcup\{E : E \in ILFS(M)$ such that $C \cdot E \subseteq A\}$.

Clearly, $(A\!:\!B) \in ILFS(R)$ and $(A\!:\!C) \in ILFS(M)$. Further, if $A, B \in ILFM(M)$ and $C \in ILFI(R)$, then $(A\!:\!B) \in ILFI(R)$ and $(A\!:\!C) \in ILFM(M)$.

Theorem 2.6. [19] *For $A, B \in ILFS(M)$ and $C \in ILFS(R)$. Then we have*

(i) $(A\!:\!B) \cdot B \subseteq A$ *;*

(ii) $C \cdot (A\!:\!C) \subseteq A$ *;*

(iii) $C \cdot B \subseteq A \Leftrightarrow C \subseteq (A\!:\!B) \Leftrightarrow B \subseteq (A\!:\!C)$.

Definition 2.7. [6, 20] For a non-constant $A \in ILFI(R)$, A is termed as intuitionistic L-fuzzy prime (respectively, primary) ideal of R if for any $a_{(p,q)}, b_{(t,s)} \in ILFP(R)$ such that $a_{(p,q)}b_{(t,s)} \in A$ inferred that $a_{(p,q)} \in A$ or $b_{(t,s)} \in A$ (or respectively $a_{(p,q)} \in A$ or $b^n_{(t,s)} \in A$, for some $n \in \mathbb{N}$).

Definition 2.8. [25] For a non-constant $A \in ILFI(R)$, A is termed as intuitionistic L-fuzzy 2-absorbing ideal of R if for any $a_{(p,q)}, b_{(t,s)}, c_{(u,v)} \in ILFP(R)$ such that $a_{(p,q)}b_{(t,s)}c_{(u,v)} \in A$ inferred that $a_{(p,q)}b_{(t,s)} \in A$ or $b_{(t,s)}c_{(u,v)} \in A$ or $a_{(p,q)}c_{(u,v)} \in A$.

Definition 2.9. [21, 22] For a non-constant $A \in ILFI(M)$, A is termed as intuitionistic L-fuzzy prime (respectively, primary) submodule of M if for any $r_{(t,s)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$ such that $r_{(t,s)}x_{(p,q)} \in A$ inferred that $x_{(p,q)} \in A$ or $r_{(t,s)} \in (A \colon \chi_M)$ (or respectively, $x_{(p,q)} \in A$ or $r^n_{(t,s)} \in (A \colon \chi_M)$, for some $n \in \mathbb{N}$).

Theorem 2.10. [21] *(a) Let N be a prime submodule of M and α a prime element in L. If A is an ILFS of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \textit{if } x \in N \\ \alpha, & \textit{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \textit{if } y \in N \\ \alpha^{'}, & \textit{otherwise}. \end{cases}$$

for all $x \in M$, where $\alpha^{'}$ is complement of α in L. Then A is an intuitionistic L-fuzzy prime submodule (ILFPSM) of M.
(b)Conversely, any intuitionistic L-fuzzy prime submodule can be obtained as in (a).

Corollary 2.11. [21] *If A is an intuitionistic L-fuzzy prime submodule of M, then $(A : \chi_M)$ is an intuitionistic L-fuzzy prime ideal of R.*

Definition 2.12. [9,21] An element $\alpha \in L, 1 \neq \alpha$, is called a prime element in L if for all $p; s \in L$ if $p \wedge s \leq \alpha$ implies $p \leq \alpha$ or $s \leq \alpha$.

Definition 2.13. [19] An R-module M is called an intuitionistic L-fuzzy multiplication module if and only if for each intuitionistic L-fuzzy submodule A of M there exists a intuitionistic L-fuzzy ideal C of R with $C(0_R) = (1, 0)$ such that $A = C\chi_M$. One can easily show that if $A = C\chi_M$ then $A = (A : \chi_M)\chi_M$.

3 Intuitionistic L-fuzzy classical prime submodules

In this section, we introduce the notion of intuitionistic L-fuzzy classical prime submodules of a R-module M, which is a generalisation of intuitionistic L-fuzzy prime submodules. We recall that, a proper submodule N of an R-module M is called classical prime, if for any elements $a, b \in R$ and $x \in M$, the condition $abx \in N$ implies that $ax \in N$ or $bx \in N$. Then we provide some basic results on intuitionistic L-fuzzy classical prime submodules. Throughout this paper, we assume that M is a module over a commutative ring R with unity.

Definition 3.1. Let A be an ILFSM of an R-module M. Then A is called an intuitionistic L-fuzzy classical prime submodule (ILFCPSM) of M if for any $a_{(p,q)}, b_{(t,s)} \in ILFP(R)$ and $x_{(u,v)} \in ILFP(M)$ $(a, b \in R, x \in M, p, q, r, s, u, v \in L)$, we have $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$ implies that either $a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$.

Theorem 3.2. *Let A be an intuitionistic L-fuzzy classical prime submodule of an R-module M. Then for every $\alpha, \beta \in L$ with $A_{(\alpha,\beta)} \neq M$, $A_{(\alpha,\beta)}$ is a classical prime submodule of M.*

Proof. Let A be an intuitionistic L-fuzzy classical prime submodule of M and suppose that $a, b \in R, x \in M$ are such that $abx \in A_{(\alpha,\beta)}$. Then $\mu_A(abx) \geq \alpha$, $\nu_A(abx) \leq \beta$ implies $\mu_{(abx)_{(\alpha,\beta)}}(abx) = \alpha \leq \mu_A(abx)$ and $\nu_{(abx)_{(\alpha,\beta)}}(abx) = \beta \geq \nu_A(abx)$ and so we have $(abx)_{(\alpha,\beta)} \subseteq A$, i.e., $a_{(\alpha,\beta)}b_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$. Since A is an ILFCPSM of M, we have $(ax)_{(\alpha,\beta)} = a_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$ or $(bx)_{(\alpha,\beta)} = b_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$. Thus $ax \in A_{(\alpha,\beta)}$ or $bx \in A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a classical prime submodule of M. $\square$

Corollary 3.3. *If A is an intuitionistic L-fuzzy classical prime submodule of M, then*

$$A_* = \{x \in M : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$$

is a classical prime submodule of M.

Proof. Since A is a non-constant ILFCPSM of M, then $A_* \neq M$. Now the result follows from Theorem (3.2). $\square$

Theorem 3.4. *Let N be a classical prime submodule of M and α be a prime element of L and α' be the complement of α in L. If $A \in ILFS(M)$ defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha', & \text{otherwise} \end{cases}.$$

for all $x \in M$, then A is an intuitionistic L-fuzzy classical prime submodule of M.

Proof. Since N is a classical prime submodule of M and $N \neq M$, therefore A is a non-constant ILFSM of M. Suppose that $a_{(p,q)}, b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$, i.e., $(abx)_{(p\wedge s\wedge u, q\vee t\vee v)} \subseteq A$. Then $p\wedge s\wedge u = \mu_{(abx)_{(p\wedge s\wedge u,q\vee t\vee v)}}(abx) \leq \mu_A(abx)$ and $q \vee t \vee v = \nu_{(abx)_{(p\wedge s\wedge u,q\vee t\vee v)}}(abx) \geq \nu_A(abx)$. If $a_{(p,q)}x_{(u,v)} \nsubseteq A$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A$, then $p \wedge u = \mu_{(ax)_{(p\wedge u,q\vee v)}}(ax) \nleq \mu_A(ax)$, $q \vee v = \nu_{(ax)_{(p\wedge u,q\vee v)}}(ax) \ngeq \nu_A(ax)$, we have $\mu_A(ax) = \alpha, \nu_A(ax) = \alpha'$ and so $ax \notin N$. Similarly $s \vee u = \mu_{(bx)_{(s\wedge u,t\vee v)}}(bx) \nleq \mu_A(bx)$, $t \vee v = \nu_{(ax)_{(s\wedge u,t\vee v)}}(bx) \ngeq \nu_A(bx)$, we have $\mu_A(bx) = \alpha, \nu_A(bx) = \alpha'$ and so $bx \notin N$. Since α is assumed to be a prime element of L, so $p \wedge s \wedge u \nleq \alpha, q \vee t \vee v \ngeq \alpha'$. Since N is a classical prime submodule of M, we have $abx \notin N$. Consequently $\mu_A(abx) = \alpha, \nu_A(abx) = \alpha'$. Therefore, $p \wedge s \wedge u \leq \alpha, q \vee t \vee v \geq \alpha'$, which is a contradiction. Hence A is an intuitionistic L-fuzzy classical prime submodule of M. $\square$

Remark 3.5. Every intuitionistic L-fuzzy prime submodule is an intuitionistic L-fuzzy classical prime submodule. But the converse, in general is not true, see the following example.

Example 3.6. Let R be an integral domain and P a non-zero prime ideal of R. Then for the free R-module $M = R \oplus R$, the submodule $\{0\} \oplus P$ is a classical prime submodule, which is not a prime submodule. Let α be any prime element in L and α' be its complement in L.

Define the ILFS A of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \{0\} \oplus P \\ \alpha, & \text{otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in \{0\} \oplus P \\ \alpha', & \text{otherwise} . \end{cases}$$

for all $x \in M$, then by Theorem (3.4), A is an intuitionistic L-fuzzy classical prime submodule of M which is not an intuitionistic L-fuzzy prime submodule.

Lemma 3.7. (1) *Let A be an ILFSM of M. Then A is an ILFCPSM of M if and only if for each $x_{(u,v)} \in ILFP(M)$ such that $x_{(u,v)} \notin A$, $(A : x_{(u,v)})$ is an ILFPI of R.*

(2) *Let $\{A_i : i \in \Lambda\}$ be a family of ILFCPSMs of M such that for each $x_{(u,v)} \notin \bigcap_{i\in\Lambda} A_i$, $\{(A_i : x_{(u,v)}) : i \in \Lambda\}$ is a chain of ILFIs of R. Then $\bigcap_{i\in\Lambda} A_i$ is an ILFCPSM of M.*

(3) *If $\{A_i : i \in \Lambda\}$ be a family of ILFPSMs of M such that $\{(A_i : \chi_M) : i \in \Lambda\}$ is a chain of ILFIs of R. Then $\bigcap_{i\in\Lambda} A_i$ is an ILFCPSM of M.*

Proof. (1) Firstly, assume that A be an ILFCPSM of M and $x_{(u,v)} \in ILFP(M)$ such that $x_{(u,v)} \notin A$. To show that $(A : x_{(u,v)})$ is an ILFPI of R. Suppose that $C, D \in ILFI(R)$ such that $CD \subseteq (A : x_{(u,v)})$ and $C \nsubseteq (A : x_{(u,v)})$, i.e., $CDx_{(u,v)} \subseteq A$ and $Cx_{(u,v)} \nsubseteq A$. As A be an ILFCPSM of M. So, we have $Dx_{(u,v)} \subseteq A$, i.e., $D \subseteq (A : x_{(u,v)})$. Hence $(A : x_{(u,v)})$ is an ILFPI of R.

Conversely, assume that $(A : x_{(u,v)})$ be an ILFPI of R and $x_{(u,v)} \notin A$. Suppose that $a_{(p,q)}, b_{(t,s)} \in ILFP(R)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$ imply $a_{(p,q)}b_{(t,s)} \in (A : x_{(u,v)})$. As $(A : x_{(u,v)})$ is an ILFPI of R imply that $a_{(p,q)} \in (A : x_{(u,v)})$ or $b_{(t,s)} \in (A : x_{(u,v)})$, i.e., $a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$. This implies that A is an ILFCPSM of M.

(2) Assume that $a_{(p,q)}, b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq \bigcap_{i\in\Lambda} A_i \subseteq A_i$. This implies $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_i, \forall i \in \Lambda$. As each A_i is an ILFCPSM of M, so $a_{(p,q)}x_{(u,v)} \subseteq A_i$ or $b_{(t,s)}x_{(u,v)} \subseteq A_i$ for each $i \in \Lambda$. Suppose that $a_{(p,q)}x_{(u,v)} \nsubseteq \bigcap_{i\in\Lambda} A_i$ and $b_{(t,s)}x_{(u,v)} \nsubseteq \bigcap_{i\in\Lambda} A_i$. Hence $a_{(p,q)}x_{(u,v)} \nsubseteq A_k$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_l$ for some $k, l \in \Lambda$. In this case $a_{(p,q)} \notin (A_k : x_{(u,v)})$ and $b_{(t,s)} \notin (A_l : x_{(u,v)})$. Since $x_{(u,v)} \notin \bigcap_{i\in\Lambda} A_i$, we can assume that $(A_k : x_{(u,v)}) \subseteq (A_l : x_{(u,v)})$. Therefore, $a_{(p,q)}x_{(u,v)} \nsubseteq A_k$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_k$. while $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_k$. This contradict the assumption that A_k is an ILFCPSM of M. Hence $\bigcap_{i\in\Lambda} A_i$ is an ILFCPSM of M.

(3) Suppose for some ILFPs $a_{(p,q)}, b_{(t,s)}$ of R and $x_{(u,v)}$ of M we have $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq \bigcap_{i\in\Lambda} A_i$. So $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_i, \forall i \in \Lambda$. Assume that $a_{(p,q)}x_{(u,v)} \nsubseteq \bigcap_{i\in\Lambda} A_i$ and $b_{(t,s)}x_{(u,v)} \nsubseteq \bigcap_{i\in\Lambda} A_i$, then $a_{(p,q)}x_{(u,v)} \nsubseteq A_k$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_l$ for some $k, l \in \Lambda$. In this case $a_{(p,q)} \notin (A_k : \chi_M)$ and $b_{(t,s)} \notin (A_l : \chi_M)$. We can assume that $(A_k : \chi_M) \subseteq (A_l : \chi_M)$. By Corollary (2.11), $(A_k : \chi_M)$ is an ILFPI of R. Therefore, $a_{(p,q)}b_{(t,s)} \notin (A_k : \chi_M)$. Also, $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_k$. As A_k is an ILFPSM of M. It follows that $a_{(p,q)}x_{(u,v)} \subseteq A_k$ or $b_{(t,s)}x_{(u,v)} \subseteq A_k$ which is a contradiction. Hence $\bigcap_{i\in\Lambda} A_i$ is an ILFCPSM of M. □

Theorem 3.8. *An intuitionistic L-fuzzy submodule A of an R-module M, where $A \neq \chi_M$, is an intuitionistic L-fuzzy classical prime submodule if and only if $(A\colon\chi_M)$ is an intuitionistic L-fuzzy prime ideal of R.*

Proof. Let $A(\neq \chi_M)$ be an intuitionistic L-fuzzy classical prime submodule of M. Let $a_{(p,q)}$, $b_{(t,s)} \in ILFP(R)$ such that $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$. This imply $a_{(p,q)}b_{(t,s)}\chi_M \subseteq A$. As A is ILFCPSM, therefore we get $a_{(p,q)}\chi_M \subseteq A$ or $b_{(t,s)}\chi_M \subseteq A$. Which further gives $a_{(p,q)} \in (A\colon\chi_M)$ or $b_{(t,s)} \in (A\colon\chi_M)$. Hence A is an intuitionistic L-fuzzy prime ideal of R.

Conversely, for every ILFSM $A(\neq \chi_M)$ of an R-module M we have that $(A : \chi_M)$ is an intuitionistic L-fuzzy prime ideal of R. Suppose that $a_{(p,q)}b_{(t,s)} \in ILFP(R)$ and $x_{(u,v)} \in ILFP(M)$ we have $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$. As $(A : \chi_M)$ is an ILFPI of R, so either $a_{(p,q)} \subseteq (A\colon\chi_M)$ or $b_{(t,s)} \subseteq (A\colon\chi_M)$. Also because $x_{(u,v)} \subseteq \chi_M$ implies $(A\colon\chi_M) \subseteq (A\colon x_{(u,v)})$. Therefore, we have $a_{(p,q)} \in (A : x_{(u,v)})$ or $b_{(t,s)} \in (A : x_{(u,v)})$. Which further imply that $a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$. Hence A is an ILFCPSM of M. □

In the sequel, we will find a relationship between intuitionistic L-fuzzy prime submodules and intuitionistic L-fuzzy classical prime submodules.

Theorem 3.9. *Let M be an R-module and A a non-constant ILFSM of M. Then A is an intuitionistic L-fuzzy prime submodule of M if and only if A is intuitionistic L-fuzzy primary submodule and intuitionistic L-fuzzy classical prime submodule.*

Proof. If A is an intuitionistic L-fuzzy prime submodule of M, then the result is obvious. Now suppose that A be an intuitionistic L-fuzzy primary submodule as well as intuitionistic L-fuzzy classical prime submodule of M. Consider $a_{(p,q)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}x_{(u,v)} \subseteq A$ and $x_{(u,v)} \notin A$. As A be an intuitionistic L-fuzzy primary submodule, so there exists some positive integer n such that for each $y_{(u_1,v_1)} \in ILFP(M \setminus A_*)$ (where u_1, v_1 are chosen so that $y_{(u_1,v_1)} \notin A$) we have $a^n_{(p,q)} \in (A : y_{(u_1,v_1)})$. Since A is an ILFCSM of M, then by Lemma (3.7)(1) $(A : y_{(u_1,v_1)})$ is an ILFPI of R, then $a_{(p,q)} \in (A : y_{(u_1,v_1)})$. Hence for each $y_{(u_1,v_1)} \in ILFP(M)$, we have $a_{(p,q)}y_{(u_1,v_1)} \in A$, i.e., $a_{(p,q)}\chi_M \in A$ and so $a_{(p,q)} \in (A\colon\chi_M)$. This implies that A is an ILFPSM of M. □

Proposition 3.10. *Let M be a multiplication R-module and let $A \in ILFSM(M)$, where $A \neq \chi_M$. If $(A : \chi_M)$ is an intuitionistic L-fuzzy prime ideal of R, then A is an intuitionistic L-fuzzy classical submodule of M.*

Proof. Let $(A\colon\chi_M)$ be an ILFPI of R. Consider $a_{(p,q)}, b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$. This imply $\langle a_{(p,q)}\rangle\langle b_{(t,s)}\rangle\langle x_{(u,v)}\rangle \subseteq A$. Now $\langle x_{(u,v)}\rangle$ is an ILFSM of M so we can write $\langle x_{(u,v)}\rangle = C\chi_M$, for some ILFI C of R. Therefore, $\langle a_{(p,q)}\rangle\langle b_{(t,s)}\rangle C\chi_M \subseteq A$, i.e., $\langle a_{(p,q)}\rangle(\langle b_{(t,s)}\rangle C) \subseteq (A\colon\chi_M)$. As $(A\colon\chi_M)$ is an ILFPI of R, we have $\langle a_{(p,q)}\rangle \subseteq (A\colon\chi_M)$ or $\langle b_{(t,s)}\rangle C \subseteq (A\colon\chi_M)$.
$\Rightarrow \langle a_{(p,q)}\rangle\langle x_{(u,v)}\rangle \subseteq (A\colon\chi_M)\chi_M = A$ or $\langle b_{(t,s)}\rangle\langle x_{(u,v)}\rangle = \langle b_{(t,s)}\rangle C\chi_M \subseteq (A\colon\chi_M)\chi_M = A$
$\Rightarrow a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$. Hence A is an ILFCPSM of M. □

Remark 3.11. From Theorem (3.10) and Proposition (3.8) we notice that in the case of multiplication R-module M, every intuitionistic L-fuzzy classical submodule is an intuitionistic L-fuzzy prime submodule. Thus, in the case of the multiplication module M, the two notions of intuitionistic L-fuzzy classical prime submodules and intuitionistic L-fuzzy prime submodules coincide.

Theorem 3.12. *Let $f : M \to M^{'}$ be a homomorphism of R-modules.*

1. *Suppose that f is a monomorphism. If B is an ILFCPSM of $M^{'}$ with $f^{-1}(B) \neq \chi_M$, then $f^{-1}(B)$ is an ILFCPSM of M.*
2. *Suppose that f is an isomorphism. If A is an ILFCPSM of M, then $f(A)$ is an ILFCPSM of $M^{'}$.*

Proof. (i) Suppose that B is an ILFCPSM of $M^{'}$ with $f^{-1}(B) \neq \chi_M$. Let $a_{(p,q)}, b_{(t,s)} \in ILFP(R)$, $x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq f^{-1}(B)$
$\Rightarrow (abx)_{(p\wedge s\wedge u, q\vee t\vee v)} \subseteq f^{-1}(B)$
$\Rightarrow \mu_{f^{-1}(B)}(abx) \geq p \wedge s \wedge u$ and $\nu_{f^{-1}(B)}(abx) \leq q \vee t \vee v$
$\Rightarrow \mu_B(f(abx)) \geq p \wedge s \wedge u$ and $\nu_B(f(abx)) \leq q \vee t \vee v$
$\Rightarrow \mu_B(abf(x)) \geq p \wedge s \wedge u$ and $\nu_B(abf(x)) \leq q \vee t \vee v$
$\Rightarrow (abf(x))_{(p\wedge s\wedge u, q\vee t\vee v)} \subseteq B$
$\Rightarrow a_{(p,q)}b_{(t,s)}(f(x))_{(u,v)} \subseteq B$.
As B is an ILFCPSM of $M^{'}$ this implies $a_{(p,q)}(f(x))_{(u,v)} \subseteq B$ or $b_{(t,s)}(f(x))_{(u,v)} \subseteq B$
$\Rightarrow (af(x))_{(p\wedge u, q\vee v)} \subseteq B$ or $(bf(x))_{(t\wedge u, s\vee v)} \subseteq B$
$\Rightarrow (f(ax))_{(p\wedge u, q\vee v)} \subseteq B$ or $(f(bx))_{(t\wedge u, s\vee v)} \subseteq B$
$\Rightarrow \mu_B(f(ax)) \geq p \wedge u$, $\nu_B(f(ax)) \leq q \vee v)$ or $\mu_B(f(bx)) \geq t \wedge u$, $\nu_B(f(bx)) \leq s \vee v$
$\Rightarrow \mu_{f^{-1}(B)}(ax) \geq p \wedge u$, $\nu_{f^{-1}(B)}(ax) \leq q \vee v$ or $\mu_{f^{-1}(B)}(bx) \geq t \wedge u$, $\nu_{f^{-1}(B)}(bx) \leq s \vee v$
$\Rightarrow (ax)_{(p\wedge u, q\vee v)} \subseteq f^{-1}(B)$ or $(bx)_{(t\wedge u, s\vee v)} \subseteq f^{-1}(B)$
$\Rightarrow a_{(p,q)}x_{(u,v)} \subseteq f^{-1}(B)$ or $b_{(t,s)}x_{(u,v)} \subseteq f^{-1}(B)$.
Hence $f^{-1}(B)$ is an ILFCPSM of M.

(ii) Assume that A is an ILFCPSM of M. Let $a_{(p,q)}, b_{(t,s)} \in ILFP(R)$, $y_{(u_1,v_1)} \in ILFP(M^{'})$. As f is an isomorphism so we have unique $x \in M$ such that $f(x) = y$. Suppose that $a_{(p,q)}b_{(t,s)}y_{(u_1,v_1)} \subseteq f(A)$ so that $a_{(p,q)}b_{(t,s)}(f(x))_{(u_1,v_1)} \subseteq f(A)$. This implies that $(f(abx))_{(p\wedge s\wedge u_1, q\vee t\vee v_1)} \subseteq f(A)$, i.e., $\mu_{f(A)}(f(abx)) \geq p \wedge s \wedge u_1$ and $\nu_{f(A)}(f(abx)) \leq q \vee t \vee v_1$. As f is an isomorphism, then we have $\mu_A(abx) \geq p \wedge s \wedge u_1$ and $\nu_A(abx) \leq q \vee t \vee v_1$. This implies $(abx)_{(p\wedge s\wedge u_1, q\vee t\vee v_1)} \subseteq A$, i.e., $a_{(p,q)}b_{(t,s)}x_{(u_1,v_1)} \subseteq A$. As A is an ILFCPSM of M, therefore either $a_{(p,q)}x_{(u_1,v_1)} = (ax)_{(p\wedge u_1, q\vee v_1)} \subseteq A$ or $b_{(t,s)}x_{(u_1,v_1)} = (bx)_{(t\wedge u_1, s\vee v_1)} \subseteq A$ which further imply $f((ax)_{(p\wedge u_1, q\vee v_1)}) \subseteq f(A)$ or $f((bx)_{(t\wedge u_1, s\vee v_1)}) \subseteq f(A)$. Thus $a_{(p,q)}(f(x))_{(u_1,v_1)} \subseteq f(A)$ or $b_{(t,s)}(f(x))_{(u_1,v_1)} \subseteq f(A)$ and so $a_{(p,q)}y_{(u_1,v_1)} \subseteq f(A)$ or $b_{(t,s)}y_{(u_1,v_1)} \subseteq f(A)$. Hence $f(A)$ is an ILFCPSM of $M^{'}$. □

4 Intuitionistic L-fuzzy 2-absorbing submodules

In this section, we introduce the concepts of intuitionistic L-fuzzy 2-absorbing submodules and intuitionistic L-fuzzy strongly 2-absorbing submodules. We give some basic properties of these classes of intuitionistic L-fuzzy submodules and then investigate the interplay between 2-absorbing submodules and intuitionistic L-fuzzy 2-absorbing submodules.

Definition 4.1. Let A be a non-constant ILFSM of B. A is called an intuitionistic L-fuzzy 2-absorbing submodule (ILF2ASM) of B if for any ILFPs $a_{(p,q)}, b_{(t,s)}$ of R and $x_{(u,v)}$ of M $(a, b \in R, x \in M, p, q, r, s, u, v \in L)$, $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$ implies that $a_{(p,q)}b_{(t,s)} \subseteq (A : B)$ or $a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$. A is called an ILF2ASM of M if it is an ILF2ASM of χ_M.

Definition 4.2. Let A be an ILFSM of M. Then A is said to be an intuitionistic L-fuzzy strongly 2- absorbing submodule (ILFS2ASM) of M if it is non-constant and whenever $C, D \in ILFI(R)$ and $E \in ILFSM(M)$ with $CDE \subseteq A$, then $CD \subseteq (A : \chi_M)$ or $CE \subseteq A$ or $DE \subseteq A$.

Theorem 4.3. (1) *Every intuitionistic L-fuzzy classical prime submodule of M is an intuitionistic L-fuzzy 2-absorbing submodule.*

(2) *Every intuitionistic L-fuzzy prime submodule of M is an intuitionistic L-fuzzy strongly 2-absorbing submodule.*

(3) *Every intuitionistic L-fuzzy strongly 2-absorbing submodule of M is a intuitionistic L-fuzzy 2-absorbing submodule.*

Proof. (1) and (2) are mmediate consequences of the definition.

(3) Let A be an intuitionistic L-fuzzy strongly 2-absorbing submodule of M. Assume that $a_{(p,q)}, b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ with $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$. Then by Definition (2.4) $\langle a_{(p,q)}\rangle\langle b_{(t,s)}\rangle\langle x_{(u,v)}\rangle = \langle a_{(p,q)}b_{(t,s)}x_{(u,v)}\rangle \subseteq A$. Since A is an intuitionistic L-fuzzy strongly 2-absorbing submodule, we have $\langle a_{(p,q)}b_{(t,s)}\rangle = \langle a_{(p,q)}\rangle\langle b_{(t,s)}\rangle \subseteq (A : \chi_M)$ or $\langle a_{(p,q)}x_{(u,v)}\rangle = \langle a_{(p,q)}\rangle\langle x_{(u,v)}\rangle \subseteq A$ or $\langle b_{(t,s)}x_{(u,v)}\rangle = \langle b_{(t,s)}\rangle\langle x_{(u,v)}\rangle \subseteq A$. Therefore, $a_{(p,q)}b_{(t,s)}\chi_M \subseteq A$ or $a_{(p,q)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}x_{(u,v)} \subseteq A$, i.e., A is an intuitionistic L-fuzzy 2-absorbing submodule of M. □

By Theorem (4.3), every intuitionistic L-fuzzy prime submodule is an intuitionistic L-fuzzy 2-absorbing submodule, but the converse is not necessarily true. See the following example:

Example 4.4. Consider $M = R = \mathbb{Z}$. Then M is a $\mathbb{Z}$-module. Let p and q be a pair of distinct prime numbers, and set $J = pq\mathbb{Z}$ is a 2-absorbing submodule of M. Now define an IFS A of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in pq\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in pq\mathbb{Z} \\ 1, & \text{otherwise} \end{cases}.$$

Then A is an intuitionistic L-fuzzy 2-absorbing submodule of M. Moreover $A_* = J$ is a 2-absorbing submodule of M that is not a prime submodule of M. Hence A is not an intuitionistic L-fuzzy prime submodule of M.

Theorem 4.5. *If A be an ILF2ASM of B, then $A_{(\alpha,\beta)}$ is a 2-absorbing submodule of $B_{(\alpha,\beta)}$ for every $\alpha, \beta \in L$ such that $A_{(\alpha,\beta)} \neq B_{(\alpha,\beta)}$.*

Proof. Let $abx \in A_{(\alpha,\beta)}$ for some $a, b \in R, x \in M$. In this case $\mu_A(abx) \geq \alpha, \nu_A(abx) \leq \beta$. We get $a_{(\alpha,\beta)}b_{(\alpha,\beta)}x_{(\alpha,\beta)} = (abx)_{(\alpha,\beta)} \subseteq A$. As A is an intuitionistic L-fuzzy 2-absorbing submodule of M, we have $a_{(\alpha,\beta)}b_{(\alpha,\beta)}B \subseteq A$ or $a_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$ or $b_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$.

If $(abx)_{(\alpha,\beta)} \subseteq A$, then for every $w \in abB_{(\alpha,\beta)}$, we have $w = aby$ for some $y \in B_{(\alpha,\beta)}$. Then $\mu_B(y) \geq \alpha, \nu_B(y) \leq \beta$. Now $\alpha = \alpha \wedge \mu_B(y) \leq \sup_{w=abx}\{\alpha \wedge \mu_B(x)\} = \mu_{(ab)_{(\alpha,\beta)}B}(w) \leq \mu_A(w)$. Similarly, $\beta = \beta \vee \nu_B(y) \geq \inf_{w=abx}\{\beta \vee \nu_B(x)\} = \nu_{(ab)_{(\alpha,\beta)}B}(w) \geq \nu_A(w)$. Therefore, $\mu_A(w) \geq \alpha, \nu_A(w) \leq \beta$ imply $w \in A_{(\alpha,\beta)}$ and so $abB_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$, i.e., $ab \in (A_{(\alpha,\beta)} : B_{(\alpha,\beta)})$.

If $(ax)_{(\alpha,\beta)} = a_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$, then $ax \in A_{(\alpha,\beta)}$. Similarly, if $(bx)_{(\alpha,\beta)} = b_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$, then $bx \in A_{(\alpha,\beta)}$. This implies that $A_{(\alpha,\beta)}$ is a 2-absorbing submodule of $B_{(\alpha,\beta)}$. □

Corollary 4.6. *If A is an intuitionistic L-fuzzy 2-absorbing submodule of M, then A_* is a 2-absorbing submodule of M.*

Proof. The result follows from Theorem (4.5) since A is an intuitionistic L-fuzzy 2-absorbing submodule; hence it is a non-constant intuitionistic L-fuzzy submodule of M and so $A_* \neq M$. □

Definition 4.7. Let $\alpha \in L\backslash\{1\}$. Then α is called a 2-absorbing element of L if $p \wedge s \wedge u \leq \alpha$ implies that $p \wedge s \leq \alpha$ or $s \wedge u \leq \alpha$ or $p \wedge u \leq \alpha$ for all $p, s, u \in L$.

Theorem 4.8. *Assume that N is a 2-absorbing submodule of M and let α be a 2-absorbing element of L and $\alpha^{'}$ be its complement in L. If A is an ILFS of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha^{'}, & \text{otherwise} \end{cases}$$

for all $x \in M$, then A is an intuitionistic L-fuzzy 2-absorbing submodule of M.

Proof. Assume that N is a 2-absorbing submodule of M. Then N is a proper submodule of M. Therefore, A is a non-constant intuitionistic L-fuzzy submodule of M. Suppose that $a_{(p,q)}$, $b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$ but $a_{(p,q)}x_{(u,v)} \nsubseteq A$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A$. In this case $\mu_A(ax) \geq \alpha, \nu_A(ax) \leq \alpha^{'}$. Therefore, $ax \notin N$. Similarly, we get $bx \notin N$. Moreover from $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$, we have $p \wedge s \wedge u = \mu_{(abx)_{(p\wedge s\wedge u, s\vee t\vee v)}}(abx) \leq \mu_A(abx)$ imply that $\mu_A(abx) \geq p \wedge s \wedge u$. Similarly, we get $\nu_A(abx) \leq s \vee t \vee v$.

If $\mu_A(abx) = 1, \nu_A(abx) = 0$, then from $abx \in N$, $ax \notin N, bx \notin N$, we get $ab \in (N : M)$ since N is a 2-absorbing submodule of M, then $\mu_A(abm) = \alpha, \nu_A(abm) = \alpha^{'}, \forall m \in M$. Now we have $p \wedge s = \mu_{a_{(p,q)}b_{(t,s)}\chi_M}(abm) \leq \mu_A(abm)$ and $q \vee t = \nu_{a_{(p,q)}b_{(t,s)}\chi_M}(abm) \geq \nu_A(abm)$.

If $\mu_A(abx) = \alpha, \nu_A(abx) = \alpha^{'}$, then from $p \wedge s \wedge u \leq \alpha, p \wedge u \nleq \alpha, s \wedge u \nleq \alpha$. Since α is a 2-absorbing element of L, we get $p \wedge s \leq \alpha$. In this case we have $\mu_{a_{(p,q)}b_{(t,s)}\chi_M}(w) = p \wedge s \leq \alpha \leq \mu_A(w)$. Similarly, we can show that $\nu_{a_{(p,q)}b_{(t,s)}\chi_M}(w) \geq \alpha^{'} \geq \nu_A(w)$, for all $w \in M$. Therefore, $a_{(p,q)}b_{(t,s)} \in (A : \chi_M)$, that is A is an intuitionistic L-fuzzy 2-absorbing submodule of M. □

Theorem 4.9. *The intersection of two intuitionistic L-fuzzy prime submodules of an R-module M is an intuitionistic L-fuzzy 2-absorbing submodule.*

Proof. Let A_1, A_2 be two ILFPSMs of an R-module M. Suppose that $A = A_1 \cap A_2$. Consider $a_{(p,q)}, b_{(t,s)} \in ILFP(R), x_{(u,v)} \in ILFP(M)$ such that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A$ but $a_{(p,q)}x_{(u,v)} \nsubseteq A$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A$. Then $a_{(p,q)}x_{(u,v)} \nsubseteq A_1$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_1$ and $a_{(p,q)}x_{(u,v)} \nsubseteq A_2$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_2$, but these are impossible as A_1, A_2 are ILFPSMs of M. So suppose $a_{(p,q)}x_{(u,v)} \nsubseteq A_1$ and $b_{(t,s)}x_{(u,v)} \nsubseteq A_2$. Since $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A = A_1 \cap A_2$ implies that $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_1$ and $a_{(p,q)}b_{(t,s)}x_{(u,v)} \subseteq A_2$. Then $a_{(p,q)}b_{(t,s)} \in (A_1 : \chi_M)$ and $a_{(p,q)}b_{(t,s)} \in (A_2 : \chi_M)$ and so $a_{(p,q)}b_{(t,s)} \in (A_1 : \chi_M) \cap (A_2 : \chi_M) = (A_1 \cap A_2 : \chi_M) = (A : \chi_M)$. Hence $A = A_1 \cap A_2$ is an ILF2ASM of M. □

Remark 4.10. The intersection of two intuitionistic L-fuzzy 2-absorbing submodules need not be an intuitionistic L-fuzzy 2-absorbing submodule.

Example 4.11. Consider $M = R = \mathbb{Z}$. Then M is a $\mathbb{Z}$-module. Let $p < q < r$ be three distinct prime numbers, and set $J = pq\mathbb{Z}$ and $J' = pr\mathbb{Z}$ are 2-absorbing submodules of M. Now define IFSs A and B of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in pq\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in pq\mathbb{Z} \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0.5, & \text{if } x \in pr\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in pr\mathbb{Z} \\ 1, & \text{otherwise} \end{cases}.$$

Then A and B are intuitionistic L-fuzzy 2-absorbing submodules of M (see Example (4.4)). Also, then

$$\mu_{A\cap B}(x) = \begin{cases} 0.5, & \text{if } x \in pqr\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{A\cap B}(x) = \begin{cases} 0, & \text{if } x \in pqr\mathbb{Z} \\ 1, & \text{otherwise} \end{cases},$$

where $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$, for all $x \in M$.

Here we note that $A \cap B$ is not an ILF2APSM of M.

Consider $p_{(0.25,0.5)}, q_{(0.25,0.5)}, r_{(0.25,0.5)} \in ILFP(Z)$. Now, $p_{(0.25,0.5)}q_{(0.25,0.5)}r_{(0.25,0.5)}$. Also, then $\mu_{A\cap B}(pqr) = min\{\mu_A(pqr), \mu_B(pqr)\} = min\{1, 0.5\} = 0.5 > 0.25 = \mu_{(pqr)_{(0.25,0.5)}}(pqr)$. Similarly, $\nu_{A\cap B}(pqr) = max\{\nu_A(pqr), \nu_B(pqr)\} = max\{0, 0\} = 0 < 0.5 = \nu_{(pqr)_{(0.25,0.5)}}(pqr)$. Therefore, $(pqr)_{(0.25,0.5)} \subseteq A \cap B$, i.e., $p_{(0.25,0.5)}q_{(0.25,0.5)}r_{(0.25,0.5)} \subseteq A \cap B$. But it is easy to see that $p_{(0.25,0.5)}q_{(0.25,0.5)} \nsubseteq A \cap B$ and $p_{(0.25,0.5)}r_{(0.25,0.5)} \nsubseteq A \cap B$ and $q_{(0.25,0.5)}r_{(0.25,0.5)} \nsubseteq A \cap B$.

Proposition 4.12. *If A be an intuitionistic L-fuzzy 2-absorbing submodule of an R-module M, then $(A : x_{(u,v)})$ is an intuitionistic L-fuzzy 2-absorbing ideal of R for all $x_{(u,v)} \notin A$.*

Proof. Let $x_{(u,v)} \in ILFP(M)$ such that $x_{(u,v)} \notin A$. Then $(A : x_{(u,v)})$ is a proper ILFI of R. Assume that $a_{(p,q)}b_{(t,s)}c_{(i,j)} \subseteq (A : x_{(u,v)})$ for all $a, b, c \in R, p, q, t, s, i, j \in L$. Then $a_{(p,q)}b_{(t,s)}(c_{(i,j)}x_{(u,v)}) \subseteq A$. Since A is an ILF2ASM of M, then $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M) \subseteq$

$(A : x_{(u,v)})$ or $a_{(p,q)}c_{(i,j)}x_{(u,v)} \subseteq A$ or $b_{(t,s)}c_{(i,j)}x_{(u,v)} \subseteq A$, i.e., $a_{(p,q)}b_{(t,s)} \subseteq (A : x_{(u,v)})$ or $a_{(p,q)}c_{(i,j)} \subseteq (A : x_{(u,v)})$ or $b_{(t,s)}c_{(i,j)} \subseteq (A : x_{(u,v)})$. Thus $(A : x_{(u,v)})$ is an intuitionistic L-fuzzy 2-absorbing ideal of R. $\square$

Theorem 4.13. *If A be an intuitionistic L-fuzzy 2-absorbing submodule of an R-module M, then $(A : \chi_M)$ is an intuitionistic L-fuzzy 2-absorbing ideal of R.*

Proof. Let $a_{(p,q)}b_{(t,s)}c_{(u,v)} \subseteq (A : \chi_M)$ such that $a_{(p,q)}c_{(u,v)} \nsubseteq (A : \chi_M)$ and $b_{(t,s)}c_{(u,v)} \nsubseteq (A : \chi_M)$. We show that $a_{(p,q)}b_{(t,s)} \in (A : \chi_M)$. There exists $m_{(\alpha_1,\beta_1)}, n_{(\alpha_2,\beta_2)} \in ILFP(M)$ such that $a_{(p,q)}c_{(u,v)}m_{(\alpha_1,\beta_1)} \nsubseteq A$ and $b_{(t,s)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \nsubseteq A$ but $a_{(p,q)}b_{(t,s)}c_{(u,v)}m_{(\alpha_1,\beta_1)} \subseteq A$ and $a_{(p,q)}b_{(t,s)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \subseteq A$ and so $a_{(p,q)}b_{(t,s)}(c_{(u,v)}m_{(\alpha_1,\beta_1)} + c_{(u,v)}n_{(\alpha_2,\beta_2)}) \subseteq A$.

By assumption, A is an intuitionistic L-fuzzy 2-absorbing submodule of M so that $a_{(p,q)}(c_{(u,v)}m_{(\alpha_1,\beta_1)} + c_{(u,v)}n_{(\alpha_2,\beta_2)}) \subseteq A$ or $b_{(t,s)}(c_{(u,v)}m_{(\alpha_1,\beta_1)} + c_{(u,v)}n_{(\alpha_2,\beta_2)}) \subseteq A$ or $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$.

If $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$, we are done. If $a_{(p,q)}(c_{(u,v)}m_{(\alpha_1,\beta_1)} + c_{(u,v)}n_{(\alpha_2,\beta_2)}) \subseteq A$, then $a_{(p,q)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \nsubseteq A$, since $a_{(p,q)}c_{(u,v)}m_{(\alpha_1,\beta_1)} \nsubseteq A$ implies $a_{(p,q)}c_{(u,v)}m_{(\alpha_1,\beta_1)} \subseteq A$ which is a contradiction. Now $a_{(p,q)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \nsubseteq A$ and $b_{(t,s)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \nsubseteq A$ while $a_{(p,q)}b_{(t,s)}c_{(u,v)}n_{(\alpha_2,\beta_2)} \subseteq A$, then $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$. With a same argument, we can show that if $b_{(t,s)}(c_{(u,v)}m_{(\alpha_1,\beta_1)} + c_{(u,v)}n_{(\alpha_2,\beta_2)}) \subseteq A$, then $a_{(p,q)}b_{(t,s)} \subseteq (A : \chi_M)$. This complete the proof. $\square$

5 Conclusion

In this paper, we have introduced and examined the two generalised notions of intuitionistic L-fuzzy prime submodules of an R-module M, where R is a commutative ring with unity, namely the intuitionistic L-fuzzy classical prime submodules and the intuitionistic L-fuzzy 2-absorbing submodule. It is obvious from their definitions that every intuitionistic L-fuzzy prime submodule is an intuitionistic L-fuzzy classical submodule, and that every intuitionistic L-fuzzy classical submodule is an intuitionistic L-fuzzy 2-absorbing submodule, but its converse part is not always true. We obtained the condition when the converse part is also true (see Theorem (3.9)). We have also proved that in the case of multiplication module M, the two notions of intuitionistic L-fuzzy prime submodules and intuitionistic L-fuzzy classical submodules coincide (see Remark (3.11)). Further, we have shown that the intersection of two intuitionistic L-fuzzy prime submodules of an R-module M is an intuitionistic L-fuzzy 2-absorbing submodule (see Theorem (4.9)). However, the intersection of two intuitionistic L-fuzzy 2-absorbing submodules need not be an intuitionistic L-fuzzy 2-absorbing submodule (see Example (4.11)). Besides these, many other related findings were achieved.

References

[1] Anderson, F. W., & Fuller, K. R. (1992). *Rings and Categories of Modules* (2nd edition). Springer Verlag.

[2] Atanassov, K., & Stoeva, S. (1984). Intuitionistic L-fuzzy sets. *Cybernetics and System Research*, 2, Elsevier Sci. Publ. Amsterdam, 539–540.

[3] Atani, S. E., & Farzalipour, F. (2007). On weakly prime submodules. *Tamkang Journal of Mathematics*, 38(3), 247–252.

[4] Azizi, A. (2008). On prime and weakly prime submodules. *Vietnam Journal of Mathematics*, 36(3), 315–325.

[5] Badawi, A. (2007). On 2-absorbing ideals of commutative rings. *Bulletin of the Australian Mathematical Society*, 75, 417–429.

[6] Bakhadach, I., Melliani, S., Oukessou, M. S., & Chadli, L. S. (2016). Intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal in a ring. *Notes on Intuitionistic Fuzzy Sets*, 22(2), 59–63.

[7] Behboodi, M., & Koohy, H. (2004). Weakly prime modules. *Vietnam Journal of Mathematics*, 32(2), 185–195.

[8] Behboodi, M., & Shojaee, S. H. (2010). On chains of classical prime submodules and dimension theory of modules. *Bulletin of the Iranian Mathematical Society*, 36, 149–166.

[9] Birkhoff, G. (1967). *Lattice Theory*. American Mathematical Society, Providence, Rhode Island.

[10] Darani, A. Y., & Soheilnia, F. (2011). 2-absorbing and weakly 2-absorbing submodules. *Thai Journal of Mathematics*, 9(3), 577–584.

[11] Dauns, J. (1978). Prime modules. *Journal für die reine und angewandte Mathematik*, 298, 156–181.

[12] Deschrijver, G., & Kerre, E. E. (2003). On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, 133 227–235.

[13] Goguen, J. (1967). L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18, 145–174.

[14] Jamali, M., & Nezhad, R. J. (2022). On classical weakly prime submodules. *Facta Universitatis, Series: Mathematics and Informatics*, 37(1), 17–30.

[15] Kanchan, Sharma, P. K., & Pathania, D. S. (2020). Intuitionistic L-fuzzy submodules. *Advances in Fuzzy Sets and Systems,* 25(2), 123–142.

[16] Kasch, F. (1982). *Modules and Rings*. Academic Press, London.

[17] Meena, K., & Thomas, K. V. (2011). Intuitionistic L-fuzzy subrings. *International Mathematics Forum* 6(52), 2561–2572.

[18] Palaniappan, N., Naganathan, S., & Arjunan, K. (2009). A study on Intuitionistic L-fuzzy Subgroups. *Applied Mathematics Sciences*, 3(53), 2619–2624.

[19] Sharma, P. K. (2022). On intuitionistic fuzzy multiplication modules. *Annals of Fuzzy Mathematics and Informatics*, 23(3), 295–309.

[20] Sharma, P. K. (2022). On intuitionistic fuzzy primary ideal of a ring. *Pan-American Journal of Mathematics*, 1, 1–11.

[21] Sharma, P. K., & Kanchan. (2018). On intuitionistic L-fuzzy prime submodules. *Annals of Fuzzy Mathematics and Informatics*, 16(1), 87–97.

[22] Sharma, P. K., & Kanchan. (2020). On intuitionistic L-fuzzy primary and P-primary submodules. *Malaya Journal of Matematik*, 8(4), 1417–1426.

[23] Sharma, P. K., & Kanchan. (2021). On the topological structure of spectrum of intuitionistic L-fuzzy prime submodules. *Annals of Fuzzy Mathematics and Informatics*, 21(2), 195–215.

[24] Wang, G. J., & He, Y. Y. (2000). Intuitionistic fuzzy sets and L-fuzzy sets. *Fuzzy Sets and Systems*, 110, 271–274.

[25] Yavuz, S., Onar, S., Sonmez, D., & Ersoy, B. A. (2017). Intuitionistic fuzzy 2-absorbing primary ideals of commutative rings. *Journal of Hyperstructures*, 6, 56–71.